

Banach Lattices with the Positive Dunford-Pettis Relatively Compact Property

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Abstract: The paper is devoted to such Banach lattices E that every Dunford-Pettis and weakly null sequence $(x_n) \subset E$ with disjoint terms is norm null (the positive Dunford-Pettis relatively compact property). It is established that a Banach lattice E has the positive Dunford-Pettis relatively compact property if and only if its almost Dunford-Pettis subsets are L-weakly compact. Consequently, we derive the following result: Banach lattices with the property that their almost Dunford-Pettis subsets are relatively compact, are precisely the discrete KB-spaces.

Key words: The positive Dunford-Pettis relatively compact property, almost Dunford-Pettis completely continuous operator, almost Dunford-Pettis set, Banach lattice.

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1. INTRODUCTION

Banach spaces in which Dunford-Pettis sets are relatively compact were introduced by G. Emmanuele [11]. Such spaces are said to have the Dunford-Pettis relatively compact property (DPrCP). Recently, Y. Wen and J. Chen [15] introduced a weak version of Dunford-Pettis operators, called Dunford-Pettis completely continuous (DPcc) operators, that is, operators which carry Dunford-Pettis and weakly null sequences to norm null ones. They showed that, an operator T on a Banach space X is DPcc if, and only if it carries each Dunford-Pettis set $A \subset X$ to a relatively compact one. It follows that a Banach space X has the DPrCP iff all Dunford-Pettis and weakly null sequences in X are norm null. Clearly, reflexive, as well as Schur spaces, have the DPrCP. If a Banach space is a dual space, then it has the DPrCP iff it has the weak Radon-Nikodym property [11]. Discrete KB-spaces are examples of Banach lattices with the DPrCP [8, Corollary 3.10].

In this note we study a weak version of the DPrCP in the Banach lattice setting, the positive Dunford-Pettis relatively compact property (Definition 3.2).

The introduction of this property is motivated by the fact that it leads to a new reformulation (Question 2) of an open question posed by K. Bouras in [8].

QUESTION 1. In a KB-space, is every almost Dunford-Pettis set relatively weakly compact (resp. L-weakly compact)?

In the recent paper [9], the authors showed that the answer of the first part of Question 1 is affirmative. In Proposition 3.14, we shall give an original and short proof based on a known result of Meyer-Nieberg [12, Theorem 2.5.3].

However, the second part of Question 1 remains open and it is equivalent by Theorem 3.15, to the following question.

QUESTION 2. Does every KB-space have the positive Dunford-Pettis relatively compact property?

We shall see that the converse of Question 2 is always true (Remark 3.5). But, we are not able to answer Question 2 and it remains open.

It is also worth noting from Proposition 3.3 that, if is affirmative the answer of each of the two equivalent questions Question 1 and Question 2, then so is the answer of the following open problem posed by W. Wnuk in 1994.

PROBLEM. ([16, p 234]) Does every KB-space with the weak Dunford-Pettis property have the positive Schur property?

To study the positive Dunford-Pettis relatively compact property, we introduce a new class of operators, called almost Dunford-Pettis completely continuous operators (Definition 3.1), that characterize the latter property through the identity operator, and present at the same time the disjoint version of DPcc operators. Afterwards, we give some characterizations of this class of operators (Theorems 3.9, 3.12) and hence we derive those of the positive Dunford-Pettis relatively compact property of a Banach lattice (Theorem 3.15). Next, we look, under some sufficient conditions, at the relationship between this class of operators and that of DPcc operators (Corollary 3.20). Finally, we study in the last section the duality property for almost Dunford-Pettis completely continuous operators and we derive a new result concerning the relative compactness of almost DP sets in Banach lattices (Theorem 4.3).

2. NOTATION AND KNOWN RESULTS

The term “operator” between two Banach spaces refers to a bounded linear mapping. X, Y will denote real Banach spaces, and E, F, G will denote real Banach lattices. The notation B_X is used for the closed unit ball of X . $sol(A)$ denotes the solid hull of a subset A of a Banach lattice. The positive cone of E will be denoted by E^+ and $A^+ := A \cap E^+$ will denote the positive part of the subset $A \subset E$. For every norm bounded subsets $A \subset E, B \subset E', \rho_A : E' \rightarrow R^+$ and $\rho_B : E \rightarrow R^+$ are the lattice seminorms defined respectively by $\rho_A(f) = \sup \{|f| |x| : x \in A\}$ and $\rho_B(x) = \sup \{|f| |x| : f \in B\}$. Let

$$E^a := \{x \in E : \text{every monotone sequence in } [0, |x|] \text{ is convergent}\}.$$

It follows from [12, Proposition 2.4.10] that E^a is the maximal ideal in E on which the induced norm is order continuous. The order ideal generated by a positive element $x \in E$ is

$$E_x := \{y \in E : \exists \lambda > 0 \text{ with } |y| \leq \lambda x\}.$$

$(E_x, \|\cdot\|_\infty)$ is an AM-space with $B_{E_x} = [-x, x]$ ([2, Theorem 4.21]), where the norm $\|\cdot\|_\infty$ is defined by $\|y\|_\infty := \inf \{\lambda > 0 : |y| \leq \lambda x\}$. Furthermore, the canonical embedding $E_x \rightarrow E$ is continuous [14, p 102]. It is well known (see [4, Theorem 2.13]) that the class of Banach lattices whose duals are discrete contains that of Banach lattices satisfying the following condition of discreteness:

$$E_x \text{ has a discrete dual for every } x \in E^+. \tag{*}$$

Let us recall that a norm bounded subset $A \subset E$ is said to be

- (1) approximately order bounded, if it is approximately order bounded set with respect to the norm of E in the sense of [12, Remark p 73], that is, for every $\varepsilon > 0$ there exists $u \in E^+$ such that $A \subset [-u, u] + \varepsilon B_E$, equivalently, for every $\varepsilon > 0$, there exists $u \in E^+$ such that $\|(|x| - u)^+\| \leq \varepsilon$ for each $x \in A$.
- (2) L-weakly compact, if $\|x_n\| \rightarrow 0$ for every disjoint sequence (x_n) in $sol(A)$. The following characterizations (see [12, Propositions 3.6.2 and 3.6.3]) of an L-weakly compact sets $A \subset E, B \subset E'$ will be used throughout this paper.
 - (a) $\rho_A(f_n) \rightarrow 0$ for every disjoint norm bounded sequence $(f_n) \subset E'$.

