EXTRACTA MATHEMATICAE Vol. **29**, Núm. 1 - 2, 19 – 33 (2014)

Uniformly Bounded Superposition Operators in the Space of Functions of Bounded n-Dimensional Φ-Variation

M. Bracamonte[1], J. Ereú[1], J. Giménez[2], N. Merentes[3]

[1] *Departamento de Matemáticas, Universidad Centroccidental Lisandro Alvarado, Barquisimeto, Venezuela*

[2] *Departamento de Matemáticas Universidad de Los Andes, Mérida, Venezuela*

[3] *Escuela de Matemáticas, Universidad Central de Venezuela, Caracas, Venezuela*

mireyabracamonte@ucla.edu.ve, jereu@ucla.edu.ve, jgimenez@ula.ve, nmerucv@gmail.com

Presented by Manuel González Received August 25, 2013

Abstract: We prove that if a superposition operator maps a subset of the space of all functions of n-dimensional bounded Φ-variation in the sense of Riesz, into another such space, and is uniformly bounded, then the non-linear generator $h(x, y)$ of this operator must be of the form $h(x, y) = A(x)y + B(x)$ where, for every x, $A(x)$ is a linear map.

Key words: Nemytskij operator, n-dimensional Φ-variation, φ-function.

AMS *Subject Class.* (2010): 26B30, 26B35.

1. Introduction

Given two (non-empty) sets A and an B, the notation B^A will stand for the set of all functions from A to B. As usual, if M is a normed space, $\mathcal{L}(M)$ denotes the set of all bounded operators on M.

Let A, B and C be non-empty sets. If $h : A \times C \to B$ is a given function, $X \subset C^A$ and $Y \subset B^A$ are linear spaces then, the nonlinear superposition (Nemytskij) operator $\mathbf{H} : X \to Y$, generated by the function h, is defined as

$$(\mathbf{H}f)(\mathbf{t}) := h(\mathbf{t}, f(\mathbf{t})), \quad \mathbf{t} \in A.$$

This operator plays a central role in many mathematical fields (e.g. in the theory of nonlinear integral equations), by its applications to a variety of nonlinear problems, and has been studied thoroughly. Apart from conditions for the mere inclusion $\mathbf{H}(X) \subset Y$, the boundedness or the continuity of $\mathbf{H}$ (cf. [2]), another important problem has been widely investigated:

to find conditions on the generating function h in the case in which $(X, d_X), (Y, d_Y)$ are also metric spaces and the superposition operator $\mathbf{H}$ is uniformly continuous or satisfies some global or local Lipschitz condition of the form

$$d_Y(\boldsymbol{H}(f_1), \boldsymbol{H}(f_2)) \leq \alpha\, d_X(f_1, f_2), \quad f_1,\, f_2 \in X,$$

cf. e.g., [10, 11, 5].

In this paper we will investigate a problem related to this last situation.

Throughout this paper the letter n denotes a positive integer. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be points in $\mathbb{R}^n$. We will use the notation $\mathbf{a} < \mathbf{b}$ to mean that $a_i < b_i$ for each $i = 1, ..., n$ and accordingly we define $\mathbf{a} = \mathbf{b}$, $\mathbf{a} \leq \mathbf{b}$, $\mathbf{a} \geq \mathbf{b}$ and $\mathbf{a} > \mathbf{b}$. If $\mathbf{a} < \mathbf{b}$, the set $\mathbf{J} := [\mathbf{a}, \mathbf{b}] = \prod_{i=1}^{n} [a_i, b_i]$ will be called an *n-dimensional closed interval.*

Given an n-dimensional closed interval $\mathbf{J}$, a metric vector space M and Φ a φ-function, the space of all functions, defined on $\mathbf{J}$ of n-dimensional bounded Φ-variation will be denoted by $BRV_\Phi^n(\mathbf{J}; M)$. Suppose that N is another vector metric space, $\mathcal{C}$ is a convex subset of M, Ψ is another φ-function and $h : \mathbf{J} \times \mathcal{C} \to N$ is a given function. In this paper we prove that if the superposition operator $\mathbf{H}$, generated by h, maps the set $\{f \in BRV_\Phi^n(\mathbf{J}; \mathcal{M}) : f(\mathbf{J}) \subset \mathcal{C})\}$ into $BRV_\Psi^n(\mathbf{J}; N)$ and is *uniformly bounded*, in the sense introduced in [12], then there is a linear operator $A \in L(M, N)$ and a function $B \in N^{\mathbf{J}}$ such that

$$h(\mathbf{x}, y) = A(\mathbf{x})y + B(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{J}, \quad y \in \mathcal{C}.$$

This is a counterpart of a result of Matkowski proved in [12] for the space of Lipschitz continuous functions.

