Non-additive Lie centralizer of strictly upper triangular matrices

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Abstract: Let \( F \) be a field of zero characteristic, let \( N_n(F) \) denote the algebra of \( n \times n \) strictly upper triangular matrices with entries in \( F \), and let \( f : N_n(F) \rightarrow N_n(F) \) be a non-additive Lie centralizer of \( N_n(F) \), that is, a map satisfying that \( f([X,Y]) = [f(X),Y] \) for all \( X,Y \in N_n(F) \). We prove that \( f(X) = \lambda X + \eta(X) \) where \( \lambda \in F \) and \( \eta \) is a map from \( N_n(F) \) into its center \( Z(N_n(F)) \) satisfying that \( \eta([X,Y]) = 0 \) for every \( X,Y \) in \( N_n(F) \).

Key words: Lie centralizer, strictly upper triangular matrices, commuting map.

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1. Introduction

Consider a ring \( R \). An additive mapping \( T : R \rightarrow R \) is called a left (respectively right) centralizer if \( T(ab) = T(a)b \) (respectively \( T(ab) = aT(b) \)) for all \( a,b \in R \). The map \( T \) is called a centralizer if it is a left and a right centralizer. The characterization of centralizers on algebras or rings has been a widely discussed subject in various areas of mathematics.

In [13] Zalar proved the following interesting result: if \( R \) is a 2-torsion free semiprime ring and \( T \) is an additive mapping such that \( T(a^2) = T(a)a \) (or \( T(a^2) = aT(a) \)), then \( T \) is a centralizer. Vukman [12] considered additive maps satisfying similar conditions, namely \( 2T(a^2) = T(a)a + aT(a) \) for any \( a \in R \), and showed that if \( R \) is a 2-torsion free semiprime ring then \( T \) is also a centralizer. Since then, the centralizers have been intensively investigated by many mathematicians (see, e.g., [3, 4, 5, 6, 8]).

Let \( R \) be a ring. An additive map \( f : R \rightarrow R \), is called a Lie centralizer of \( R \) if

\[
[f([x,y])] = [f(x),y] \quad \text{for all } x,y \in R,
\]

where \([x,y]\) is the Lie product of \( x \) and \( y \).

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The inspiration of this paper comes from the articles [1, 5, 7] in which the authors deal with the Lie centralizer maps of triangular algebras and rings. In this note we will consider non-additive Lie centralizers on strictly upper triangular matrices over a field of zero characteristic.

Throughout this article, \( F \) is a field of zero characteristic. Let \( M_n(F) \) and \( N_n(F) \) denote the algebra of all \( n \times n \) matrices and the algebra of all \( n \times n \) strictly upper triangular matrices over \( F \), respectively. We use \( \text{diag}(a_1, a_2, \ldots, a_n) \) to represent a diagonal matrix with diagonal \( (a_1, a_2, \ldots, a_n) \) where \( a_i \in F \). The set of all \( n \times n \) diagonal matrices over \( F \) is denoted by \( D_n(F) \).

Let \( I_n \) be the identity in \( M_n(F) \), \( J = \sum_{i=1}^{n-1} E_{i,i+1} \) and \( \{ E_{ij} : 1 \leq i, j \leq n \} \) the canonical basis of \( M_n(F) \), where \( E_{ij} \) is the matrix with 1 in the \((i, j)\) position and zeros elsewhere. By \( C_{N_n(F)}(X) \) we will denote the centralizer of the element \( X \) in the ring \( N_n(F) \).

The notation \( f : N_n(F) \to N_n(F) \) means a non-additive map satisfying \( f([X,Y]) = [f(X),Y] \) for all \( X,Y \in N_n(F) \).

Notice that it is easy to check that \( Z(N_n(F)) = FE_{1n} \).

The main result in this paper is the following:

**Theorem 1.1.** Let \( F \) be a field of zero characteristic. If \( f : N_n(F) \to N_n(F) \) is a non-additive Lie centralizer then there exists \( \lambda \in F \) and a map \( \eta : N_n(F) \to Z(N_n(F)) \) satisfying \( \eta([X,Y]) = 0 \) for every \( X,Y \in N_n(F) \) such that \( f(X) = \lambda X + \eta(X) \) for all \( X \in N_n(F) \).