- (b) For every $\varepsilon > 0$ there exists $u \in (E^a)^+$ such that $A \subset [-u, u] + \varepsilon B_E$, that is, A is an approximately order bounded set in E^a .
 - (c) $\rho_B(x_n) \rightarrow 0$ for every disjoint bounded sequence $(x_n) \subset E$.
- (3) (almost) Dunford-Pettis, if each (disjoint) weakly null sequence $(f_n) \subset E'$ converges uniformly to zero in A , that is, $\sup_{x \in A} |f_n(x)| \rightarrow 0$ as $n \rightarrow \infty$.

An operator $T : X \rightarrow Y$ is called Dunford-Pettis, if T carries each weakly null sequence $(x_n) \subset X$ to an norm null one in Y , equivalently, T carries relatively weakly compact subsets of X to relatively compact ones in Y . In his thesis [13], Sanchez considered a weak version of the class of Dunford-Pettis operators, called almost Dunford-Pettis operators, and this class of operators was investigated later by Wnuk [16]. An operator T from a Banach lattice E into a Banach space X is said to be almost Dunford-Pettis if the sequence $(\|T(x_n)\|)$ converges to 0 for every weakly null sequence (x_n) consisting of pairwise disjoint elements in E . It follows from [6, Theorem 2.2] that T is almost Dunford-Pettis if and only if $\|T(x_n)\| \rightarrow 0$ for every weakly null sequence (x_n) in E^+ . Moreover,

- (1) an operator $T : X \rightarrow E$ is L-weakly compact, if T maps the closed unit ball of X to an L-weakly compact set in E .
- (2) an operator $T : E \rightarrow X$ is order weakly compact, if T maps each order bounded set in E to a relatively weakly compact set in X .
- (3) a Banach lattice E is said to have the (positive) Schur property if the identity operator I on E is (almost) Dunford-Pettis, that is, $\|x_n\| \rightarrow 0$ for every (disjoint, or equivalently positive) weakly null sequence (x_n) in E .
- (4) a Banach lattice E is said to have the (weak) Dunford-Pettis property if $f_n(x_n) \rightarrow 0$ for all (disjoint) weakly null sequences $(x_n) \subset E$, and all weakly null sequences $(f_n) \subset E'$, equivalently, each relatively weakly compact set in E is (almost) Dunford-Pettis set (see for more details [7, 16]).

Finally, we refer the reader for unexplained terminology on Banach lattice theory and positive operators to the monographs [2, 12].

3. ALMOST DUNFORD-PETTIS COMPLETELY CONTINUOUS OPERATORS

DEFINITION 3.1. An operator T from a Banach lattice E into a Banach space X is said to be almost Dunford-Pettis completely continuous (aDPcc in short) if $\|T(x_n)\| \rightarrow 0$ for each Dunford-Pettis and weakly null sequence (x_n) in E consisting of disjoint terms.

DEFINITION 3.2. A Banach lattice E is said to have the positive Dunford-Pettis relatively compact property (PDPrCP in short) if $\|x_n\| \rightarrow 0$ provided that the sequence $(x_n) \subset E$ is Dunford-Pettis and disjoint weakly null, that is, the identity operator on E is aDPcc.

It is obvious that the positive Schur property, as well as the DPrCP, implies the PDPrCP. However, as in a Banach lattice E with the (weak) Dunford-Pettis property, each (disjoint) weakly null sequence $(x_n) \subset E$ is Dunford-Pettis, we note easily the following result.

PROPOSITION 3.3. *A Banach lattice E has the (positive) Schur property if and only if E has both the (positive) DPrCP and the (weak) Dunford-Pettis property.*

We are now in a position to distinguish through the following example the PDPrCP from the positive Schur property and the DPrCP, and hence we show that the class of operators aDPcc contains strictly that of DPcc (resp. almost Dunford-Pettis) operators.

EXAMPLE 3.4. Let $E = L^1[0, 1] \oplus L^2[0, 1]$. Then, E has the PDPrCP property but not the positive Schur property nor the DPrCP. Indeed, since the Banach lattice $L^1([0, 1])$ has the Dunford-Pettis property without the Schur property, it follows that $L^1([0, 1])$ has the positive Schur (PDPrCP) property without the DPrCP. On the other hand, the reflexive Banach lattice $L^2([0, 1])$ has the DPrCP (PDPrCP) without the positive Schur property.

Remark 3.5. A Banach lattice E with the PDPrCP must be a KB-space. In fact, since the Banach lattice c_0 does not have the PDPrCP (because it has the weak Dunford-Pettis property without the positive Schur property), it follows that E contains no lattice copy of c_0 . Therefore, E is a KB-space (see [2, Theorem 4.61]).

The following lemmas will be useful in the rest of this paper.

