Uniformly Bounded Superposition Operators in the Space of Functions of Bounded \( n \)-Dimensional \( \Phi \)-Variation

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Abstract: We prove that if a superposition operator maps a subset of the space of all functions of \( n \)-dimensional bounded \( \Phi \)-variation in the sense of Riesz, into another such space, and is uniformly bounded, then the non-linear generator \( h(x, y) \) of this operator must be of the form \( h(x, y) = A(x)y + B(x) \) where, for every \( x \), \( A(x) \) is a linear map.

Key words: Nemytskij operator, \( n \)-dimensional \( \Phi \)-variation, \( \varphi \)-function.

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1. Introduction

Given two (non-empty) sets \( A \) and an \( B \), the notation \( B^A \) will stand for the set of all functions from \( A \) to \( B \). As usual, if \( M \) is a normed space, \( \mathcal{L}(M) \) denotes the set of all bounded operators on \( M \).

Let \( A, B \) and \( C \) be non-empty sets. If \( h : A \times C \to B \) is a given function, \( X \subset C^A \) and \( Y \subset B^A \) are linear spaces then, the nonlinear superposition (Nemytskij) operator \( H : X \to Y \), generated by the function \( h \), is defined as

\[
(Hf)(t) := h(t, f(t)), \quad t \in A.
\]

This operator plays a central role in many mathematical fields (e.g. in the theory of nonlinear integral equations), by its applications to a variety of nonlinear problems, and has been studied thoroughly. Apart from conditions for the mere inclusion \( H(X) \subset Y \), the boundedness or the continuity of \( H \) (cf. [2]), another important problem has been widely investigated:
to find conditions on the generating function \( h \) in the case in which 
\((X, d_X), (Y, d_Y)\) are also metric spaces and the superposition operator \( H \) is 
uniformly continuous or satisfies some global or local Lipschitz condition of
the form
\[
d_Y(H(f_1), H(f_2)) \leq \alpha d_X(f_1, f_2), \quad f_1, f_2 \in X,
\]
cf. e.g., [10, 11, 5].

In this paper we will investigate a problem related to this last situation.
Throughout this paper the letter \( n \) denotes a positive integer. Let 
\( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_n) \) be points in \( \mathbb{R}^n \). We will use the notation \( a < b \) to mean that \( a_i < b_i \) for each \( i = 1, \ldots, n \) and accordingly we define 
\( a = b, a \leq b, a \geq b \) and \( a > b \). If \( a < b \), the set 
\( J := [a, b] = \prod_{i=1}^{n} [a_i, b_i] \) will be called an \( n \)-dimensional closed interval.

Given an \( n \)-dimensional closed interval \( J \), a metric vector space \( M \) and \( \Phi \) a
\( \varphi \)-function, the space of all functions, defined on \( J \) of
\( n \)-dimensional bounded \( \Phi \)-variation will be denoted by 
\( BRV^n_{\Phi}(J; M) \). Suppose that \( N \) is another 
vector metric space, \( C \) is a convex subset of \( M \), \( \Psi \) is another
\( \varphi \)-function and \( h : J \times C \to N \) is a given function. In this paper we prove that if the superposition 
operator \( H \), generated by \( h \), maps the set 
\( \{ f \in BRV^n_{\Phi}(J; M) : f(J) \subset C \} \) into 
\( BRV^n_{\Psi}(J; N) \) and is uniformly bounded, in the sense introduced in [12], then 
there is a linear operator \( A \in L(M, N) \) and a function \( B \in N^J \) such that
\[
h(x, y) = A(x)y + B(x), \quad x \in J, \quad y \in C.
\]

This is a counterpart of a result of Matkowski proved in [12] for the space of
Lipschitz continuous functions.

2. Functions of bounded \( n \)-dimensional \( \Phi \)-variation

In this section we present the definition and main basic aspects of the
notion of \( n \)-dimensional \( \Phi \)-variation for functions defined on \( n \)-dimensional closed intervals of \( \mathbb{R}^n \), that take values on a metric semigroup, as introduced
in [3]. This generalization of the notion of bounded variation for functions of
several variables is inspired in the works of Chistyakov and Talalyan [6, 13].
Here, we also combine the notions of variations given by Vitali ([14]) and later
generalized by Hardy and Krause (cf. [4, 7]).