Notice that the converse is trivially true: every map \( f(X) = \lambda X + \eta(X) \) with \( \eta \) satisfying the condition in Theorem 1.1 is a (non-additive) Lie centralizer.

2. **Proofs**

Let’s start with some basic properties of Lie centralizers.

**Lemma 2.1.** Let \( f \) be a non-additive Lie centralizer of \( N_n(F) \). Then:

1. \( f(0) = 0 \);
2. for every \( X,Y \in N_n(F) \), we have \( f([X,Y]) = [X,f(Y)] \);
(3) \( f \) is a commuting map, i.e., \( f(X)X = Xf(X) \) for all \( X \in N_n(\mathcal{F}) \).

Proof. To prove (1) it suffices to notice that
\[
f(0) = f([0, 0]) = [f(0), 0] = 0.
\]

(2) Observe that if \( f([X, Y]) = [f(X), Y] \), then we have
\[
f(XY - YX) = f(X)Y - Yf(X).
\]
Interchanging \( X \) and \( Y \) in the above identity, we have
\[
f(YX - XY) = f(Y)X - Xf(Y).
\]
Replacing \( X \) with \(-X\) in the above relation, we arrive at
\[
f(XY - YX) = Xf(Y) - f(Y)X\text{ which can be written as } f([X, Y]) = [X, f(Y)].
\]
From (1) one also gets (3):
\[
[f(X), X] = f([X, X]) = f(0) = 0.
\]

Remark 2.1. Let \( f \) be a non-additive Lie centralizer of \( N_n(\mathcal{F}) \) and \( X \in C_N(\mathcal{F})(Y) \). Then \( f(X) \in C_N(\mathcal{F})(Y) \). Indeed, if \( X \in C_N(\mathcal{F})(Y) \), then \( [X, Y] = 0 \) and
\[
0 = f(0) = f([X, Y]) = [f(X), Y].
\]

Lemma 2.2. Let \( f \) be a non-additive Lie centralizer of \( N_n(\mathcal{F}) \). Then:

(1) \( f \left( \sum_{i=1}^{n-1} a_i E_{i,i+1} \right) = \sum_{i=1}^{n-1} b_i E_{i,i+1} \);

(2) there exists \( \lambda \in \mathcal{F} \) such that \( f(J) = \lambda J \).

Proof. Let \( D_0 = \sum_{i=1}^{n} (n - i) E_{i,i} \).

(1) Consider \( A \in M_n(\mathcal{F}) \). It is well known that \([D_0, A] = A\) if and only if \( A = \sum_{i=1}^{n-1} a_i E_{i,i+1} \).

Hence, if \( A = \sum_{i=1}^{n-1} a_i E_{i,i+1} \), we have \([D_0, A] = A\). Thus \( f([D_0, A]) = [D_0, f(A)] = f(A) \). Therefore \( f(A) = \sum_{i=1}^{n-1} b_i E_{i,i+1} \).
(2) As in (1), consider \( A = \sum_{i=1}^{n-1} a_i E_{i,i+1} \) for some \( a_i \in \mathcal{F} \). Then \([J, A] = 0\) if and only if \( A = aJ \) for some \( a \in \mathcal{F} \).

Indeed, \( f(J) = \sum_{i=1}^{n-1} a_i E_{i,i+1} \) by (1). Thus, \( 0 = f(0) = f([J, J]) = [J, f(J)] \).

Hence, there exists \( \lambda \in \mathcal{F} \) such that \( f(J) = \lambda J \). \( \blacksquare \)

We will need the following lemma.

**Lemma 2.3.** (Lemma 2.1, [14]) Suppose that \( \mathcal{F} \) is an arbitrary field. If \( G, H \in UT_n(\mathcal{F}) \) are such that \( g_{i,i+1} = h_{i,i+1} \neq 0 \) for all \( 1 \leq i \leq n - 1 \), then \( G \) and \( H \) are conjugated in \( UT_n(\mathcal{F}) \).

Here \( UT_n(\mathcal{F}) \) is the multiplicative group of \( n \times n \) upper triangular matrices with only 1’s in the main diagonal. From the lemma above we obtain the following corollary.