2. Functions of bounded n-dimensional Φ-variation

In this section we present the definition and main basic aspects of the notion of n-dimensional Φ-variation for functions defined on n-dimensional closed intervals of $\mathbb{R}^n$, that take values on a *metric semigroup*, as introduced in [3]. This generalization of the notion of bounded variation for functions of several variables is inspired in the works of Chistyakov and Talalyan [6, 13]. Here, we also combine the notions of variations given by Vitali ([14]) and later generalized by Hardy and Krause (cf. [4, 7]).

Definition 2.1. A metric semigroup is a structure $(M, d, +)$ where $(M, +)$ is an abelian semigroup and d is a translation invariant metric on M.

In particular, the triangle inequality implies that, for all $u, v, p, q \in M$,

$$\begin{aligned} d(u,v) &\leq d(p,q) + d(u+p, v+q), \quad \text{and} \\ d(u+p, v+q) &\leq d(u,v) + d(p,q). \end{aligned} \tag{2.1}$$

In this paper we will use the following standard notation: $\mathbb{N}$ (resp. $\mathbb{N}_0$) denotes the set of all positive integers (resp. non-negative integers) and a typical point of $\mathbb{R}^n$ is denoted as $\mathbf{x} = (x_1, x_2, ..., x_n) := (x_i)_{i=1}^n$; but, the canonical unit vectors of $\mathbb{R}^n$ are denoted by $\mathbf{e_j}$ $(j = 1, 2, \ldots, n)$; that is, $\mathbf{e_j} := (e_r^{(j)})_{r=1}^n$ where, $e_r^{(j)} := \begin{cases} 0 & \text{if } r \neq j \\ 1 & \text{if } r = j \end{cases}$.

The zero n-tuple $(0, 0, \ldots, 0)$ will be denoted by $\mathbf{0}$, and by $\mathbf{1}$ we will denote the n-tuple $\mathbf{1} = (1, 1, \ldots, 1)$.

If $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, with $\alpha_j \in \mathbb{N}_0$, is a n-tuple of non-negative integers then we call α a *multi-index* ([1]).

If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ we use the notation $\mathbf{a} < \mathbf{b}$ to mean that $x_i < y_i$ for each $i = 1, ..., n$ and similarly are defined $\mathbf{a} = \mathbf{b}$, $\mathbf{a} \leq \mathbf{b}$, $\mathbf{a} \geq \mathbf{b}$ and $\mathbf{a} > \mathbf{b}$. If $\mathbf{a} < \mathbf{b}$, the set $[\mathbf{a}, \mathbf{b}] = \prod_{i=1}^{n} [a_i, b_i]$ will be called a *n-dimensional closed interval.* The euclidean volume of an n-dimensional closed interval will be denoted by Vol $[\mathbf{a}, \mathbf{b}]$; that is, Vol $[\mathbf{a}, \mathbf{b}] = \prod_{i=1}^{n} (b_i - a_i)$.

In addition, for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}_0^n$ and $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ we will use the notations

$$|\alpha| := \alpha_1 + \alpha_2 + ... + \alpha_n \qquad \text{and} \qquad \alpha\mathbf{x} := (\alpha_1 x_1, \alpha_2 x_2, ..., \alpha_n x_n).$$

We will denote by $\mathcal{N}$ the set of all strictly increasing continuous convex functions $\Phi : [0, +\infty) \to [0, +\infty)$ such that $\Phi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \Phi(t) = +\infty$.

$\mathcal{N}_\infty$ the set of all functions $\Phi \in \mathcal{N}$, for which the Orlicz condition (also called ∞_1 condition) holds: $\lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty$. Functions from $\mathcal{N}$ are often called φ-functions.

One says that a function $\Phi \in \mathcal{N}$ satisfies a condition Δ_2, and writes $\Phi \in \Delta_2$, if there are constants $K > 2$ and $t_0 > 0$ such that

$$\Phi(2t) \leq K\Phi(t) \quad \text{for all} \quad t \geq t_0. \tag{2.2}$$

Now we define two important sets.

$$\begin{aligned}\mathcal{E}(n) &:= \{\theta \in \mathbb{N}_0^n : \theta \leq \mathbf{1} \text{ and } |\theta| \text{ is even }\}\\ \mathcal{O}(n) &:= \{\theta \in \mathbb{N}_0^n : \theta \leq \mathbf{1} \text{ and } |\theta| \text{ is odd}\}.\end{aligned}$$

Notice that these sets are related in a one to one fashion; indeed, if $\theta = (\theta_1, ..., \theta_n) \in \mathcal{E}(n)$ then we can define $\widetilde{\theta} := (1-\theta_1, \theta_2, \ldots, \theta_n) \in \mathcal{O}(n)$, and this operation is clearly invertible.

