On LS-Category of a Family of Rational Elliptic Spaces II

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Abstract: Let X be a finite type simply connected rationally elliptic CW-complex with Sullivan minimal model $(\Lambda V, d)$ and let $k \geq 2$ the biggest integer such that $d = \sum_{i \geq k} d_i$ with $d_i(V) \subseteq \Lambda^i V$. If $(\Lambda V, d_k)$ is moreover elliptic then $\text{cat}(\Lambda V, d) = \text{cat}(\Lambda V, d_k) = \dim(V_{\text{even}})(k - 2) + \dim(V_{\text{odd}})$. Our work aims to give an almost explicit formula of LS-category of such spaces in the case when $k \geq 3$ and when $(\Lambda V, d_k)$ is not necessarily elliptic.

Key words: Elliptic spaces, Lusternik-Schnirelman category, Toomer invariant.


1. Introduction

The Lusternik-Schirelmann category (c.f. [7]), $\text{cat}(X)$, of a topological space $X$ is the least integer $n$ such that $X$ can be covered by $n + 1$ open subsets of $X$, each contractible in $X$ (or infinity if no such $n$ exists). It is an homotopy invariant (c.f. [3]). For $X$ a simply connected CW complex, the rational L-S category, $\text{cat}_0(X)$, introduced by Félix and Halperin in [2] is given by $\text{cat}_0(X) = \text{cat}(X_Q) \leq \text{cat}(X)$.

In this paper, we assume that $X$ is a simply connected topological space whose rational homology is finite dimensional in each degree. Such space has a Sullivan minimal model $(\Lambda V, d)$, i.e. a commutative differential graded algebra coding both its rational homology and homotopy (cf. §2).

By [1, Definition 5.22] the rational Toomer invariant of $X$, or equivalently of its Sullivan minimal model, denoted by $e_0(\Lambda V, d)$, is the largest integer $s$ for which there is a non trivial cohomology class in $H^*(\Lambda V, d)$ represented by a cocycle in $\Lambda^{\geq s} V$, this coincides in fact with the Toomer invariant of the fundamental class of $(\Lambda V, d)$. As usual, $\Lambda^s V$ denotes the elements in $\Lambda V$ of “wordlength” $s$. For more details [1], [3] and [14] are standard references.

In [4] Y. Félix, S. Halperin and J.M. Lemaire showed that for Poincaré duality spaces, the rational L-S category coincides with the rational Toomer
invariant $e_0(X)$, and in [9] A. Murillo gave an expression of the fundamental class of $(\Lambda V, d)$ in the case where $(\Lambda V, d)$ is a pure model (cf. §2).

Let then $(\Lambda V, d)$ be a Sullivan minimal model. The differential $d$ is decomposable, that is, $d = \sum_{i \geq k} d_i$, with $d_i(V) \subseteq \Lambda^i V$ and $k \geq 2$.

Recall first that in [8] the authors gave the explicit formula $\text{cat}(\Lambda V, d) = \text{dim} V^{\text{odd}} + (k - 2) \text{dim} V^{\text{even}}$ in the case when $(\Lambda V, d_k)$ is also elliptic.

The aim of this paper is to consider another class of elliptic spaces whose Sullivan minimal model $(\Lambda V, d)$ is such that $(\Lambda V, d_k)$ is not necessarily elliptic. To do this we filter this model by

$$F^p = \Lambda^i p V = \bigoplus_{i = (k-1)p} \Lambda^i V. \tag{1}$$

This gives us the main tool in this work, that is the following convergent spectral sequence (cf. §3):

$$H^{p,q}(\Lambda V, \delta) \Rightarrow H^{p+q}(\Lambda V, d). \tag{2}$$

Notice first that, if dim$(V) < \infty$ and $(\Lambda V, d)$ has finite dimensional cohomology, then $(\Lambda V, d)$ is elliptic. This gives a new family of rationally elliptic spaces.

In the first step, we shall treat the case under the hypothesis assuming that $H^N(\Lambda V, \delta)$ is one dimensional, being $N$ the formal dimension of $(\Lambda V, d)$ (cf. [5]). For this, we will combine the method used in [8] and a spectral sequence argument using (2). We then focus on the case where dim$H^N(\Lambda V, \delta) \geq 2$. Our first result reads:

**Theorem 1.** If $(\Lambda V, d)$ is elliptic, $(\Lambda V, d_k)$ is not elliptic and $H^N(\Lambda V, \delta) = \mathbb{Q}, \alpha$ is one dimensional, then

$$\text{cat}_0(X) = \text{cat}(\Lambda V, d) = \sup \{ s \geq 0, \alpha = [\omega_0] \text{ with } \omega_0 \in \Lambda^s V \}.$$  