**Definition 2.1.** A metric semigroup is a structure \((M, d, +)\) where
\((M, +)\) is an abelian semigroup and \( d \) is a translation invariant metric on
\( M \).
In particular, the triangle inequality implies that, for all \( u, v, p, q \in M \),
\[
\begin{align*}
d(u,v) & \leq d(p,q) + d(u+p,v+q), \\
d(u+p,v+q) & \leq d(u,v) + d(p,q).
\end{align*}
\tag{2.1}
\]

In this paper we will use the following standard notation: \( \mathbb{N} \) (resp. \( \mathbb{N}_0 \)) denotes the set of all positive integers (resp. non-negative integers) and a typical point of \( \mathbb{R}^n \) is denoted as \( x = (x_1, x_2, \ldots, x_n) := \left( x_i ^{\mathbb{N}}_{i=1} \right) \); but, the canonical unit vectors of \( \mathbb{R}^n \) are denoted by \( \mathbf{e}_j \) \( (j = 1, 2, \ldots, n) \); that is, \( \mathbf{e}_j := \left( \mathbf{e}_r ^{(j)} \right)_{r=1}^{n} \) where, \( \mathbf{e}_r ^{(j)} := \begin{cases} 0 & \text{if } r \neq j \\ 1 & \text{if } r = j \end{cases} \).

The zero \( n \)-tuple \( (0,0,\ldots,0) \) will be denoted by \( \mathbf{0} \), and by \( \mathbf{1} \) we will denote the \( n \)-tuple \( \mathbf{1} = (1,1,\ldots,1) \).

If \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), with \( \alpha_j \in \mathbb{N}_0 \), is a \( n \)-tuple of non-negative integers then we call \( \alpha \) a multi-index \((1)\).

If \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \) we use the notation \( \mathbf{a} < \mathbf{b} \) to mean that \( x_i < y_i \) for each \( i = 1, \ldots, n \) and similarly are defined \( \mathbf{a} = \mathbf{b}, \mathbf{a} \leq \mathbf{b} \) and \( \mathbf{a} > \mathbf{b} \). If \( \mathbf{a} < \mathbf{b} \), the set \( [\mathbf{a}, \mathbf{b}] = \prod_{i=1}^{n} [a_i, b_i] \) will be called a \( n \)-dimensional closed interval.

The euclidean volume of an \( n \)-dimensional closed interval will be denoted by \( \text{Vol} \left[ \mathbf{a}, \mathbf{b} \right] \); that is, \( \text{Vol} \left[ \mathbf{a}, \mathbf{b} \right] = \prod_{i=1}^{n} (b_i - a_i) \).

In addition, for \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n \) and \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) we will use the notations
\[
|\alpha| := \alpha_1 + \alpha_2 + \ldots + \alpha_n \quad \text{and} \quad \alpha \mathbf{x} := (\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n).
\]

We will denote by \( \mathcal{N} \) the set of all strictly increasing continuous convex functions \( \Phi : [0, +\infty) \to [0, +\infty) \) such that \( \Phi(t) = 0 \) if and only if \( t = 0 \) and \( \lim_{t \to \infty} \Phi(t) = +\infty \).

\( \mathcal{N}_\infty \) the set of all functions \( \Phi \in \mathcal{N} \), for which the Orlicz condition (also called \( \infty_1 \) condition) holds: \( \lim_{t \to \infty} \frac{\Phi(t)}{t} = +\infty \). Functions from \( \mathcal{N} \) are often called \( \varphi \)-functions.

One says that a function \( \Phi \in \mathcal{N} \) satisfies a condition \( \Delta_2 \), and writes \( \Phi \in \Delta_2 \), if there are constants \( K > 2 \) and \( t_0 > 0 \) such that
\[
\Phi(2t) \leq K \Phi(t) \quad \text{for all} \quad t \geq t_0.
\tag{2.2}
\]
Now we define two important sets.

\[ E(n) := \{ \theta \in \mathbb{N}_0^n : \theta \leq 1 \text{ and } |\theta| \text{ is even} \} \]

\[ O(n) := \{ \theta \in \mathbb{N}_0^n : \theta \leq 1 \text{ and } |\theta| \text{ is odd} \} \]

Notice that these sets are related in a one to one fashion; indeed, if \( \theta = (\theta_1, \ldots, \theta_n) \in E(n) \) then we can define \( \tilde{\theta} := (1 - \theta_1, \theta_2, \ldots, \theta_n) \in O(n) \), and this operation is clearly invertible.

In what follows \( M \) is supposed to be a metric semigroup and \([a, b]\) an \( n\)-dimensional closed interval.