In what follows M is supposed to be a metric semigroup and $[\mathbf{a}, \mathbf{b}]$ an n-dimensional closed interval.

Definition 2.2. ([7, 6, 14]) Given a function $f : [\mathbf{a}, \mathbf{b}] \to M$, we define the *$n$-dimensional Vitali difference* of f over an n-dimensional closed interval $[\mathbf{x}, \mathbf{y}] \subseteq [\mathbf{a}, \mathbf{b}]$, by

$$\Delta_n(f, [\mathbf{x}, \mathbf{y}]) := d\Big(\sum_{\theta\in\mathcal{E}(n)} f(\theta\,\mathbf{x} + (1-\theta)\mathbf{y}), \sum_{\theta\in\mathcal{O}(n)} f(\theta\,\mathbf{x} + (1-\theta)\mathbf{y})\Big). \quad (2.3)$$

This difference is also called *mixed difference* and it is usually associated to the names of Vitali, Lebesgue, **H**ardy, Krause, Fréchet and De la Vallée Poussin ([7, 6, 8]).

Now, in order to define the Φ-variation of a function $f : [\mathbf{a}, \mathbf{b}] \to M$, we consider *net* partitions of $[\mathbf{a}, \mathbf{b}]$; that is, partitions of the kind

$$\xi = \xi_1 \times \xi_2 \times ... \times \xi_n \quad \text{with} \quad \xi_i := \{t_j^{(i)}\}_{j=0}^{k_i}, \;\; i = 1, \ldots, n. \quad (2.4)$$

where, $\{k_i\}_{i=1}^n \subset \mathbb{N}$ and for each i, ξ_i is a partition of $[a_i, b_i]$. The set of all net partitions of an n-dimensional closed interval $[\mathbf{a}, \mathbf{b}]$ will be denoted by $\pi([\mathbf{a}, \mathbf{b}])$.

A point in a net partition ξ is called a *node* ([13]) and it is of the form

$$\mathbf{t}_\alpha := (t_{\alpha_1}^{(1)}, t_{\alpha_2}^{(2)}, t_{\alpha_3}^{(3)}, ..., t_{\alpha_n}^{(n)})$$

where $\mathbf{0} \leq \alpha = (\alpha_i)_{i=1}^n \leq \kappa$, with $\kappa := (k_i)_{i=1}^n$.

For the sake of simplicity in notation, we will simply write $\xi = \{\mathbf{t}_\alpha\}$, to refer to all the nodes that form a given partition ξ.

A cell of an n-dimensional closed interval $[\mathbf{a}, \mathbf{b}]$ is an n-dimensional closed subinterval of the form $[\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha]$, for $\mathbf{0} < \alpha \leq \kappa$.

Note that

$$\begin{aligned}\mathbf{t_0} &= (t_0^{(1)}, t_0^{(2)}, ..., t_0^{(n)}) = (a_1, a_2, ..., a_n) \text{ and}\\ \mathbf{t}_\kappa &= (t_{k_1}^{(1)}, t_{k_2}^{(2)}, ..., t_{k_n}^{(n)}) = (b_1, b_2, ..., b_n).\end{aligned}$$

DEFINITION 2.3. Let $f : [\mathbf{a}, \mathbf{b}] \to M$ and $\Phi \in \mathcal{N}$. The Φ-*variation, in the sense of Vitali-Riesz* of f is defined as

$$\rho^n_\Phi(f, [\mathbf{a}, \mathbf{b}]) := \sup_{\xi \in \pi[\mathbf{a},\mathbf{b}]} \rho^n_\Phi(f, [\mathbf{a}, \mathbf{b}], \xi). \tag{2.5}$$

where

$$\rho^n_\Phi(f, [\mathbf{a}, \mathbf{b}], \xi) := \sum_{1 \le \alpha \le \kappa} \Phi\left(\frac{\Delta_n\left(f, [\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha]\right)}{\mathrm{Vol}\,[\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha]} \right) \mathrm{Vol}\,[\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha].$$

We now need to define *the truncation* of a point, of an n-dimensional closed interval and of a function, by a given multi-index $\mathbf{0} < \eta \le \mathbf{1}$. Notice that in this case, the entries of a such η are either 0 or 1.