LEMMA 3.6. Let E be a Banach lattice and let B be a norm bounded subset of E' . Let $(x_n) \subset B_E^+$ and $(h_n) \subset B^+$ be respectively two sequences such that

- (i) $h(x_n) \rightarrow 0$ for all $h \in B$, and
- (ii) $h_n(x_n) > \varepsilon$ for some $\varepsilon > 0$ and for all n .

then, there exist a subsequence (z_n) of (x_n) and a disjoint sequence (g_n) in $(\text{sol}(B))^+$ such that $g_n(z_n) > \frac{\varepsilon}{2}$ for all n sufficiently large.

Proof. By the first part of the assumption, an easy inductive argument shows that there exists a subsequence (z_n) of (x_n) and a subsequence (f_n) of (h_n) such that

$$\left(4^n \sum_{i=1}^n f_i\right)(z_{n+1}) < \frac{1}{n} \quad \forall n \geq 1.$$

Put $f = \sum_{i=1}^{\infty} 2^{-i} f_i$ and $g_{n+1} = (f_{n+1} - 4^n \sum_{i=1}^n f_i - 2^{-n} f)^+$ for $n \geq 1$. The sequence (g_n) is disjoint ([2, Lemma 4.35]) and since $0 \leq g_n \leq f_n$ then, (g_n) is included in $(\text{sol}(B))^+$. Now, by the second part of the assumption we have for every n

$$\begin{aligned} g_{n+1}(z_{n+1}) &\geq \left(f_{n+1} - 4^n \sum_{i=1}^n f_i - 2^{-n} f\right)(z_{n+1}) \\ &\geq \varepsilon - \frac{1}{n} - 2^{-n} f(z_{n+1}). \end{aligned}$$

Since $2^{-n} f(z_{n+1}) \rightarrow 0$, we have $g_{n+1}(z_{n+1}) > \frac{\varepsilon}{2}$ for all n sufficiently large. ■

LEMMA 3.7. If A is a relatively weakly compact and almost Dunford-Pettis subset of a Banach lattice E , and (x_n) is a disjoint sequence in $\text{sol}(A)$, then (x_n) is a Dunford-Pettis and weakly null sequence in E . In particular, if (x_n) is an almost Dunford-Pettis and disjoint weakly null sequence, then the positive sequences $(|x_n|)$, (x_n^+) , (x_n^-) are all Dunford-Pettis and weakly null.

Proof. By [2, Theorem 4.34], (x_n) is well a weakly null sequence in E . Assume by way of contradiction that (x_n) is not a Dunford-Pettis sequence. Then, there exists a weak null sequence (f_n) in E' so that $|f_n||x_n| > \varepsilon$ for some $\varepsilon > 0$ and for all n . On the other hand, since $|x_n| \xrightarrow{w} 0$ ([2, Theorem 4.34]),

and if we denote by B the solid hull of the subset $\{f_n : n \in N\}$, then a simple application of Lemma 3.6, shows that there exist a subsequence (y_n) of (x_n) and a disjoint sequence (g_n) in B^+ such that $g_n|y_n| > \frac{\varepsilon}{2}$ holds for all n greater than some natural integer n_0 . Since $(y_n) \subset \text{sol}(A)$, there exists a sequence $(z_n) \subset A$ satisfying $|y_n| \leq |z_n|$ for all n , and hence $g_n(|z_n|) > \frac{\varepsilon}{2}$ for all $n \geq n_0$. In view of

$$g_n|z_n| = \sup\{|h(z_n)| : |h| \leq g_n\}$$

for each $n \geq n_0$ there exists some $|h_n| \leq g_n$ such that $|h_n(z_n)| > \frac{\varepsilon}{2}$. This shows that there exists a strictly increasing sequence $(n_k)_{k=1}^\infty$ of \mathbb{N} with $|h_{n_k}| \leq g_{n_k}$ and $|h_{n_k}(z_{n_k})| > \frac{\varepsilon}{2}$ for all k . Note that the sequence (g_n) (and hence $(h_{n_k})_{k=1}^\infty$) is disjoint weakly null in E' ([2, Theorem 4.34]). The latter contradicts the almost Dunford-Pettis property of the set A . Therefore, (x_n) is a Dunford-Pettis and weakly null sequence as desired. ■

Our first main result shows that operators aDPcc enjoy the following lattice approximations.

THEOREM 3.8. *Let $T : E \rightarrow X$ be an aDPcc operator from a Banach lattice into a Banach space. Then, given a relatively weakly compact and almost Dunford-Pettis subset $A \subset E$ and $\varepsilon > 0$, there exists $u \in E^+$ such that*

$$\|T(|x| - u)^+\| \leq \varepsilon$$

holds for every x in $\text{sol}(A)$. If in addition $T : E \rightarrow F$ is a positive operator between two Banach lattices, then $T(A)$ is an approximately order bounded set in F , that is, for every $\varepsilon > 0$ there exists $v \in F^+$ such that

$$\|(|Tx| - v)^+\| \leq \varepsilon$$

holds for all $x \in A$.

Proof. By Lemma 3.7, every disjoint sequence $(y_n) \subset \text{sol}(A)$ is Dunford-Pettis and weakly null, and hence as T is aDPcc, $\|T(y_n)\| \rightarrow 0$. Thus, it follows from [2, Theorem 4.36], that there exists some $u \in E^+$ lying in the ideal generated by A such that $\|T(|x| - u)^+\| < \varepsilon$ for all $x \in \text{sol}(A)$. For the second part, let $u \in E^+$ be such that $\|T(|x| - u)^+\| < \frac{\varepsilon}{2}$ for all $x \in \text{sol}(A)$,

and put $v = 2Tu$. Then for every $x \in A$ we have

$$\begin{aligned} |Tx| - v &\leq T|x| - v \\ &= Tx^+ - Tu + Tx^- - Tu \\ &\leq Tx^+ - T(x^+ \wedge u) + Tx^- - T(x^- \wedge u) \\ &= T((x^+ - u)^+) + T((x^- - u)^+) \end{aligned}$$

from which we get

$$(|Tx| - v)^+ \leq T((x^+ - u)^+) + T((x^- - u)^+)$$

and thus

$$\|(|Tx| - v)^+\| \leq \|T((x^+ - u)^+)\| + \|T((x^- - u)^+)\| < \varepsilon$$

holds for all $x \in A$. This completes the proof. ■

In the next result, we show that we can replace (resp. restrict) sequences appearing in the definition of aDPcc operators by positive (resp. disjoint positive) sequences.