**Definition 2.2.** ([7, 6, 14]) Given a function \( f : [a, b] \to M \), we define the \( n\)-dimensional Vitali difference of \( f \) over an \( n\)-dimensional closed interval \([x, y]\subseteq [a, b] \), by

\[
\Delta_n(f, [x, y]) := d\left( \sum_{\theta \in E(n)} f(\theta x + (1 - \theta)y), \sum_{\theta \in O(n)} f(\theta x + (1 - \theta)y) \right). \tag{2.3}
\]

This difference is also called mixed difference and it is usually associated to the names of Vitali, Lebesgue, Hardy, Krause, Fréchet and De la Vallée Poussin ([7, 6, 8]).

Now, in order to define the \( \Phi \)-variation of a function \( f : [a, b] \to M \), we consider net partitions of \([a, b]\); that is, partitions of the kind

\[ \xi = \xi_1 \times \xi_2 \times \ldots \times \xi_n \quad \text{with} \quad \xi_i := \{ t_{j}^{(i)} \}_{j \in \mathbb{N}_0}^{k_i}, \quad i = 1, \ldots, n. \tag{2.4} \]

where, \( \{k_i\}_{i=1}^{n} \subseteq \mathbb{N} \) and for each \( i \), \( \xi_i \) is a partition of \([a_i, b_i]\). The set of all net partitions of an \( n\)-dimensional closed interval \([a, b]\) will be denoted by \( \pi([a, b]) \).

A point in a net partition \( \xi \) is called a node ([13]) and it is of the form

\[ t_\alpha := (t_1^{(1)}, t_2^{(2)}, t_3^{(3)}, \ldots, t_n^{(n)}) \]

where \( 0 \leq \alpha = (\alpha_i)_{i=1}^{n} \leq \kappa \), with \( \kappa := (k_i)_{i=1}^{n} \).

For the sake of simplicity in notation, we will simply write \( \xi = \{ t_\alpha \} \), to refer to all the nodes that form a given partition \( \xi \).

A cell of an \( n\)-dimensional closed interval \([a, b]\) is an \( n\)-dimensional closed subinterval of the form \([t_{\alpha-1}, t_\alpha]\), for \( 0 < \alpha \leq \kappa \).

Note that

\[ t_0 = (t_0^{(1)}, t_0^{(2)}, \ldots, t_0^{(n)}) = (a_1, a_2, \ldots, a_n) \quad \text{and} \quad t_\kappa = (t_\kappa^{(1)}, t_\kappa^{(2)}, \ldots, t_\kappa^{(n)}) = (b_1, b_2, \ldots, b_n). \]
Definition 2.3. Let \( f : [a, b] \to M \) and \( \Phi \in \mathcal{N} \). The \( \Phi \)-variation, in the sense of Vitali-Riesz of \( f \) is defined as

\[
\rho_\Phi^\Phi(f, [a, b]) := \sup_{\xi \in [a, b]} \rho_\Phi^\Phi(f, [a, b], \xi).
\]

(2.5)

where

\[
\rho_\Phi^\Phi(f, [a, b], \xi) := \sum_{1 \leq \alpha \leq \kappa} \Phi\left( \frac{\Delta_n(f, [t_{\alpha-1}, t_\alpha])}{\text{Vol}[t_{\alpha-1}, t_\alpha]} \right) \text{Vol}[t_{\alpha-1}, t_\alpha].
\]

We now need to define the truncation of a point, of an \( n \)-dimensional closed interval and of a function, by a given multi-index \( 0 < \eta \leq 1 \). Notice that in this case, the entries of \( \eta \) are either 0 or 1.

- The truncation of a point \( x \in \mathbb{R}^n \) by a multi-index \( 0 < \eta \leq 1 \), which is denoted by \( x|\eta \), is defined as the \( |\eta| \)-tuple that is obtained if we suppress from \( x \) the entries for which the corresponding entries of \( \eta \) are equal to 0. That is, \( x|\eta = (x_i : i \in \{1, 2, ..., n\}, \eta_i = 1) \). For instance, if \( x = (x_1, x_2, x_3, x_4, x_5) \) and \( \eta = (0, 1, 1, 0, 1) \) then \( x|\eta = (x_2, x_3, x_5) \).

- The truncation of an \( n \)-dimensional closed interval \([a, b]\) by a multi-index \( 0 < \eta \leq 1 \), is defined as \([a, b]|\eta := [a|\eta, b|\eta]\).

- Given a function \( f : [a, b] \to M \), a multi-index \( 0 < \eta \leq 1 \) and a point \( z \in [a, b] \), we define \( f_\eta^a : [a, b]|\eta \to M \), the truncation of \( f \) by \( la \eta \), by the formula

\[
f_\eta^a(x|\eta) := f(\eta x + (1 - \eta)z), \ x \in [a, b].
\]

Note that the function \( f_\eta^a \) depends only on the \( |\eta| \) variables \( x_i \) for which \( \eta_i = 1 \).