- The truncation of a point $\mathbf{x} \in \mathbb{R}^n$ by a multi-index $\mathbf{0} < \eta \le \mathbf{1}$, which is denoted by $\mathbf{x}\lfloor\eta$, is defined as the $|\eta|$-tuple that is obtained if we suppress from $\mathbf{x}$ the entries for which the corresponding entries of η are equal to 0. That is, $\mathbf{x}\lfloor\eta = (x_i : i \in \{1, 2, ..., n\}, \eta_i = 1)$. For instance, if $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ and $\eta = (0, 1, 1, 0, 1)$ then $\mathbf{x}\lfloor\eta = (x_2, x_3, x_5)$.
- The truncation of an n-dimensional closed interval $[\mathbf{a}, \mathbf{b}]$ by a multi-index $\mathbf{0} < \eta \le \mathbf{1}$, is defined as $[\mathbf{a}, \mathbf{b}]\lfloor\eta := [\mathbf{a}\lfloor\eta, \mathbf{b}\lfloor\eta]$.
- Given a function $f : [\mathbf{a}, \mathbf{b}] \to M$, a multi-index $\mathbf{0} < \eta \le \mathbf{1}$ and a point $\mathbf{z} \in [\mathbf{a}, \mathbf{b}]$, we define $f^{\mathbf{z}}_\eta : [\mathbf{a}, \mathbf{b}]\lfloor\eta \to M$, the truncation of f by la η, by the formula

$$f^{\mathbf{z}}_\eta(\mathbf{x}\lfloor\eta) := f(\eta\mathbf{x} + (\mathbf{1} - \eta)\mathbf{z}), \;\; x \in [\mathbf{a}, \mathbf{b}].$$

Note that the function $f^{\mathbf{z}}_\eta$ depends only on the $|\eta|$ variables x_i for which $\eta_i = 1$.

Remark 2.4. Given a function $f : [\mathbf{a}, \mathbf{b}] \to M$ and a multi-index $\eta \ne \mathbf{0}$, then the $|\eta|$-dimensional Vitali difference for $f^{\mathbf{a}}_\eta$ (cf. (2.3)), is given by

$$\Delta_{|\eta|}(f^{\mathbf{a}}_\eta, [\mathbf{x}, \mathbf{y}])$$
$$:= d\Big(\sum_{\substack{\theta \in \mathcal{E}(n) \\ \theta \le \eta}} f(\eta(\theta\mathbf{x}+(\mathbf{1}-\theta)\mathbf{y})+(\mathbf{1}-\eta)\mathbf{a}, \sum_{\substack{\theta \in \mathcal{O}(n) \\ \theta \le \eta}} f(\eta(\theta\mathbf{x}+(\mathbf{1}-\theta)\mathbf{y})+(\mathbf{1}-\eta)\mathbf{a}) \Big).$$

DEFINITION 2.5. Let $\Phi \in \mathcal{N}$ and let $(M, d, +, \cdot)$ be a metric semigroup. A function $f : [\mathbf{a}, \mathbf{b}] \to M$ is said to be of *bounded* Φ-*variation*, in the sense of Riesz, if the total Φ-variation

$$TRV_\Phi(f, [\mathbf{a}, \mathbf{b}]) := \sum_{\mathbf{0} \neq \eta \leq \mathbf{1}} \rho_\Phi^{|\eta|}(f_\eta^{\mathbf{a}}, [\mathbf{a}, \mathbf{b}]\lfloor \eta), \tag{2.6}$$

is finite. The set of all functions f that satisfy $TRV_\Phi(f, [\mathbf{a}, \mathbf{b}]) < +\infty$ will be denoted by $RV_\Phi^n([\mathbf{a}, \mathbf{b}]; M)$.

3. THE NORMED SPACE $BRV_\Phi^n([\mathbf{a}, \mathbf{b}]; M)$

So far, our consideration of metric semigroups as targets sets suffices adequately to define a notion of n-dimensional variation; however, as we need to study a *superposition operator problem* between linear spaces in which the presence of this notion is desired, it will be necessary to ask for additional structure on the target set M. The one that we will considerate is that of *vectorial metric space.*

DEFINITION 3.1. By a metric vector space (MVS) we will understand a topological vector space $(\mathcal{M}, \tau)$ in which the topology τ is induced by a metric d that satisfies the following conditions:

1. d is a translation invariant metric.
2. $d(\alpha a, \alpha b) \leq |\alpha|\, d(a, b)$ for any $\alpha \in \mathbb{R}$ and $a, b \in \mathcal{M}$.