THEOREM 3.9. *For every operator T from a Banach lattice E into a Banach space X , the following assertions are equivalent:*

- (1) T is aDPcc operator;
- (2) for every almost Dunford-Pettis and weakly null sequence (x_n) in E^+ , we have $\|T(x_n)\| \rightarrow 0$;
- (3) for every Dunford-Pettis and weakly null sequence (x_n) in E^+ , we have $\|T(x_n)\| \rightarrow 0$;
- (4) for every Dunford-Pettis and disjoint weakly null sequence (x_n) in E^+ , we have $\|T(x_n)\| \rightarrow 0$.

Proof. (1) \Rightarrow (2) Let (x_n) be an almost Dunford-Pettis and weakly null sequence (x_n) in E^+ and let $\varepsilon > 0$. By Theorem 3.8 there exists some $u \in E^+$ satisfying $\|T(x_n - u)^+\| < \varepsilon$ for all n . On the other hand, it follows easily from [5, Theorem 2.5] and [2, Theorem 5.57] that the aDPcc property

of T implies its order weak compactness, and thus by [12, Corollary 3.4.9] $\|T(x_n \wedge u)\| \rightarrow 0$. Hence, from

$$\begin{aligned} \|T(x_n)\| &= \|T(x_n - u)^+ + T(x_n \wedge u)\| \\ &\leq \|T(x_n - u)^+\| + \|T(x_n \wedge u)\| \\ &\leq \varepsilon + \|T(x_n \wedge u)\| \end{aligned}$$

we show that $\limsup \|T(x_n)\| \leq \varepsilon$. As $\varepsilon > 0$ is arbitrary, we have $\|T(x_n)\| \rightarrow 0$ as desired.

(2) \Rightarrow (3) and (3) \Rightarrow (4) are obvious.

(4) \Rightarrow (1) Let (x_n) be a Dunford-Pettis and disjoint weakly null sequence in E . By Lemma 3.7 (x_n^+) and (x_n^-) are both Dunford-Pettis and disjoint weakly null in E^+ . This shows from the inequality $\|T(x_n)\| \leq \|T(x_n^+)\| + \|T(x_n^-)\|$ that $\|T(x_n)\| \rightarrow 0$ and therefore T is an aDPcc operator. ■

A simple application of [10, Corollaries 2.6 and 2.7] and Theorem 3.9, gives another characterization of positive aDPcc operators.

COROLLARY 3.10. *For a positive operator $T : E \rightarrow F$ between Banach lattices, the following assertions hold:*

- (1) T is an aDPcc if and only if $f_n(Tx_n) \rightarrow 0$ for every Dunford-Pettis (resp. almost DP) and weakly null sequence (x_n) in E^+ and every disjoint norm bounded sequence $(f_n) \subset (F')^+$.
- (2) T' is an aDPcc if and only if $f_n(Tx_n) \rightarrow 0$ for every Dunford-Pettis (resp. almost DP) and weakly null sequence (f_n) in $(F')^+$ and every disjoint bounded sequence $(x_n) \subset E^+$.

Contrary to DPcc operators (see example of [15, p 9]), positive aDPcc operators satisfy the domination property as shown in the following corollary.

COROLLARY 3.11. *If a positive operator $S : E \rightarrow F$ between Banach lattices is dominated by an aDPcc operator then, S itself is aDPcc.*

Proof. Let S and T be an operators from E into F such that $0 \leq S \leq T$ and T is aDPcc. Let (x_n) be a Dunford-Pettis and weak null sequence in E^+ . Since $0 \leq S(x_n) \leq T(x_n)$ and $\|T(x_n)\| \rightarrow 0$, then $\|S(x_n)\| \rightarrow 0$. This proves that S is an aDPcc operator. ■

The following result gives some set and operator characterizations of positive aDPcc operators between two Banach lattices when the range space has order continuous norm.

THEOREM 3.12. *Let E and F be two Banach lattices. For a positive operator $T : E \rightarrow F$, consider the following statements:*

- (1) T is aDPcc;
- (2) T carries each relatively weakly compact and almost Dunford-Pettis subset of E to an approximately order bounded subset of F ;
- (3) T carries each relatively weakly compact and almost Dunford-Pettis subset of E to an L -weakly compact subset of F ;
- (4) for each operator S from an arbitrary Banach space X into E , the product TS is L -weakly compact provided that S is weakly compact and the adjoint S' is an almost Dunford-Pettis operator.

Then, (3) \Rightarrow (4) \Rightarrow (1) \Rightarrow (2) and if the norm of F is order continuous, then the assertions (1)-(4) are equivalent.

Proof. (3) \Rightarrow (4) Obvious.