Remark 2.4. Given a function \( f : [a, b] \to M \) and a multi-index \( \eta \neq 0 \), then the \( |\eta| \)-dimensional Vitali difference for \( f_\eta^a \) (cf. (2.3)), is given by

\[
\Delta_{|\eta|}(f_\eta^a, [x, y]) := d\left( \sum_{\theta \in \mathcal{E}(n)} f(\eta(\theta x + (1 - \theta)y) + (1 - \eta)a, \sum_{\theta \in \mathcal{O}(n)} f(\eta(\theta x + (1 - \theta)y) + (1 - \eta)a) \right).
\]
Definition 2.5. Let $\Phi \in \mathcal{N}$ and let $(M, d, +, \cdot)$ be a metric semigroup. A function $f : [a, b] \to M$ is said to be of bounded $\Phi$-variation, in the sense of Riesz, if the total $\Phi$-variation

$$TRV_\Phi(f, [a, b]) := \sum_{0 \neq \eta \leq 1} \rho_\Phi^{|\eta|}(f^a_\eta, [a, b]|\eta),$$

is finite. The set of all functions $f$ that satisfy $TRV_\Phi(f, [a, b]) < +\infty$ will be denoted by $RV_\Phi^n([a, b]; M)$.

3. The normed space $BRV_\Phi^n([a, b]; M)$

So far, our consideration of metric semigroups as targets sets suffices adequately to define a notion of $n$-dimensional variation; however, as we need to study a superposition operator problem between linear spaces in which the presence of this notion is desired, it will be necessary to ask for additional structure on the target set $M$. The one that we will considerate is that of vectorial metric space.

Definition 3.1. By a metric vector space (MVS) we will understand a topological vector space $(M, \tau)$ in which the topology $\tau$ is induced by a metric $d$ that satisfies the following conditions:

1. $d$ is a translation invariant metric.
2. $d(\alpha a, \alpha b) \leq |\alpha| d(a, b)$ for any $\alpha \in \mathbb{R}$ and $a, b \in M$.

Note that, in particular, any MVS is a metric semigroup. In what follows $\mathcal{M}$ is supposed to be a MVS and $[a, b]$ an $n$-dimensional closed interval.

Remark 3.2. It readily follows from 2.1 that given two functions $f, g : [a, b] \to M$, a multi-index $\eta \neq 0$ and an $n$-dimensional closed interval $[x, y] \subset [a, b]$, then the $|\eta|$-dimensional Vitali difference (c.f. (2.3)) of the truncation $(f + g)^a_\eta$ satisfies the inequality

$$\Delta_{|\eta|}\left(f^a_\eta + g^a_\eta, [x, y]\right) \leq \Delta_{|\eta|}\left(f^a_\eta, [x, y]\right) + \Delta_{|\eta|}\left(g^a_\eta, [x, y]\right). \quad (3.1)$$

Lemma 3.3. The functional $TRV_\Phi(\cdot, [a, b])$ is convex.

Proof. The lemma is consequence of (3.1) and of the fact that $\Phi$ is a nondecreasing convex function. \qed
Theorem 3.4. The class $RV^n_\Phi([a,b];\mathcal{M})$ is symmetric and convex.

Proof. That $RV^n_\Phi([a,b];\mathcal{M})$ is symmetric is consequence of property (2) (since $d(-a,-b) \leq d(a,b)$) of Definition 3.1 while convexity follows from Lemma 3.3.

As a consequence of Theorem 3.4, the linear space generated by the set $RV^n_\Phi([a,b];\mathcal{M})$ is

$$\{ f : [a,b] \to \mathcal{M} : \lambda f \in RV^n_\Phi([a,b];\mathcal{M}) \text{ for some } \lambda > 0 \},$$

which, we will call, the space of functions of bounded $\Phi$-variation in the sense of Vitali-Hardy-Riesz and will denote as $BRV^n_\Phi([a,b];\mathcal{M})$.

Lemma 3.5. The set

$$\Lambda := \{ f \in BRV^n_\Phi([a,b];\mathcal{M}) : TRV_\Phi(f,[a,b]) \leq 1 \}$$

is a convex, balanced and absorbent subset of $BRV^n_\Phi([a,b];\mathcal{M})$.