Let us explain in what follow, the algorithm that gives the first inequality,

$$\text{cat}(\Lambda V, d) \geq \sup \{ s \geq 0, \alpha = [\omega_0] \text{ with } \omega_0 \in \Lambda^s V \} := r.$$  

i) Initially we fix a representative $\omega_0 \in \Lambda^s V$ of the fundamental class $\alpha$ with $r$ being the largest $s$ such that $\omega_0 \in \Lambda^s V$. 
ii) A straightforward calculation gives successively:
\[ \omega_0 = \omega_0^0 + \omega_1^0 + \cdots + \omega_l^0 \]
with
\[ \omega_i^0 = (\omega_i^{0,0}, \omega_i^{0,1}, \ldots, \omega_i^{0,k-2}) \in \Lambda^{(k-1)(p+i)}V \oplus \Lambda^{(k-1)(p+i)+1}V \]
\[ \oplus \cdots \oplus \Lambda^{(k-1)(p+i)+k-2}V. \]

Using \( \delta(\omega_0) = 0 \) we obtain
\[ d\omega_0 = a_0^0 + a_0^1 + \cdots + a_0^{l+1} \]
with
\[ a_i^0 = (a_i^{0,0}, a_i^{0,1}, \ldots, a_i^{0,k-2}) \in \Lambda^{(k-1)(p+i)}V \oplus \Lambda^{(k-1)(p+i)+1}V \]
\[ \oplus \cdots \oplus \Lambda^{(k-1)(p+i)+k-2}V. \]

iii) We take the largest integer satisfying the inequality:
\[ t \leq \frac{1}{2(k-1)} (N - 2(k-1)(p+l) - 2k + 5). \]
Since \( d^2 = 0 \), it follows that \( a_0^2 = \delta(b_2) \) for some
\[ b_2 \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2)-(k-1)+j}V. \]

iv) We continue with \( \omega_1 = \omega_0 - b_2. \)

v) By the imposition iii), the algorithm leads to a representative \( \omega_{t+l-1} \in \Lambda^{\geq r}V \) of the fundamental class of \((\Lambda V, d)\) and then \( e_0(\Lambda V, d) \geq r. \)

Now, \( \dim(V) < \infty \) imply \( \dim H^N(\Lambda V, \delta) < \infty. \) Notice also that the filtration (1) induces on cohomology a graduation such that \( H^N(\Lambda V, \delta) = \bigoplus_{p+q=N} H^p(\Lambda V, \delta). \) There is then a basis \( \{\alpha_1, \ldots, \alpha_m\} \) of \( H^N(\Lambda V, \delta) \) with \( \alpha_j \in H^{p_jq_j}(\Lambda V, \delta), (1 \leq j \leq m). \) Denote by \( \omega_{0j} \in \Lambda^{\geq r}V \) a representative of the generating class \( \alpha_j \) with \( r_j \) being the largest \( s_j \) such that \( \omega_{0j} \in \Lambda^{\geq s_j}V. \) Here \( p_j \) and \( q_j \) are filtration degrees and \( r_j = \{p_j(k-1), \ldots, p_j(k-1)+(k-2)\}. \)

The second step in our program is given as follow:

**Theorem 2.** If \((\Lambda V, d)\) is elliptic and \( \dim H^N(\Lambda V, \delta) = m \) with basis \( \{\alpha_1, \ldots, \alpha_m\}, \) then, there exists a unique \( p_j, \) such that
\[ \text{cat}_0(X) = \sup \{s \geq 0, \ \alpha_j = [\omega_{0j}] \text{ with } \omega_{0j} \in \Lambda^{\geq s}V \} := r_j. \]
Remark 1. The previous theorem gives us also an algorithm to determine LS-category of any elliptic Sullivan minimal model \((\Lambda V, d)\). Knowing the largest integer \(k \geq 2\) such that \(d = \sum_{i \geq k} d_i\) with \(d_i(V) \subseteq \Lambda^i V\) and the formal dimension \(N\) (this one is given in terms of degrees of any basis elements of \(V\)), one has to check for a basis \(\{\alpha_1, \ldots, \alpha_m\}\) of \(H^N(\Lambda V, \delta)\) (which is finite dimensional since \(\dim(V) < \infty\)). The NP-hard character of the problem into question, as it is proven by L. Lechuga and A. Murillo (cf [12]), sits in the determination of the unique \(j \in \{1, \ldots, m\}\) for which a represents cocycle \(\omega_{0j}\) of \(\alpha_j\) survives to reach the \(E_\infty\) term in the spectral sequence (2).

2. Basic facts

We recall here some basic facts and notation we shall need.

A simply connected space \(X\) is called rationally elliptic if \(\dim H^*(X, \mathbb{Q}) < \infty\) and \(\dim(X) \otimes \mathbb{Q} < \infty\).

A commutative graded algebra \(H\) is said to have formal dimension \(N\) if \(H^p = 0\) for all \(p > N\), and \(H^N \neq 0\). An element \(0 \neq \omega \in H^N\) is called a fundamental class.

A Sullivan algebra ([3]) is a free commutative differential graded algebra \((\Lambda V, d)\) where \(\Lambda V = \text{Exterior}(V_{\text{odd}}) \otimes \text{Symmetric}(V_{\text{even}})\) generated by the graded \(\mathbb{K}\)-vector space \(V = \bigoplus_{i=0}^{\infty} V^i\) which has a well ordered basis \(\{x_\alpha\}\) such that \(dx_\alpha \in \Lambda^i V\). Such algebra is said minimal if \(\deg(x_\alpha) < \deg(x_\beta)\) implies \(\alpha < \beta\). If \(V^0 = V^1 = 0\) this is equivalent to saying that \(d(V) \subseteq \bigoplus_{i=2}^{\infty} \Lambda^i V\).