(4) \Rightarrow (1) Let (x_n) be an almost Dunford-Pettis and weakly null sequence of E^+ and let $(f_n) \subset (E')^+$ be a disjoint norm bounded sequence. We shall show by Corollary 3.10 that $f_n(Tx_n) \rightarrow 0$. To this end, consider the operator

$$S : \ell^1 \longrightarrow E, \quad (\lambda_n)_n \longmapsto \sum_{n=1}^{\infty} \lambda_n x_n$$

and note that its adjoint S' is defined by

$$S' : E' \longrightarrow \ell^\infty, \quad f \longmapsto \{f(x_n)\}_{n \geq 1}.$$

From [2, Theorem 5.26] S is well weakly compact and by the almost Dunford-Pettis property of (x_n) , S' is an almost Dunford-Pettis operator. According to our hypothesis TS is L -weakly compact and so $A = TS(B_{\ell^1})$ is L -weakly compact set in F . This shows that

$$\begin{aligned} |f_n(Tx_n)| &= |f_n(TSe_n)| \leq |f_n| |TSe_n| \\ &\leq \sup\{|f_n||y| : y \in A\} = \varrho_A(f_n) \longrightarrow 0 \end{aligned}$$

as desired.

(1) \Rightarrow (2) Follows from Theorem 3.8.

(2) \Rightarrow (3) As the norm of F is order continuous, it follows that the set $T(A)$ is approximately order bounded in $F = F^a$, i.e., $T(A)$ is L-weakly compact in F . This completes the proof. ■

In the following example, we show that the order continuity of the norm of F is essential to have the equivalence of the assertions in the above theorem.

EXAMPLE 3.13. Let $E = L^1[0, 1]$, $F = c$. By [17, Theorem 2] there exist two operators T, S with $0 \leq S \leq T$, T is Dunford-Pettis (aDPcc) but S is not. From Corollary 3.11 S is well aDPcc. Now, by [2, Theorem 5.81], let A be a relatively weakly compact subset of E so that $S(A)$ is not relatively norm compact, and note that A is Dunford-Pettis since E has the Dunford-Pettis property as an AL-space (see [2, Theorem 5.85]). Now, assume that $S(A)$ is an L-weakly compact set in F and let $\varepsilon > 0$. Let $v \in (F^a)^+$ so that $S(A) \subset [-v, v] + \varepsilon B_F$. Since F is discrete then F^a is so. Therefore, [1, Corollary 21.13] implies the norm compactness of $[-v, v]$ and hence the relative norm compactness of $S(A)$. This contradiction shows that $S(A)$ is not L-weakly compact.

Now, to derive a characterization of the PDPrCP of a Banach lattice, we need to state the following proposition in which we see that the answer of the first part of Question 1 is affirmative.

PROPOSITION 3.14. *A Banach lattice E is a KB-space if, and only if, each almost Dunford-Pettis set in E is relatively weakly compact.*

Proof. The “if” part is trivial as in this case E contains no lattice copy of c_0 .

For the “only if” part, let A be an almost Dunford-Pettis subset of E . According to [12, Theorem 2.5.3], it suffices to show that $\varrho_A(f_n) \rightarrow 0$ for every order bounded disjoint sequence (f_n) in E' . Otherwise, there exist an order bounded disjoint sequence $(f_n) \subset E'$ and a sequence $(x_n) \subset A$ so that $|f_n|(|x_n|) > \varepsilon$ for some $\varepsilon > 0$ and for all n . Now, by [2, Theorem 1.23] one can find a sequence $(g_n) \subset E'$ such that

$$|g_n| \leq |f_n| \text{ and } g_n(x_n) = |f_n|(|x_n|) > \varepsilon$$

hold for all n . Note that the sequence (f_n) is weakly null [2, p 192] and hence so is the disjoint sequence (g_n) [2, Theorem 4.34]. This contradicts the almost Dunford-Pettis property of A and the proof is finished. ■

THEOREM 3.15. *For a Banach lattice E , the following statements are equivalent:*

- (1) E has the PDPrcP;
- (2) $\|x_n\| \rightarrow 0$ for every almost Dunford-Pettis and weakly null sequence (x_n) of E^+ ;
- (3) $\|x_n\| \rightarrow 0$ for every Dunford-Pettis and weakly null sequence (x_n) of E^+ ;
- (4) $\|x_n\| \rightarrow 0$ for every Dunford-Pettis and disjoint weakly null sequence (x_n) of E^+ ;
- (5) $f_n(x_n) \rightarrow 0$ for every Dunford-Pettis and weakly null sequence (x_n) of E^+ and every disjoint norm bounded sequence $(f_n) \subset (E')^+$.
- (6) E is a KB-space and each almost Dunford-Pettis subset of E is approximately order bounded;
- (7) each almost Dunford-Pettis subset of E is L -weakly compact;
- (8) each operator S from an arbitrary Banach space X into E is L -weakly compact provided that the adjoint S' is an almost Dunford-Pettis operator.

Proof. The equivalence of the assertions (1)-(5) follows from Theorem 3.9 and Corollary 3.10.

(1) \Rightarrow (6) Follows from Theorem 3.12 combined with Remark 3.5 and Proposition 3.14.

(6) \Rightarrow (7) Follows from Theorem 3.12 combined with Proposition 3.14.

(7) \Rightarrow (8) Obvious.