**Corollary 2.1.** Let \( \mathcal{F} \) be a field. For every \( A = \sum_{1 \leq i < j \leq n} a_{ij} E_{ij} \), where \( a_{i,i+1} \neq 0 \) for every \( i = 1, \ldots, n-1 \), there exists \( B \in T_n(\mathcal{F}) \) such that \( B^{-1} AB = J \) and \( T_n(\mathcal{F}) \) is the ring of upper triangular matrices.

**Proof.** Let \( A \) be a matrix in \( N_n(\mathcal{F}) \) of the mentioned form. Then \( I_n + A \) is a unitriangular matrix. Let’s notice first that there exists \( B_1 \in D_n(\mathcal{F}) \) such that \( (B_1^{-1} AB_1)_{i,i+1} = 1 \) for all \( i \in \mathbb{N} \). We can construct \( B_1 \in D_n(\mathcal{F}) \) recursively by:

\[
(B_1)_{11} = 1, \quad (B_1)_{i,i+1} = (B_1)_{ii} \cdot (A_{i,i+1})^{-1} \quad \text{for} \quad i \geq 1.
\]

Consider the matrix \( I_n + B_1^{-1} AB_1 \in UT_n(\mathcal{F}) \). The unitriangular matrices \( I_n + J \) and \( I_n + B_1^{-1} AB_1 \) fulfill the condition in Lemma 2.3. Hence, there exists \( B_2 \in UT_n(\mathcal{F}) \) such that

\[
I_n + J = B_2^{-1} (I_n + B_1^{-1} AB_1) B_2.
\]

Then \( J = B_2^{-1} (B_1^{-1} AB_1) B_2 \). Taking \( B = B_1 B_2 \in T_n(\mathcal{F}) \), we get \( J = B^{-1} AB \) as wanted. \( \blacksquare \)

**Lemma 2.4.** Let \( A = \sum_{i<j} a_{ij} E_{ij} \) be a matrix in \( N_n(\mathcal{F}) \) with \( a_{i,i+1} \neq 0 \) for every \( i = 1, \ldots, n-1 \). Then there exists \( \lambda_A \in \mathcal{F} \) such that \( f(A) = \lambda_A A \).
Proof. Since $A = \sum_{1 \leq i < j \leq n} a_{ij}E_{ij}$, where $a_{i,i+1} \neq 0$, there exists $T \in T_n(\mathcal{F})$ such that $TAT^{-1} = J$ by the previous corollary. Define $h : N_n(\mathcal{F}) \to N_n(\mathcal{F})$ by $h(X) = Tf(T^{-1}XT)T^{-1}$. Then $h$ is a non-additive Lie centralizer. Indeed, for all $A, B \in N_n(\mathcal{F})$ we have:

$$h([A,B]) = Tf(T^{-1}[A,B]T)T^{-1} = Tf(T^{-1}(AB - BA)T)T^{-1} = Tf(T^{-1}ATT^{-1}BT - T^{-1}BTT^{-1}AT)T^{-1} = Tf([T^{-1}AT, T^{-1}BT])T^{-1} = T(f(T^{-1}AT),T^{-1}BT)T^{-1} = T(f(T^{-1}AT)T^{-1}B - BTf(T^{-1}AT))T^{-1} = [Tf(T^{-1}AT)T^{-1}, B] = [h(A), B].$$

Hence, $h(J) = \lambda_A J$ by Lemma 2.2. Then

$$Tf(A)T^{-1} = Tf(T^{-1}(TAT^{-1})T)T^{-1} = h(J) = \lambda_A J = \lambda_A TAT^{-1}. $$

Multiplying the left and right sides by $T^{-1}$ and $T$ respectively yields $f(A) = \lambda_A A$. |

Now we wish to extend Lemma 2.4 to all elements of $N_n(\mathcal{F})$. In order to do this, let’s introduce the following set:

$$\mathcal{S} = \{ B = (b_{ij}) \in N_n(\mathcal{F}) : b_{i,i+1} \neq 0 \quad \forall \ i = 1, \ldots, n-1 \}. $$

This set has an important property that is established below.