A Sullivan model ([3]) for a commutative differential graded algebra \((A, d)\) (cdga for short) \((\Lambda V, d)\) (where \(\Lambda V = \text{Exterior}(V_{\text{odd}}) \otimes \text{Symmetric}(V_{\text{even}})\)) generated by the graded \(\mathbb{K}\)-vector space \(V = \bigoplus_{i=0}^{\infty} V^i\) which has a well ordered basis \(\{x_\alpha\}\) such that \(dx_\alpha \in \Lambda^i V\). Such algebra is said minimal if \(\deg(x_\alpha) < \deg(x_\beta)\) implies \(\alpha < \beta\). If \(V^0 = V^1 = 0\) this is equivalent to saying that \(d(V) \subseteq \bigoplus_{i=2}^{\infty} \Lambda^i V\).

A Sullivan model ([3]) for a commutative differential graded algebra \((A, d)\) is a quasi-isomorphism (morphism inducing isomorphism in cohomology) \((\Lambda V, d) \rightarrow (A, d)\) with source, a Sullivan algebra. If \(H^0(A) = K\), \(H^1(A) = 0\) and \(\dim(H^i(A, d)) < \infty\) for all \(i \geq 0\), then, [6, Th.7.1], this minimal model exists. If \(X\) is a topological space any minimal model of the polynomial differential forms on \(X\), \(A_{PL}(X)\), is said a Sullivan minimal model of \(X\).

\((\Lambda V, d)\) (or \(X\)) is said elliptic, if both \(V\) and \(H^*(\Lambda V, d)\) are finite dimensional graded vector spaces (see for example [3]).

A Sullivan minimal model \((\Lambda V, d)\) is said to be pure if \(d(V_{\text{even}}) = 0\) and \(d(V_{\text{odd}}) \subset \Lambda V_{\text{even}}\). For such one, A. Murillo [9] gave an expression of a cocycle representing the fundamental class of \(H(\Lambda V, d)\) in the case where \((\Lambda V, d)\) is elliptic. We recall this expression here:

Assume \(\dim V < \infty\), choose homogeneous basis \(\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_m\}\)
of $V^{\text{even}}$ and $V^{\text{odd}}$ respectively, and write
\[dy_j = a_j^1 x_1 + a_j^2 x_2 + \cdots + a_j^{n-1} x_{n-1} + a_j^n x_n, \quad j = 1, 2, \ldots, m,\]
where each $a_j^i$ is a polynomial in the variables $x_i, x_{i+1}, \ldots, x_n$, and consider the matrix,
\[
A = \begin{pmatrix}
a_1^1 & a_1^2 & \cdots & a_1^n \\
a_2^1 & a_2^2 & \cdots & a_2^n \\
\vdots & \vdots & & \vdots \\
a_m^1 & a_m^2 & \cdots & a_m^n
\end{pmatrix}.
\]

For any $1 \leq j_1 < \cdots < j_n \leq m$, denote by $P_{j_1 \cdots j_n}$ the determinant of the matrix of order $n$ formed by the columns $i_1, i_2, \ldots, i_n$ of $A$:
\[
\begin{pmatrix}
a_{j_1}^1 \\
a_{j_2}^1 \\
\vdots \\
a_{j_n}^1
\end{pmatrix}
\]
\[
\begin{pmatrix}
a_{j_1}^n \\
a_{j_2}^n \\
\vdots \\
a_{j_n}^n
\end{pmatrix}
\]

Then (see [9]) if $\dim H^*(AV, d) < \infty$, the element $\omega \in AV$,
\[
\omega = \sum_{1 \leq j_1 < \cdots < j_n \leq m} (-1)^{j_1 + \cdots + j_n} P_{j_1 \cdots j_n} y_1 \cdots \hat{y}_{j_1} \cdots \hat{y}_{j_n} \cdots y_m, \quad (3)
\]
is a cocycle representing the fundamental class of the cohomology algebra.

### 3. Our spectral sequence

Let $(AV, d)$ be a Sullivan minimal model, where $d = \sum_{i \geq k} d_i$ with $d_i(V) \subseteq \Lambda^i V$ and $k \geq 2$. We first recall the filtration given in the introduction:
\[
F^p = \Lambda^{\geq (k-1)p} V = \bigoplus_{i=(k-1)p}^\infty \Lambda^i V. \quad (4)
\]

$F^p$ is preserved by the differential $d$ and satisfies $F^p(AV) \otimes F^q(AV) \subseteq F^{p+q}(AV)$, $\forall p, q \geq 0$, so it is a filtration of differential graded algebras. Also, since
$F^0 = \Lambda V$ and $F^{p+1} \subseteq F^p$ this filtration is decreasing and bounded, so it induces a convergent spectral sequence. Its $0^{th}$-term is

$$E_0^{p,q} = \left( \frac{F^p}{F^{p+1}} \right)^{p+q} = \left( \frac{\Lambda^{\geq (k-1)p}V}{\Lambda^{\geq (k-1)(p+1)}V} \right)^{p+q}.$$