(8) \Rightarrow (1) It is enough to repeat the same arguments of the implication (4) \Rightarrow (1) of Theorem 3.12. ■

COROLLARY 3.16. *If the bidual E'' of a Banach lattice E is a KB-space then, E has the PDPrcP.*

Proof. Let $(x_n) \subset E^+$, $(f_n) \subset (E')^+$ be respectively a Dunford-Pettis and weakly null sequence and a disjoint norm bounded sequence. As the norm of E'' is order continuous ([2, Theorem 4.69]), then by [12, Theorem 2.4.14] $f_n \xrightarrow{w} 0$. From the Dunford-Pettis property of (x_n) we see that $f_n(x_n) \rightarrow 0$. Now, the desired conclusion follows from Theorem 3.15. ■

Remark 3.17. The equivalences (1) \Leftrightarrow (7) and (1) \Leftrightarrow (8) in Theorem 3.15 present respectively a generalisation of Theorem 2.9 and Corollary 2.12 of [8]. Indeed, by Corollaries 3.16 and 4.2, each of the two assumptions “(i) E'' has an order continuous norm” and “(ii) E is a dual KB-space” implies the PDPrC of the Banach lattice E . However, the PDPrC of E does not necessarily imply (i) (resp. (ii)). For instance, the Banach lattice $E = \ell^1(\ell_n^\infty)$ is a dual KB-space (and hence it has the PDPrC by Corollary 4.2) whose bidual E'' fails to have an order continuous norm (see [12, p 95]). On the other hand, $L^1[0, 1]$ has the PDPrC but it does not have a predual.

Next, we show that the PDPrC of Banach lattices is an hereditary property in the sense of the following theorem.

THEOREM 3.18. *If a Banach lattice E has the PDPrC, then every closed sublattice of E has the PDPrC.*

Proof. Let E_0 be a closed sublattice of E . Consider an operator $T : X \rightarrow E_0$ defined on an arbitrary Banach space such that the adjoint T' is almost Dunford-Pettis. Let $i : E_0 \rightarrow E$ be the canonical lattice embedding. Now, using the characterization by positive weakly null sequences of almost DP operators (see [6, Theorem 2.2]) it follows that $(i \circ T)' = T' \circ i'$ is likewise almost DP. As E has the PDPrC, it follows from Theorem 3.15 that $i \circ T : X \rightarrow E$ is L-weakly compact, and hence clearly T is also L-weakly compact. Now, again by Theorem 3.15 E_0 has the PDPrC. This ends the proof. ■

Let us recall that the lattice operations in a Banach lattice E are said to be sequentially weakly continuous if for every $(x_n) \subset E$, $x_n \xrightarrow{w} 0$ implies $|x_n| \xrightarrow{w} 0$. In our following result we state a sufficient conditions, under which, a positive operator between two Banach lattices that factors through a Banach lattice with sequentially weakly continuous lattice operations is an operator DPcc.

THEOREM 3.19. *Consider the scheme of positive operators between Banach lattices $E \xrightarrow{T} F \xrightarrow{S} G$ such that the lattice operations in F are sequentially weakly continuous. If T is an aDPcc operator and S is order weakly compact, then the product ST is an operator DPcc.*

Proof. Let $(x_n) \subset E$ be a Dunford-Pettis and weakly null sequence. We shall see that $\|ST(x_n)\| \rightarrow 0$. To this end, let $\varepsilon > 0$. As T is aDPcc operator then, by Theorem 3.8, pick some $u \in F^+$ such that

$$\|(|Tx_n| - u)^+\| \leq \varepsilon$$

holds for all n . Now, from the inequality

$$\begin{aligned} |ST(x_n)| &\leq S|T(x_n)| \\ &= S(|Tx_n| - u)^+ + S(|Tx_n| \wedge u) \end{aligned}$$

we see that

$$\begin{aligned} \|ST(x_n)\| &\leq \|S\| \|(|Tx_n| - u)^+\| + \|S(|Tx_n| \wedge u)\| \\ &\leq \|S\| \varepsilon + \|S(|Tx_n| \wedge u)\| \end{aligned}$$

holds for all n . Or, as the lattice operations in F are sequentially weakly continuous, then the sequence $(|T(x_n)| \wedge u)$ is order bounded weakly null in F^+ . It follows from [12, Corollary 3.4.9] that $\|S(|T(x_n)| \wedge u)\| \rightarrow 0$. This shows that $\limsup \|ST(x_n)\| \leq \|S\| \varepsilon$. As $\varepsilon > 0$ is arbitrary, we infer that $\|ST(x_n)\| \rightarrow 0$ as desired. ■

COROLLARY 3.20. *Let E and F be two Banach lattices such that F is discrete with order continuous norm. Then a positive operator $T : E \rightarrow F$ is aDPcc if and only if it is DPcc.*

Proof. Assume that $T : E \rightarrow F$ is an aDPcc operator. As F is discrete with order continuous norm then, it follows from [12, Proposition 2.5.23] that the lattice operations in F are sequentially weakly continuous. Furthermore, since the norm of F is order continuous then, the identity operator $I : F \rightarrow F$ is order weakly compact. Now, it follows from Theorem 3.19, that $T = IT$ is a DPcc operator as desired. ■

It is well known that for the class of discrete Banach lattices, the Schur and the positive Schur properties are the same. In the following corollary we prove a similar result for the DPrCP. Also, the same fact holds for Banach lattices satisfying the condition of discreteness (*). The detail follows.

COROLLARY 3.21. *Let E be a Banach lattice such that*

- (i) E is discrete, or
- (ii) E_x has a discrete dual for each $x \in E^+$.

Then, E has the PDPrCP if and only if it has the DPrCP.

Proof. (i) Follows from Corollary 3.20 by noting that the PDPrC implies the order continuity of the norm.