Proof. To prove convexity suppose that $f,g \in \Lambda$ and let $\alpha,\beta$ be non-negative real numbers such that $\alpha + \beta = 1$. Then $TRV_\Phi(f,[a,b]) \leq 1$, $TRV_\Phi(g,[a,b]) \leq 1$ and by Lemma 3.3

$$TRV_\Phi(\alpha f + \beta g, [a,b]) \leq \alpha TRV_\Phi(f, [a,b]) + \beta TRV_\Phi(g, [a,b])$$

$$\leq \alpha + \beta = 1.$$

Hence $\Lambda$ is convex.

On the other hand, from Definition 2.3 it readily follows that if $f_0 \equiv 0$ then $TRV_\Phi(f_0,[a,b]) = 0$, thus $f_0 \in \Lambda$ and therefore, by virtue of the convexity property of $\Lambda$ just proved, $\Lambda$ is balanced. Finally, the fact that $\Lambda$ is absorbent follows from property (2) of Definition 3.1 and the convexity of $\Phi$.

By virtue of Lemma 3.5, the Minkowski Functional of $\Lambda$

$$p_\Lambda(f) := \inf \{ t > 0 : TRV_\Phi\left(\frac{f}{t}, [a,b]\right) \leq 1 \},$$

defines a seminorm on $BRV^n_\Phi([a,b];\mathcal{M})$, and therefore

$$\| f \| := \| f \|_{BRV^n_\Phi([a,b];\mathcal{M})} := d(f(a),0) + p_\Lambda(f)$$

defines a norm on $BRV^n_\Phi([a,b];\mathcal{M})$. 

Lemma 3.6. Let $f \in BRV^n_{\Phi}([a, b]; \mathcal{M})$,

(i) If $\|f\| \neq 0$ then $TRV_{\Phi}(f/\|f\|, [a, b]) \leq 1$;
(ii) if $0 \neq \|f\| \leq 1$ then $TRV_{\Phi}(f, [a, b]) \leq \|f\|.$

Proof. (i) From definition 3.2 $p_{\Lambda}(f) \leq \|f\|$. If $p_{\Lambda}(f) < \|f\|$ then there is $\xi \in \Lambda$ such that $p_{\Lambda}(f) < \xi \leq \|f\|$ and $TRV_{\Phi}(f/\xi, [a, b]) \leq 1$. So, since $\Lambda$ is absorbent, $f/\|f\| \in \Lambda$.

If $p_{\Lambda}(f) = \|f\|$, then there is a sequence $t_n \in \Lambda$ such that $t_n \to \|f\|$ and $TRV_{\Phi}(f/t_n, [a, b]) \leq 1$.

It follows, by the continuity of the functional $TRV_{\Phi}(\cdot, [a, b])$, that

$$TRV_{\Phi}(f/\|f\|, [a, b]) \leq 1.$$

(ii) follows from (i) and the convexity of $TRV_{\Phi}(\cdot, [a, b])$. 

Remark 3.7. It follows from Lemma 3.6 (i) that if $p_{\Lambda}(f) \neq 0$ and $t > \|f\|$ then $t \in \Lambda$.

4. Composition operator in the space $BRV^n_{\Phi}([a, b]; \mathcal{M})$

In this section we state and prove the main results of this paper concerning the action of a superposition operator between spaces of functions of bounded n-dimensional $\Phi$-variation. For the sake of clarity of exposition, we will denote the norm of $BRV^n_{\Phi}([a, b]; \mathcal{M})$ by $\|\cdot\|_{(\Phi, \mathcal{M})}$.

Theorem 4.1. Suppose that $[a, b] \subseteq \mathbb{R}^n$ is an n-dimensional closed interval, and that $\Phi, \Psi$ are $\varphi$-functions. Let $\mathcal{M}$ and $\mathcal{N}$ be linear metric spaces, $\mathcal{C} \subseteq \mathcal{M}$ a convex and closed set with non empty interior and let $h : [a, b] \times \mathcal{C} \to \mathcal{N}$ be a continuous function. If the Nemytskij operator $H$, generated by the function $h$, applies the set $K = \{f \in BRV^n_{\Phi}([a, b]; \mathcal{M}) : f([a, b]) \subseteq \mathcal{C}\}$ into $BRV^n_{\Psi}([a, b]; \mathcal{N})$ and satisfies the inequality

$$||H(f_1) - H(f_2)||_{(\Psi, \mathcal{N})} \leq \gamma \left(||f_1 - f_2||_{(\Phi, \mathcal{M})}\right) \quad f_1, f_2 \in K,$$

(4.1)
for some function \( \gamma : [0, \infty) \rightarrow [0, \infty) \), then there are functions \( A : [a, b] \rightarrow \mathcal{L}(\mathcal{M}, \mathcal{N}) \) and \( B \in \mathcal{N}^{[a, b]} \) such that

\[
h(x, u) = A(x)u + B(x), \quad x \in [a, b], \quad u \in \mathcal{C}.
\]

If \( 0 \in \mathcal{C} \), then \( B \in BRV^0([a, b] ; \mathcal{N}) \).