Hence, we have the identification:

$$E_0^{p,q} = \left( \Lambda^{(k-1)p}V \oplus \Lambda^{(k-1)p+1}V \oplus \cdots \oplus \Lambda^{(k-1)p+k-2}V \right)^{p+q},$$

with the product given by:

$$(u_0, u_1, \ldots, u_{k-2}) \otimes (u'_0, u'_1, \ldots, u'_{k-2}) = (v_0, v_1, \ldots, v_{k-2})$$

for all $(u_0, u_1, \ldots, u_{k-2}), (u'_0, u'_1, \ldots, u'_{k-2}) \in E_0^{p,q}$ with $v_m = \sum_{i+j=m} u_i u'_j$ and $m = 0, \ldots, k - 2$.

The differential on $E_0$ is zero, hence $E_1^{p,q} = E_0^{p,q}$ and so the identification above gives the following diagram:

$$E_1^{p,q} \xrightarrow{\delta} (\Lambda^{(k-1)p}V \oplus \Lambda^{(k-1)p+1}V \oplus \cdots \oplus \Lambda^{(k-1)p+k-2}V)^{p+q}$$

with $\delta$ defined as follows,

$$\delta(u_0, u_1, \ldots, u_{k-2}) = (w_k, w_{k+1}, \ldots, w_{2k-2}) \quad \text{with} \quad w_{k+j} = \sum_{i+j' = j} d_{k+i} u_{i'}.$$ 

Let $E_1^p = \bigoplus_{q \geq 0} E_1^{p,q}$ and $E_1^* = \bigoplus_{p \geq 0} E_1^{p,*} = \Lambda V$ as a graded vector space. In this general situation, the $1^{st}$-term is the graded algebra $\Lambda V$ provided with a differential $\delta$, which is not necessarily a derivation on the set of generators. That is, $(\Lambda V, \delta)$ is a commutative differential graded algebra, but it is not a Sullivan algebra. This gives, consequently, our spectral sequence:

$$E_2^{p,q} = H^{p,q}(\Lambda V, \delta) \Rightarrow H^{p+q}(\Lambda V, d).$$

Once more, using this spectral sequence, the algorithm completed by proves of claims that will appear, will give the appropriate generating class of $H^N(\Lambda V, \delta)$ that survives to the $\infty$ term. Accordingly, the explicit formula of LS category for this general case, is expressed in terms of the greater filtering degree of a represent of this class.
4. Proof of the main results

4.1. Proof of Theorem 1. Recall that \((\Lambda V, d)\) is assumed elliptic, so that, \(\text{cat}(\Lambda V, d) = c_0(\Lambda V, d)\) \([4]\). Notice also that the subsequent notations imposed us sometimes to replace a sum by some tuple and vice-versa.

4.1.1. The first inequality. In what follows, we put:

\[ r = \sup\{s \geq 0, \alpha = [\omega_0] \text{ with } \omega_0 \in \Lambda^{\geq s} V\}. \]

Denote by \(p\) the least integer such that \(p(k - 1) \leq r < (p + 1)(k - 1)\) and let then \(\omega_0 \in \Lambda^{> r} V\). We have

\[ \omega_0 \in (\Lambda^{(k-1)p} V \oplus \cdots \oplus \Lambda^{(k-1)p+k-2} V) \]
\[ \oplus (\Lambda^{(k-1)p+k-1} V \oplus \cdots \oplus \Lambda^{(k-1)p+2k-3} V) \]
\[ \oplus \cdots \]

Since \(|\omega_0| = N\) and \(\dim V < \infty\), there is an integer \(l\) such that

\[ \omega_0 = \omega_0^0 + \omega_0^1 + \cdots + \omega_0^l \]

with \(\omega_0^0 \neq 0\) and \(\forall i = 0, \ldots, l, \omega_0^i = (\omega_0^{i,0}, \omega_0^{i,1}, \ldots, \omega_0^{i,k-2}) \in \Lambda^{(k-1)(p+i)} V \oplus \cdots \oplus \Lambda^{(k-1)(p+i)+k-2} V.\)

We have successively:

\[ \delta(\omega_0^i) = \delta\left(\omega_0^{i,0}, \omega_0^{i,1}, \ldots, \omega_0^{i,k-2}\right) \]
\[ = \left( d_k \omega_0^{i,0}, \sum_{i'+i''=1} d_{k+i'} \omega_0^{i',i''}, \sum_{i'+i''=2} d_{k+i'} \omega_0^{i',i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'} \omega_0^{i',i''} \right), \]

\[ \delta(\omega_0) = \sum_{i=0}^l \delta\left(\omega_0^{i,0}, \omega_0^{i,1}, \ldots, \omega_0^{i,k-2}\right) \]
\[ = \sum_{i=0}^l \left( d_k \omega_0^{i,0}, \sum_{i'+i''=1} d_{k+i'} \omega_0^{i',i''}, \sum_{i'+i''=2} d_{k+i'} \omega_0^{i',i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'} \omega_0^{i',i''} \right). \]
Also, we have \( d\omega_0 = d\omega^0_0 + d\omega^1_0 + \cdots + d\omega^l_0 \), with:

\[
d\omega^0_0 = \left( \omega^{0,0}_0, \omega^{0,1}_0, \ldots, \omega^{0,k-2}_0 \right)
= \left( d_k\omega^{0,0}_0, \sum_{i' + i'' = 1} d_{k+i'}\omega^{0,i''}_0, \ldots, \sum_{i' + i'' = k-2} d_{k+i'}\omega^{0,i''}_0 \right) + \cdots
\in \left( \bigoplus_{k'=k-1}^{2k-3} \Lambda^{(k-1)p+k'} V \right) \oplus \cdots
\]

\[
d\omega^1_0 = \left( \omega^{1,0}_0, \omega^{1,1}_0, \ldots, \omega^{1,k-2}_0 \right)
= \left( d_k\omega^{1,0}_0, \sum_{i' + i'' = 1} d_{k+i'}\omega^{1,i''}_0, \ldots, \sum_{i' + i'' = k-2} d_{k+i'}\omega^{1,i''}_0 \right) + \cdots
\in \left( \bigoplus_{k'=2k-2}^{3k-4} \Lambda^{(k-1)p+k'} V \right) \oplus \cdots
\]

\[
\vdots
\]

\[
d\omega^i_0 = \left( \omega^{i,0}_0, \omega^{i,1}_0, \ldots, \omega^{i,k-2}_0 \right)
= \left( d_k\omega^{i,0}_0, \sum_{i' + i'' = 1} d_{k+i'}\omega^{i,i''}_0, \ldots, \sum_{i' + i'' = k-2} d_{k+i'}\omega^{i,i''}_0 \right) + \cdots
\in \left( \bigoplus_{k'=(k-1)p+(i+2)k-(i+3)}^{(k-1)p+(i+1)k-(i+1)} \Lambda^{(k-1)p+k'} V \right) \oplus \cdots
\]

Therefore

\[
d\omega_i = \sum_{i=0}^l \left( d_k\omega^{i,0}_0, \sum_{i' + i'' = 1} d_{k+i'}\omega^{i,i''}_0, \ldots, \sum_{i' + i'' = k-2} d_{k+i'}\omega^{i,i''}_0 \right) + \sum_{k'=2k-2}^{l} \left( d_{2k-2}\omega_0^{i,1} + (d_{2k-2} + d_{2k-3})\omega_0^{i,2} + \cdots + (d_{2k-2} + d_{2k-3} + \cdots + d_{k+1})\omega_0^{i,k-2} \right)
\]
that is:

\[ d\omega_0 = \delta(\omega_0) + \sum_{i=0}^l \left( d_{2k-2} \omega_0^{i,1} + (d_{2k-2} + d_{2k-3}) \omega_0^{i,2} + \cdots + (d_{2k-2} + \cdots + d_{k+1}) \omega_0^{i,k-2} \right) + \sum_{k' > 2k-2} d_{k'} \omega_0. \]

As \( \delta(\omega_0) = 0 \), we can rewrite:

\[ d\omega_0 = a_0^0 + a_3^0 + \cdots + a_{t+l}^0 \quad \text{with} \quad a_i^0 = (a_{i,0}^0, a_{i,1}^0, \ldots, a_{i,k-2}^0) \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+i)+j} V. \]

Note also that \( t \) is a fixed integer. Indeed, the degree of \( a_{t+l}^0 \) is greater than or equal to \( 2((k-1)(p+t+l) + k - 2) \) and it coincides with \( N + 1 \), \( N \) being the formal dimension of \( (\Lambda V, d) \).

Then

\[ N + 1 \geq 2((k-1)(p+t+l) + k - 2). \]

Hence

\[ t \leq \frac{1}{2(k-1)} (N - 2(k-1)(p+t+l) + 5 - 2k). \]

In what follows, we take \( t \) the largest integer satisfying this inequality.

Now, we have:

\[ d^2 \omega_0 = da_2^0 + da_3^0 + \cdots + da_{t+l}^0 \]
\[ = d(a_{2,0}^0, a_{2,1}^0, \ldots, a_{2,k-2}^0) + \sum_{i=0}^l d(a_{3,0}^0, a_{3,1}^0, \ldots, a_{3,k-2}^0) + \cdots + \sum_{k' > 2k-2} d_{k'} \omega_0 \]

with

\[ d(a_{2,0}^0, a_{2,1}^0, \ldots, a_{2,k-2}^0) = d_k(a_{2,0}^0, a_{2,1}^0, \ldots, a_{2,k-2}^0) + d_{k+1}(a_{2,0}^{0,0}, a_{2,1}^{0,1}, \ldots, a_{2,k-2}^{0,k-2}) + \cdots \]
\[ = \left( d_k a_{2,0}^0, \sum_{i' + i'' = 1} d_{k+i'} a_{2,i''}^0, \ldots, \sum_{i' + i'' = k-2} d_{k+i'} a_{2,i''}^0 \right) + \left( d_{2k-1} a_{2,0}^0 + d_{2k-2} a_{2,1}^0 + \cdots \right) + \cdots \]
\[ d(a_0^0, a_1^0, \ldots, a_{k-2}^0) = d_k(a_0^0, a_1^0, \ldots, a_{k-2}^0) + d_{k+1}(a_0^0, a_1^0, \ldots, a_{k-2}^0) + \cdots = \left( d_k a_0^0, \sum_{i'+i''=1} d_{k+i'} a_3^{i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'} a_3^{i''} \right) + (d_{2k-1} a_3^0 + d_{2k-2} a_3^1 + \cdots) + \cdots \]