(ii) Assume that E has the PDPrC and let (x_n) be a Dunford-Pettis and weakly null sequence of E . Put $A = \{x_n : n \in \mathbb{N}\}$, and let $\varepsilon > 0$. Theorem 3.15 implies the existence of $x \in E^+$ such that $A \subset [-x, x] + \varepsilon B_E$. On the other hand, the canonical embedding $i : E_x \rightarrow E$ is a weakly compact operator and hence so is $i' : E' \rightarrow (E_x)'$. Then $i'(B_{E'})$ is a relatively weakly compact subset of $(E_x)'$. Since $(E_x)'$ is an AL-space, it follows that $(E_x)'$ has the positive Schur and hence the Schur property since $(E_x)'$ is discrete. Therefore $i'(B_{E'})$ is a relatively compact subsets of $(E_x)'$. This shows that the operator i' (and then i) is compact. Thus, $[-x, x] = i(B_{E_x})$ is compact for the norm of E . This shows that A is a relatively compact set in E . Therefore, $\|x_n\| \rightarrow 0$ and we are done. ■

4. DUALITY PROPERTY OF ADPCC OPERATORS

As for DPcc operators, the class of aDPcc operators does not satisfy the duality property. In fact, the identity operator of the Banach lattice ℓ^1 is aDPcc (because ℓ^1 has the Schur (PDPrC) property); but its adjoint, which is the identity operator of the Banach lattice ℓ^∞ , is not aDPcc (because ℓ^∞ does not have the PDPrC). Conversely, the identity operator of the Banach lattice ℓ^∞ is not aDPcc, but its adjoint, which is the identity operator of the Banach lattice $(\ell^\infty)'$, is aDPcc (Corollary 4.2).

In the following theorem we give necessary and sufficient conditions under which the adjoint of each positive aDPcc operator between Banach lattices is likewise aDPcc.

THEOREM 4.1. *For a Banach lattices E and F , the following assertions are equivalent:*

- (1) *each positive operator $T : E \rightarrow F$ has an aDPcc adjoint;*
- (2) *each positive aDPcc operator $T : E \rightarrow F$ has an aDPcc adjoint;*
- (3) *at least one of the following conditions is valid:*
 - (a) *F' has the PDPrC;*
 - (b) *the norm of E' is order continuous.*

Proof. (1) \Rightarrow (2) Obvious.

(2) \Rightarrow (3) It suffices to prove that if the norm of E' is not order continuous, then F' has the PDPrcP. By Corollary 3.10, it suffices to show that $f_n(x_n) \rightarrow 0$ for every Dunford-Pettis and weakly null sequence (f_n) of $(F')^+$ and every disjoint bounded sequence $(x_n) \subset F^+$. Since the norm of E' is not order continuous, [12, Theorem 2.4.2] implies that there exists an order bounded disjoint sequence (g_n) of $(E')^+$ satisfying $\|g_n\| = 1$ for all n . Consider the positive operators

$$S_1 : E \rightarrow \ell^1, \quad x \mapsto (g_n(x))_{n=1}^\infty,$$

$$S_2 : \ell^1 \rightarrow F, \quad (\lambda_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty \lambda_n x_n,$$

and

$$T = S_2 \circ S_1 : E \rightarrow F, \quad x \mapsto \sum_{n=1}^\infty g_n(x) x_n.$$

Since $T = S_2 \circ Id_{\ell^1} \circ S_1$ and ℓ^1 has the PDPrcP, the operator T is aDPcc and hence by hypothesis T' is aDPcc. Now, for every n we have

$$0 \leq f_n(x_n) g_n \leq \sum_{k=1}^\infty f_n(x_k) g_k = T'(f_n)$$

and hence

$$|f_n(x_n)| = \|f_n(x_n) g_n\| \leq \|T'(f_n)\| \rightarrow 0.$$

This completes the proof.

(3.a) \Rightarrow (1) Obvious.

(3.b) \Rightarrow (1) Let $T : E \rightarrow F$ be a positive operator. Let $(x_n) \subset E^+$, $(f_n) \subset (F')^+$ be respectively a disjoint bounded sequence and a Dunford-Pettis weakly null sequence. By [8, Corollary 2.15], $A = \{T'(f_n) : n \in N\}$ is an L-weakly compact set in E' . Hence

$$|f_n(Tx_n)| \leq |T'(f_n)||x_n| \leq \sup\{|g||x_n| : g \in A\} = \rho_A(x_n) \rightarrow 0.$$

Now, the desired conclusion follows from Corollary 3.10. \blacksquare

Recall that a Banach lattice E is said to be a dual Banach lattice if $E = F'$ for some Banach lattice F . As consequence of the previous theorem, dual Banach lattices with the PDPrcP are precisely those that are a KB-spaces.

COROLLARY 4.2. *A dual Banach lattice E has the PDPrCP if and only if it is a KB-space.*

Proof. It is enough to see that if E is a KB-space, then E has the PDPrCP. For this, apply Theorem 4.1 for the identity operator $I : G \rightarrow G$ where $E = G'$. ■

We are now in a position to show that the result in Proposition 3.14 is sharpened as follows.

THEOREM 4.3. *A Banach lattice E is a discrete KB-space if, and only if, each almost Dunford-Pettis set in E is relatively compact.*

Proof. The “if” part: That E is a KB-space is trivial as it is shown before. Note that each order interval of E is an almost Dunford-Pettis set (see [8, Corollary 2.2]) and hence norm compact. Now, the discreteness of E follows from [1, Corollary 21.13].

The “only if” part: Let $A \subset E$ be an almost DP set. If E is a discrete KB-space, then it is a dual [12, Exercise 5.4.E2]. Now, from Corollary 4.2, E has the PDPrCP. It follows from Theorem 3.15 that A is approximately order bounded, that is, for every $\varepsilon > 0$ there exists $u \in E^+$ so that $A \subset [-u, u] + \varepsilon B_E$. Now, since $[-u, u]$ is norm compact (again [1, Corollary 21.13]) it follows that A is relatively compact as desired. ■

We conclude this paper by giving necessary and sufficient conditions under which each positive operator between Banach lattices is aDPcc whenever its adjoint is aDPcc.