**Lemma 2.5.** Let $\mathcal{F}$ be a field. Every element of $N_n(\mathcal{F})$ can be written as a sum of at most two elements of $\mathcal{S}$.

**Proof.** If $a_{i,i+1} \neq 0$ for all $i = 1, \ldots, n-1$, then $A$ belongs to $\mathcal{S}$, so there is nothing to prove. If $A$ is not in $\mathcal{S}$, then we can define $B_1$ and $B_2$ as follows:

$$(B_1)_{ij} = \begin{cases} a_{i,i+1} - b_i & \text{if } j = i + 1, \\ a_{ij} & \text{if } j > i + 1, \end{cases} \quad (B_2)_{ij} = \begin{cases} b_i & \text{if } j = i + 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $b_i$ is an element in $\mathcal{F}$ different from $a_{i,i+1}$. It is easy to see that $B_1$, $B_2$ are in $\mathcal{S}$, and $A = B_1 + B_2$, so we wanted. |
Lemma 2.6. Let $\mathcal{F}$ be a field. For arbitrary elements $A, B$ of $N_n(\mathcal{F})$, there exists $\lambda_{A,B} \in \mathcal{F}$ such that
\[
f(A + B) = f(A) + f(B) + \lambda_{A,B} E_{1n}.
\]

Proof. For any $A, B, X$ of $N_n(\mathcal{F})$, we have
\[
[f(A + B), X] = f([A + B, X])
\]
\[
= [A + B, f(X)]
\]
\[
= [A, f(X)] + [B, f(X)]
\]
\[
= [f(A), X] + [f(B), X]
\]
\[
= [f(A) + f(B), X],
\]
which implies that $f(A + B) - f(A) - f(B) \in \mathcal{Z}(N_n(\mathcal{F}))$. Thus, there exists $\lambda_{A,B} \in \mathcal{F}$ such that $f(A + B) = f(A) + f(B) + \lambda_{A,B} E_{1n}$.

Now we can prove the main theorem.

Proof of Theorem 1.1. Let $A, B \in S$ be two non-commuting elements. By Lemma 2.4, $f(A) = \lambda_A A, f(B) = \lambda_B B$, $\lambda_A, \lambda_B \in \mathcal{F}$.

Since $f$ is a non-additive Lie centralizer, we get,
\[
f([A, B]) = [f(A), B] = \lambda_A [A, B]
\]
\[
= [A, f(B)] = \lambda_B [A, B].
\]

Then, $[A, B] \neq 0$ implies that $\lambda_A = \lambda_B$. If $A, B \in S$ commute, then we take $C \in S$ that does not commute neither with $A$ nor with $B$. As we have just seen, $\lambda_A = \lambda_C$ and $\lambda_B = \lambda_C$. So $\lambda_A = \lambda_B = \lambda$ for arbitrary elements $A, B \in S$. Given $X \in N_n(\mathcal{F})$ we know, by Lemma 2.5 that there exists $A, B \in S$ such that $X = A + B$ (we can assume that $X \notin S$). Then $f(X) - f(A) - f(B) \in \mathcal{Z}(N_n(\mathcal{F}))$ by Lemma 2.6.

That is $f(X) - \lambda_A A - \lambda_B B = f(X) - \lambda X \in \mathcal{Z}(N_n(\mathcal{F}))$ for $\lambda \in \mathcal{F}$ such that $f(A) = \lambda A$ for each $A \in S$.

We can define $\eta : N_n(\mathcal{F}) \to \mathcal{Z}(N_n(\mathcal{F}))$ such that $\eta(X) = f(X) - \lambda X$, that is, $f(X) = \lambda X + \eta(X)$.

Notice that $\eta(A) = 0$ for each $A \in S$. Furthermore, if $X, Y \in N_n(\mathcal{F})$, then
\[
f([X, Y]) = \lambda [X, Y] + \eta([X, Y]) = [f(X), Y]
\]
\[
= \lambda [X + \eta(X), Y] = \lambda [X, Y],
\]
since $\eta(X) \in \mathcal{Z}(N_n(\mathcal{F}))$.

Consequently, $\eta([X, Y]) = 0$ and Theorem 1.1 is proved.
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REFERENCES