It follows that:
\[ d^2 \omega_0 = \left( d_k a_0^0, \sum_{i'+i''=1} d_{k+i'} a_2^{i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'} a_2^{i''} \right) + (d_{2k-1} a_2^0 + d_{2k-2} a_2^1 + \cdots) + \cdots \]

Since \( d^2 \omega_0 = 0 \), we have
\[ \left( d_k a_0^0, \sum_{i'+i''=1} d_{k+i'} a_2^{i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'} a_2^{i''} \right) = \delta(a_2^0) = 0 \]

with \( a_2^0 = (a_0^0, \ldots, a_{k-2}^0) \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2)+j} V \). Consequently \( a_2^0 \) is a \( \delta \)-coboundary.

**Claim 1.** \( a_2^0 \) is an \( \delta \)-coboundary.

**Proof.** Recall first that the general \( r^{th} \)-term of the spectral sequence (6) is given by the formula:
\[ E_r^{p,q} = Z_r^{p,q} / Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q} \]
where
\[ Z_r^{p,q} = \{ x \in [F^p(\Lambda V)]_p \mid dx \in [F^{p+r}(\Lambda V)]_q \} \]
and
\[ B_{r-1}^{p,q} = d([F^{p-r}(\Lambda V)]_q) \cap F^p(\Lambda V) = d(Z_{r-1}^{p-r+1,q+r-2}). \]
Recall also that the differential \( d_r : E_r^{p,q} \to E_r^{p+r,q-r+1} \) in \( E_r^{*,*} \) is induced from the differential \( d \) of \( (\Lambda V, d) \) by the formula \( d_r([v])_r = [dv]_r \), \( v \) being any representative in \( Z_r^{p,q} \) of the class \([v]_r \) in \( E_r^{p,q} \).

We still assume that \( \dim H^N(\Lambda V, \delta) = 1 \) and adopt notations of § 4.1.1.

Notice then \( \omega_0 \in Z_2^{p,q} \) and it represents a non-zero class \([\omega_0]_2 \) in \( E_2^{p,q} \). Otherwise \( \omega_0 = \omega_0^0 + d(\omega_0^\alpha) \), where \( \omega_0^0 \in Z_1^{p+1,q-1} \) and \( \omega_0^\alpha \in B_1^{p,q} \), so that \( \alpha = [\omega_0] = [\omega_0^0 - (d - \delta)(\omega_0^\alpha)] \). But \( \omega_0^0 - (d - \delta)(\omega_0^\alpha) \in \Lambda^{z_r+1} \) is a contradiction to the definition of \( \omega_0 \). Now, using the isomorphism \( E_2^{*,*} \cong H^{*,*}(\Lambda V, \delta) \), we deduce that, \( [\omega_0]_2 \in E_2^{p,q} \) (being the only generating element) must survive to \( E_3^{p,q} \), otherwise, the spectral sequence fails to converge. Whence \( d_2([\omega_0]_2) = [a_2^0]_2 = 0 \) in \( E_2^{p+2,q-1} \), i.e., \( a_2^0 \in Z_1^{p+3,q-2} + B_1^{p+2,q-1} \). However \( a_2^0 \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2)+j} V \) and \( a_2^0 = \delta(b_2) \), so that \( a_2^0 = \delta(x) \), \( x \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+1)+j} V \). By wordlength argument, we have necessary \( a_2^0 = \delta(x) \), which finishes the proof of Claim 1.

Notice that this is the first obstruction to \([\omega_0] \) to represent a non-zero class in the term \( E_3^{*,*} \) of (6). The others will appear progressively as one advances in the algorithm.

Let then \( b_2 \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2)-(k-1)+j} V \) such that \( a_2^0 = \delta(b_2) \) and put \( \omega_1 = \omega_0 - b_2 \). Reconsider the previous calculation for it:

\[
d_{\omega_1} = d\omega_0 - db_2
= (a_2^0 + a_3^0 + \cdots + a_{k-1}^0) - (d_kb_2 + d_4b_2 + \cdots),
\]

with

\[
d_kb_2 = d_k(b_2^0, b_2^1, \ldots, b_2^{k-2}) = (d_kb_2^0, d_kb_2^1, \ldots, d_kb_2^{k-2}) \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2)+j} V,
\]

\[
d_{k+1}b_2 = d_{k+1}(b_2^0, b_2^1, \ldots, b_2^{k-2})
= (d_{k+1}b_2^0, d_{k+1}b_2^1, \ldots, d_{k+1}b_2^{k-2}) \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2)+j+1} V,
\]