THEOREM 4.4. *Let E and F be two Banach lattices such that F is a dual Banach lattice. Then the following conditions are equivalent:*

- (1) *each positive operator $T : E \rightarrow F$ is aDPcc;*
- (2) *each positive operator $T : E \rightarrow F$ is aDPcc whenever its adjoint T' is aDPcc;*
- (3) *one of the following assertions is valid:*
 - (a) *E has the PDPrCP;*
 - (b) *the norm of F is order continuous.*

Proof. (1) \Rightarrow (2) Obvious.

(2) \Rightarrow (3) It suffices to prove that if the norm of F is not order continuous, then E has the PDPrCP. We shall show that $f_n(x_n) \rightarrow 0$ for every Dunford-Pettis and weakly null sequence (x_n) of E^+ and every disjoint norm bounded sequence $(f_n) \subset (E')^+$. Consider the positive operator $S_1 : E \rightarrow \ell^\infty$ defined by $S_1(x) = (f_n(x))_n$. Since the norm of F is not order continuous, it results from [2, Theorem 4.51] that ℓ^∞ is lattice embeddable in F , i.e., there exist a lattice homomorphism $S_2 : \ell^\infty \rightarrow F$ and two positive constants M and m satisfying

$$m \|(\lambda_k)_k\|_\infty \leq \|S_2((\lambda_k)_k)\| \leq M \|(\lambda_k)_k\|_\infty$$

for all $(\lambda_k)_k \in \ell^\infty$. Put $T = S_2 \circ S_1 : E \rightarrow \ell^\infty \rightarrow F$ and note that $T' = S_1' \circ Id_{(\ell^\infty)'} \circ S_2'$ is aDPcc (because $(\ell^\infty)'$ has the PDPrCP by Corollary 4.2). From our hypothesis, T is aDPcc. Hence, since $(x_n) \subset E^+$ is a Dunford-Pettis and weakly null sequence, it follows that

$$|f_n(x_n)| \leq \|S_1(x_n)\|_\infty \leq \frac{1}{m} \|S_2 \circ S_1(x_n)\| = \frac{1}{m} \|T(x_n)\| \rightarrow 0$$

as desired.

(3.a) \Rightarrow (1) Obvious.

(3.b) \Rightarrow (1) As F is a dual Banach lattice with order continuous norm, then it has the PDPrCP by Corollary 4.2. Therefore, (1) is obviously true. ■

Remark 4.5. The condition “ F is a dual Banach lattice” is not superfluous in the above theorem. Indeed, every positive operator $T : \ell^\infty \rightarrow c$ is weakly compact and hence DPcc (since it factors through a reflexive Banach space). However, neither the assertion (a) nor (b) is valid.

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REFERENCES

- [1] C.D. ALIPRANTIS, O. BURKINSHAW, “Locally Solid Riesz Spaces”, Pure and Applied Mathematics **76**, Academic Press, New York-London, 1978.
- [2] C.D. ALIPRANTIS, O. BURKINSHAW, “Positive Operators”, Reprint of the 1985 original, Springer, Dordrecht, 2006.
- [3] B. ALTIN, On b -weakly compact operators on Banach lattices, *Taiwanese J. Math.* **11** (1) (2007), 143–150.

- [4] B. AQZZOUZ, R. NOUIRA, L. ZRAOULA, Compactness properties of operators dominated by AM-compact operators, *Proc. Amer. Math. Soc.* **135** (4) (2007), 1151–1157.
- [5] B. AQZZOUZ, K. BOURAS, Dunford-Pettis sets in Banach lattices, *Acta Math. Univ. Comenian. (New Ser.)* **81** (2) (2012), 185–196.
- [6] B. AQZZOUZ, A. ELBOUR, Some characterizations of almost Dunford-Pettis operators and applications, *Positivity* **15** (3) (2011), 369–380.
- [7] K. BOURAS, M. MOUSSA, Banach lattices with weak Dunford-Pettis property, World Academy of Science, Engineering and Technology, International Science Index 50, **5** (2) (2011), 739–743.
- [8] K. BOURAS, Almost Dunford-Pettis sets in Banach lattices, *Rend. Circ. Mat. Palermo (2)* **62** (2) (2013), 227–236.
- [9] J.X. CHEN, L. LI, On a question of Bouras concerning weak compactness of almost Dunford-Pettis sets, *Bull. Aust. Math. Soc.* **92** (1) (2015), 111–114.
- [10] P.G. DODDS, D.H. FREMLIN, Compact operators in Banach lattices, *Israel J. Math.* **34** (4) (1979), 287–320.
- [11] G. EMMANUELE, Banach spaces in which Dunford-Pettis sets are relatively compact, *Arch. Math. (Basel)* **58** (5) (1992), 477–485.
- [12] P. MEYER-NIEBERG, “Banach Lattices”, Universitext, Springer-Verlag, Berlin, 1991.
- [13] J.A. SANCHEZ, “Operators on Banach Lattices” (Spanish), Ph. D. Thesis, Complutense University, Madrid, 1985.
- [14] H.H. SCHAEFER, “Banach Lattices and Positive Operators”, Springer-Verlag, New York-Heidelberg, 1974.
- [15] Y. WEN, J. CHEN, Characterizations of Banach spaces with relatively compact Dunford-Pettis sets, *Adv. in Math. (China)*, <http://advmath.pku.edu.cn/EN/10.11845/sxjz.2014095b>
- [16] W. WNUK, Banach lattices with the weak Dunford-Pettis property, *Atti Semin. Mat. Fis. Univ. Modena* **42** (1) (1994), 227–236.
- [17] A.W. WICKSTEAD, Converses of the Dodds-Fremlin and Kalton-Saab theorems, *Math. Proc. Cambridge Philos. Soc.* **120** (1) (1996), 175–179.