**Proof.** We will show that \( h \) satisfies the Jensen equation in the second variable.

Indeed, let \( t_1 = (t_1^{(i)})_{i=1}^n \) and \( t_2 = (t_2^{(i)})_{i=1}^n \in [a, b] \), suppose further that \( t_1 \leq t_2 \), and define the functions

\[
\eta_i(t) := \begin{cases} 
0 & \text{if } a_i \leq t \leq t_1^{(i)} \\
\frac{t - t_1^{(i)}}{t_2^{(i)} - t_1^{(i)}} & \text{if } t_1^{(i)} \leq t \leq t_2^{(i)} \\
1 & \text{if } t_2^{(i)} \leq t \leq b_i.
\end{cases}
\]

Next, consider \( y_1, y_2 \in \mathcal{C}, y_1 \neq y_2 \) and define

\[
f_j(x) := \frac{1}{2} \left[ \prod_{i=1}^n \eta_i(x_i) (y_1 - y_2) + y_j + y_2 \right], \tag{4.2}
\]

for \( j = 1, 2 \), where \( x := (x_1, x_2, ..., x_n) \).

Notice that

\[
f_1(x) - f_2(x) = \frac{1}{2} \left[ \prod_{i=1}^n \eta(x_i) (y_1 - y_2) + y_1 + y_2 - \prod_{i=1}^n \eta(x_i) (y_1 - y_2) - y_2 - y_2 \right]
= \frac{y_1 - y_2}{2}.
\]

Hence \( f_1 - f_2 \) has zero \( \Phi \)-variation and

\[
\|f_1 - f_2\|_{(\Phi, \mathcal{M})} = d((f_1 - f_2)(a), 0) + p_{\Phi}(f_1 - f_2)
= d((f_1 - f_2)(a), 0) = d \left( \frac{y_1 - y_2}{2}, 0 \right)
= d \left( \frac{y_1}{2}, \frac{y_2}{2} \right) > 0.
\]

Notice further that
If $x = t$, then
\[ \prod_{i=1}^{n} \eta(t_{\alpha_{i}}) = \prod_{i=1}^{n} \frac{t_{\alpha_{i}} - t}{t_{2} - t_{1}} = 1. \]

If $x = t$ with $\alpha_{i} \neq 2$ for some $1 \leq i \leq n$ then
\[ \prod_{i=1}^{n} \eta(t_{\alpha_{i}}) = \prod_{i=1}^{n} \frac{t_{\alpha_{i}} - t}{t_{2} - t_{1}} = 0. \]

Thus, by (4.2)
\[ \begin{align*}
& \text{If } \alpha_{i} = 2 \text{ for } i = 1, 2, ..., n \text{ then } \\
& \quad f_{1}(t) := \frac{1}{2} \left[ \prod_{i=1}^{n} \eta(t_{\alpha_{i}})(y_{1} - y_{2}) + y_{1} + y_{2} \right] = y_{1}, \\
& \quad f_{2}(t) := \frac{1}{2} \left[ \prod_{i=1}^{n} \eta(t_{\alpha_{i}})(y_{1} - y_{2}) + y_{2} + y_{2} \right] = \frac{y_{1} + y_{2}}{2}. \\
\end{align*} \]

Thus, by the definition of $H$, we have
\[ \begin{align*}
& \quad Hf_{1}(t_{2}) = h(t_{2}, f_{1}(t_{2})) = h(t_{2}, y_{1}) \\
& \quad Hf_{2}(t_{2}) = h(t_{2}, f_{2}(t_{2})) = h(t_{2}, \frac{y_{1} + y_{2}}{2}) \\
& \quad Hf_{1}(t_{1}) = h(t_{1}, f_{1}(t_{1})) = h(t_{1}, \frac{y_{1} + y_{2}}{2}) \\
& \quad Hf_{2}(t_{1}) = h(t_{1}, f_{2}(t_{1})) = h(t_{1}, y_{2}), \\
\end{align*} \]
and, if $\theta$ is a non-zero multi-index different from 1
\[ \begin{align*}
& \quad Hf_{1}(\theta t_{1} + (1 - \theta)t_{2}) = h(\theta t_{1} + (1 - \theta)t_{2}, \frac{y_{1} + y_{2}}{2}) \\
& \quad Hf_{2}(\theta t_{1} + (1 - \theta)t_{2}) = h(\theta t_{1} + (1 - \theta)t_{2}, y_{2}). \\
\end{align*} \]
On the other hand, for $f_1, f_2 \in K$ we have