\[
\ldots
\]
This implies that

\[
d\omega_1 = a_2^0 + a_3^0 + \cdots + a_{t+l}^0 - \left( d_k b_2^0 + \sum_{i'+i''=1}^{i'+i''=k-2} d_{k+i'} b_2^{i''} \right)
- (d_{2k-1} b_2^0 + \cdots, \ldots) = a_2^0 - \delta(b_2) + a_3^0 - (d_{2k-1} b_2^0 + \cdots, \ldots) + \cdots
= a_3^0 - (d_{2k-1} b_2^0 + \cdots, \ldots) + \cdots,
\]

and then:

\[
d\omega_1 = a_3^1 + a_4^1 + \cdots + a_{t+l}^1, \quad \text{with} \quad a_i^1 \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+i)+j} V.
\]

So,

\[
d^2 \omega_1 = da_3^1 + da_4^1 + \cdots + da_{t+l}^1
= \left( d_k a_3^{1,0} + \sum_{i'+i''=1}^{i'+i''=k-2} d_{k+i'} a_3^{1,i''} \right)
+ \left( d_{2k-1} a_3^{1,0} + \cdots, \ldots \right) + \cdots
\]

Since \(d^2 \omega_1 = 0\), by wordlength reasons,

\[
\left( d_k a_3^{1,0} + \sum_{i'+i''=1}^{i'+i''=k-2} d_{k+i'} a_3^{1,i''} \right) = \delta(a_3^1) = 0.
\]

We claim that \(a_3^1 = \delta(b_3)\) and consider \(\omega_2 = \omega_1 - b_3\).

We continue this process defining inductively \(\omega_j = \omega_{j-1} - b_{j+1}, \ j \leq t+l-2\) such that:

\[
d\omega_j = a_{j+2}^j + a_{j+3}^j + \cdots + a_{t+l}^j, \quad \text{with} \quad a_i^j \in \bigoplus_{h=0}^{k-2} \Lambda^{(k-1)(p+i)+h} V
\]

and \(a_{j+2}^j\) a \(\delta\)-cocycle.
Claim 2. $a_{j+2}^j$ is a $\delta$-coboundary, i.e., there is

$$b_{j+2} \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+j+2)-(k-1)+j}V$$

such that $\delta(b_{j+2}) = a_{j+2}^j; 1 \leq j \leq t + l - 2$.

Proof. We proceed in the same manner as for the first claim. Indeed, we have clearly for any $1 \leq j \leq t + l - 2$, $\omega_j = \omega_{j-1} - b_{j+1} = \omega_0 - b_2 - b_3 - \cdots - b_{j+1} \in Z_p^{p,q}_{j+2}$ and it represents a non zero class $[\omega_j]_{j+2}$ in $E_{j+2}^{p,q}$ which is also one dimensional. Whence as in Claim 1, we conclude that, $a_{j+2}^j$ is a $\delta$-coboundary for all $1 \leq j \leq t + l - 2$.

Consider $\omega_{t+l-1} = \omega_t + l - 2 - b_{t+l}$, where $\delta(b_{t+l}) = a_{t+l}^{t+l-2}$. Notice that $|d\omega_{t+l-1}| = |d\omega_{t+l-2}| = N + 1$, but by the hypothesis on $t, d(\omega_{t+l-2}) = a_{t+l}^{t+l-2}$ and then

$$|d(\omega_{t+l-2} - b_{t+l})| = |a_{t+l}^{t+l-2} - \delta(b_{t+l}) - (d - \delta)b_{t+l}| = | - (d - \delta)b_{t+l} | > N + 1.$$

It follows that $d\omega_{t+l-1} = 0$, that is $\omega_{t+l-1}$ is a $d$-cocycle. But it can’t be a $d$-coboundary. Indeed suppose that $\omega_{t+l-1} = (\omega_0^0 + \omega_1^0 + \cdots + \omega_l^0) - (b_2 + b_3 + \cdots + b_{t+l})$, were a $d$-coboundary, by wordlength reasons, $\omega_0^0$ would be a $\delta$-coboundary, i.e., there is $x \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)p-(k-1)+j}V$ such that $\delta(x) = \omega_0^0$.

Then

$$\omega_0 = \delta(x) + \omega_1^0 + \cdots + \omega_l^0.$$

Since $\delta(\omega_0) = 0$, we would have $\delta(\omega_0^0 + \cdots + \omega_l^0) = 0$ and then $[\omega_0] = [\omega_0^0 + \cdots + \omega_l^0]$. But $\omega_0^0 + \cdots + \omega_l^0 \in \Lambda^{\geq r}V$, contradicts the property of $\omega_0$. Consequently $\omega_{t+l-1}$ represents the fundamental class of $(\Lambda V, d)$.

Finally, since $\omega_{t+l-1} \in \Lambda^{\geq r}V$ we have

$$e_0(\Lambda V, d) \geq r.$$

4.1.2. For the second inequality. Denote $s = e_0(\Lambda V, d)$ and let $\omega \in \Lambda^{\geq r}V$ be a cocycle representing the generating class $\alpha$ of $H^N(\Lambda V, d)$.