$$\|H(f_1) - H(f_2)\|_{(\psi, N)} \leq \gamma \left(\|f_1 - f_2\|_{(\Phi, M)}\right),$$

thus

$$p_\phi(H(f_1) - H(f_2)) \leq \|H(f_1) - H(f_2)\|_{(\psi, N)} \leq \gamma \left(\|f_1 - f_2\|_{(\Phi, M)}\right).$$

Hence, by Remark 3.7 we have

$$p_n \phi \left(\frac{H f_1 - H f_2}{\gamma(\|f_1 - f_2\|_{(\Phi, M)})}, [a, b]\right) \leq TRV \phi \left(\frac{H f_1 - H f_2}{\gamma(\|f_1 - f_2\|_{(\Phi, M)})}, [a, b]\right),$$

$$\leq TRV \phi \left(\frac{H f_1 - H f_2}{\|H f_1 - H f_2\|_{(\psi, N)}}, [a, b]\right) \leq 1.$$  \hspace{1cm} (4.3)

Thus

$$1 \geq p_n \phi \left(\frac{H(f_1) - H(f_2)}{\gamma(\|f_1 - f_2\|_{(\Phi, M)})}, [a, b]\right) \geq \Phi \left(\frac{\Delta_n(H(f_1) - H(f_2), [t_1, t_2])}{\gamma(\|f_1 - f_2\|_{(\Phi, M)}) \text{Vol}[t_1, t_2]}\right) \text{Vol}[t_1, t_2],$$

which implies

$$\Phi^{-1} \left(\frac{1}{\text{Vol}[t_1, t_2]}\right) \text{Vol}[t_1, t_2] \geq \Delta_n \left(\frac{H(f_1) - H(f_2), [t_1, t_2]}{\gamma(\|f_1 - f_2\|_{(\Phi, M)})}\right)$$

and

$$\Delta_n(H(f_1) - H(f_2), [t_1, t_2]) \leq \Phi^{-1} \left(\frac{1}{\text{Vol}[t_1, t_2]}\right) \text{Vol}[t_1, t_2] \gamma(\|f_1 - f_2\|_{(\Phi, M)}).$$  \hspace{1cm} (4.4)
Making $t_2 \to t_1$ on the left hand side of (4.4) we get
\[
\lim_{t_2 \to t_1} d \left( \sum_{\theta \leq 1} (-1)^{\|\theta\|} (H(f_1) - H(f_2))(\theta t_1 + (1 - \theta)t_2), 0 \right)
= d \left( h(t_1, y_1) - h(t_1, \frac{y_1 + y_2}{2}) \right)
+ \lim_{t_2 \to t_1} \sum_{\theta \leq 1, \theta \neq 1} (-1)^{\|\theta\|} (H(f_1) - H(f_2))(\theta t_1 + (1 - \theta)t_2), 0 \right)
= d \left( h(t_1, y_1) - h(t_1, \frac{y_1 + y_2}{2}) \right)
+ \sum_{\theta \leq 1, \theta \neq 1} (-1)^{\|\theta\|} \left[ h(\theta t_1 + (1 - \theta)t_2, \frac{y_1 + y_2}{2}) - h(\theta t_1 + (1 - \theta)t_1, y_2) \right], 0 \right)
= d \left( h(t_1, y_1) - h(t_1, \frac{y_1 + y_2}{2}) \right)
+ \sum_{\theta \leq 1, \theta \neq 1} (-1)^{\|\theta\|} \left[ h(t_1, \frac{y_1 + y_2}{2}) - h(t_1, y_2) \right].
\] (4.5)

Now, the number of $n$-tuples that contain $k$ 1s, with $k > 0$, is equal to
\[
\binom{n}{k} = \frac{n!}{(n-k)!k!},
\]
thus
\[
\sum_{\theta \leq 1, \theta \neq 0} (-1)^{\|\theta\|} \left[ h \left( t_1, \frac{y_1 + y_2}{2} \right) - h(t_1, y_2) \right]
= \left[ h \left( t_1, \frac{y_1 + y_2}{2} \right) - h(t_1, y_2) \right] \sum_{k=1}^{n} (-1)^k \binom{n}{k}
= \left[ h \left( t_1, \frac{y_1 + y_2}{2} \right) - h(t_1, y_2) \right] \left\{ \sum_{k=0}^{n} (-1)^k \binom{n}{k} - \binom{n}{0} \right\}
uniformly bounded superposition operators in \(BV\)

\[
\left[ h\left(t_1, \frac{y_1 + y_2}{2}\right) - h(t_1, y_2) \right] \left\{(-1 + 1)^n - \binom{n}{0} \right\}
\]

\[
= \left[ h\left(t_1, \frac{y_1 + y_2}{2}\right) - h(t_1, y_2) \right] \{-1\}.
\]

Hence, substituting this last identity in (4.5) we get

\[
\lim_{t_2 \to t_1} \left( \sum_{\theta \leq 1} (-1)^{[\theta]} (H(f_1) - H(f_2)) (\theta t_1 + (1 - \theta)t_2), 0 \right)
\]

\[
= d\left(h(t_1, y_1) - h(t_1, \frac{y_1 + y_2}{2})
+ \sum_{\theta \leq \frac{1}{2}} (-1)^{[\theta]} [h(t_1, \frac{y_1 + y_2}{2}) - h(t_1, y_2)], 0 \right)
\]

\[
= d\left(h(t_1, y_1) - h(t_1, \frac{y_1 + y_2}{2}) - h(t_1, \frac{y_1 + y_2}{2}) + h(t_1, y_2), 0 \right). \quad (4.6)
\]

On the other hand, the limit as \(t_2 \to t_1\) on the right side of (4.4) is zero, therefore

\[
d\left(h(t_1, y_1) - h(t_1, \frac{y_1 + y_2}{2}) - h(t_1, \frac{y_1 + y_2}{2}) + h(t_1, y_2), 0 \right) = 0
\]

or equivalently

\[
\frac{h(t_1, y_1) + h(t_1, y_2)}{2} = h\left(t_1, \frac{y_1 + y_2}{2}\right).
\]

Thus \(h(t_1, \cdot)\) is solution for the Jensen equation in \(C\) for \(t_1 \in [a, b]\).

Adapting the classical standard argument (cf. Kuczma [9], see also [12]) we conclude that there exist \(A(t_1) \in L(M, N)\) and \(B \in \mathcal{N}([a, b])\) such that

\[
h(t_1, y) = A(t_1)y + B(t_1) \quad y \in C. \quad (4.7)
\]

Finally, notice that if \(0 \in C\), then taking \(y = 0\) in (4.7), we have \(h(t, 0) = B(t)\), for \(t \in [a, b]\), which implies that \(B \in BRV_\Psi([a, b]; \mathcal{N})\). \(\blacksquare\)

Notice that condition (4.1) is a generalization of the classical Lipschitz condition; indeed, that is the case if, in particular, the function \(\gamma\) is an increasing linear function.

In [12] J. Matkowski gives the following definition
Definition 4.2. Let $Y$ and $Z$ be two metric (or normed) spaces. We say that the map $H : Y \to Z$ is uniformly bounded if, for all $t > 0$ there exists a real number $\gamma(t)$ such that for all non empty set $B \subset Y$:

$$\text{diam } B \leq t \implies \text{diam } H(B) \leq \gamma(t).$$ (4.8)

Corollary 4.3. Suppose that $[a, b] \subseteq \mathbb{R}^n$ is an $n$-dimensional closed interval, and that $\Phi, \Psi$ are $\varphi$-functions. Let $\mathcal{M}$ and $\mathcal{N}$ be linear metric spaces, $C \subseteq \mathcal{M}$ a convex and closed set with non empty interior and let $h : [a, b] \times C \to \mathcal{N}$ be a continuous function. If the Nemytskij operator $H$, generated by the function $h$, applies the set $K = \{f \in BRV^n_\Phi([a, b]; M) : f([a, b]) \subset C\}$ into $BRV^n_\Psi([a, b]; N)$ and is uniformly bounded then there are functions $A : [a, b] \to L(M, N)$ and $B \in N([a, b])$ such that

$$h(x, u) = A(x)u + B(x), \quad x \in [a, b], \quad u \in C.$$ If $0 \in C$, then $B \in BRV^n_\Psi([a, b]; N)$.

Proof. If $f_1, f_2 \in K$ then $\text{diam}(\{f_1, f_2\}) = ||f_1 - f_2||_\Phi$. Since $H$ is uniformly bounded we have

$$\text{diam } H(\{f_1, f_2\}) = ||H(\varphi) - H(\psi)||_\Psi \leq \gamma (||\varphi - \psi||_\Phi),$$
and the result readily follows from Theorem 4.1.

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References