Write $\omega = \omega_0 + \omega_1 + \cdots + \omega_t, \omega_i \in \Lambda^{a+i}V$. We deduce that:

$$d\omega = \left( \sum_{i+i' = 1} d_{k+i} \omega_{i'} + \cdots + \sum_{i+i' = k-2} d_{k+i} \omega_{i'} \right) + d_{k} \omega_{k-1} + d_{2k-1} \omega_0 + \cdots$$

$$= \delta(\omega_0, \omega_1, \cdots, \omega_{k-2}) + \cdots$$
Since \( d\omega = 0 \), by wordlength reasons, it follows that \( \delta(\omega_0, \omega_1, \ldots, \omega_{k-2}) = 0 \). If \((\omega_0, \omega_1, \ldots, \omega_{k-2})\), were a \( \delta \)-boundary, i.e., \((\omega_0, \omega_1, \ldots, \omega_{k-2}) = \delta(x)\), then

\[
\omega - dx = (\omega_0, \omega_1, \ldots, \omega_{k-2}) - \delta(x) + (\omega_{k-1} + \cdots + \omega_l) - (d - \delta)(x),
\]

so, \( \omega - dx \in \Lambda^{\geq s+k-1}V \), which contradicts the fact \( s = e_0(\Lambda V, d) \). Hence \((\omega_0, \omega_1, \ldots, \omega_{k-2})\) represents the generating class of \( H^N(\Lambda V, \delta) \). But \((\omega_0, \omega_1, \ldots, \omega_{k-2}) \in \Lambda^{\geq s}V \) implies that \( s \leq r \). Consequently, \( e_0(\Lambda V, d) \leq r \).

Thus, we conclude that
\[
e_0(\Lambda V, d) = r.
\]

### 4.2. Proof of Theorem 2.

It suffices to remark that since \((\Lambda V, d)\) is elliptic, it has Poincaré duality property and then \( \dim H^N(\Lambda V, d) = 1 \). The convergence of (6) implies that \( \dim E^*_{\infty} = 1 \). Hence there is a unique \((p, q)\) such that \( p + q = N \) and \( E^*_{\infty} = E^p_q \). Consequently only one of the generating classes \( \alpha_1, \ldots, \alpha_m \) had to survive to \( E_{\infty} \). Let \( \alpha_j \) this representative class and \((p_j, q_j)\) its pair of degrees.

**Example 1.** Let \( d = \sum_{i \geq 3} d_i \) and \((\Lambda V, d)\) be the model defined by \(V^{even} = \langle x_2, x_2^2 \rangle, V^{odd} = \langle y_5, y_7, y_7^2 \rangle\), \(dx_2 = dx_2^2 = 0, dy_5 = x_3^2, dy_7 = x_4^2\) and \(dy_7 = x_2^2 x_2^2\), in which subscripts denote degrees.

For \( k \geq 3, l \geq 0 \), we have
\[
x_2^k x_2^l = x_2^{k-3} x_2^3 x_2^d = d(y_5 x_2^{k-3} x_2^d).
\]

For \( k \geq 4, l \geq 0 \),
\[
x_2^k x_2^l = x_2^l x_2^4 x_2^{k-4} x_2^4 = d(x_2^l x_2^{k-4} y_7).
\]

Clearly we have
\[
\dim H(\Lambda V, d) < \infty \text{ and } \dim H(\Lambda V, d_3) = \infty.
\]

Using A. Murillo’s algorithm (cf. §2) the matrix determining the fundamental class is:
\[
A = \begin{pmatrix}
x_2^2 & 0 \\
0 & x_2^3 \\
x_2 x_2^2 & 0
\end{pmatrix},
\]
so, $\omega = -x_2^2x_2^3y_7 + x_2x_2^5y_5 \in \Lambda^{\geq 6}V$ is a generator of this fundamental cohomology class.

It follows that $e_0(\Lambda V, d) = 6 \neq m + n(k - 2)$.

**Example 2.** Let $d = \sum_{i \geq 3} d_i$ and $(\Lambda V, d)$ be the model defined by

\[
V^{\text{even}} = \langle x_2, x_3 \rangle, \quad V^{\text{odd}} = \langle y_5, y_9, y_9 \rangle, \quad dx_2 = dx_2 = 0, \quad dy_5 = x_2^3, \quad dy_9 = x_2^5 \text{ and } dy_9 = x_2^3x_2^2.
\]

Clearly we have

$$\dim H(\Lambda V, d) < \infty \text{ and } \dim H(\Lambda V, d_3) = \infty.$$  

Using A. Murillo’s algorithm (cf. §2) the matrix determining the fundamental class is:

\[
A = \begin{pmatrix}
  x_2^2 & 0 \\
  0 & x_2^4 \\
  x_2^2x_2^2 & 0
\end{pmatrix},
\]

so, $\omega = -x_2^2x_2^3y_7 + x_2x_2^5y_5 \in \Lambda^{\geq 7}V$ is a generator of this fundamental cohomology class.

It follows that $e_0(\Lambda V, d) = 7 \neq m + n(k - 2)$.

**References**