Note that, in particular, any MVS is a metric semigroup. In what follows $\mathcal{M}$ is supposed to be a MVS and $[\mathbf{a}, \mathbf{b}]$ an n-dimensional closed interval.

Remark 3.2. It readily follows from 2.1 that given two functions $f, g : [\mathbf{a}, \mathbf{b}] \to \mathcal{M}$, a multi-index $\eta \neq \mathbf{0}$ and an n-dimensional closed interval $[\mathbf{x}, \mathbf{y}] \subset [\mathbf{a}, \mathbf{b}]$, then the $|\eta|$-dimensional Vitali difference (c.f. (2.3)) of the truncation $(f+g)_\eta^{\mathbf{a}}$ satisfies the inequality

$$\Delta_{|\eta|}\left(f_\eta^{\mathbf{a}} + g_\eta^{\mathbf{a}}, [\mathbf{x}, \mathbf{y}]\right) \;\leq\; \Delta_{|\eta|}\left(f_\eta^{\mathbf{a}}, [\mathbf{x}, \mathbf{y}]\right) + \Delta_{|\eta|}\left(g_\eta^{\mathbf{a}}, [\mathbf{x}, \mathbf{y}]\right). \tag{3.1}$$

LEMMA 3.3. *The functional* $TRV_\Phi(\cdot, [\mathbf{a}, \mathbf{b}])$ *is convex.*

Proof. The lemma is consequence of (3.1) and of the fact that Φ is a nondecreasing convex function. ∎

THEOREM 3.4. *The class* $RV^n_\Phi([\mathbf{a},\mathbf{b}];\mathcal{M})$ *is symmetric and convex.*

Proof. That $RV^n_\Phi([\mathbf{a},\mathbf{b}];\mathcal{M})$ is symmetric is consequence of property (2) (since $d(-a,-b) \le d(a,b)$) of Definition 3.1 while convexity follows from Lemma 3.3. ∎

As a consequence of Theorem 3.4, the *linear space* generated by the set $RV^n_\Phi([\mathbf{a},\mathbf{b}];\mathcal{M})$ is

$$\{f : [\mathbf{a},\mathbf{b}] \to \mathcal{M} \;:\; \lambda f \in RV^n_\Phi([\mathbf{a},\mathbf{b}];\mathcal{M}) \text{ for some } \lambda > 0\},$$

which, we will call, the *space of functions of bounded* Φ*-variation in the sense of Vitali-Hardy-Riesz* and will denote as $BRV^n_\Phi([\mathbf{a},\mathbf{b}];\mathcal{M})$.

LEMMA 3.5. *The set*

$$\Lambda := \{f \in BRV^n_\Phi([\mathbf{a},\mathbf{b}];\mathcal{M}) \;:\; TRV_\Phi(f,[\mathbf{a},\mathbf{b}]) \le 1\}$$

is a convex, balanced and absorbent subset of $BRV^n_\Phi([\mathbf{a},\mathbf{b}];\mathcal{M})$.

Proof. To prove convexity suppose that $f,g \in \Lambda$ and let α,β be non-negative real numbers such that $\alpha+\beta = 1$. Then $TRV_\Phi(f,[\mathbf{a},\mathbf{b}]) \le 1$, $TRV_\Phi(g,[\mathbf{a},\mathbf{b}]) \le 1$ and by Lemma 3.3

$$\begin{aligned} TRV_\Phi(\alpha f+\beta g,[\mathbf{a},\mathbf{b}]) &\le \alpha TRV_\Phi(f,[\mathbf{a},\mathbf{b}]) + \beta TRV_\Phi(g,[\mathbf{a},\mathbf{b}]) \\ &\le \alpha+\beta = 1. \end{aligned}$$

Hence Λ is convex.

On the other hand, from Definition 2.3 it readily follows that if $f_0 \equiv 0$ then $TRV_\Phi(f_0,[\mathbf{a},\mathbf{b}]) = 0$, thus $f_0 \in \Lambda$ and therefore, by virtue of the convexity property of Λ just proved, Λ is balanced. Finally, the fact that Λ is absorbent follows from property (2) of Definition 3.1 and the convexity of Φ. ∎

By virtue of Lemma 3.5, the *Minkowski Functional* of Λ

$$p_\Lambda(f) := \inf\Big\{t>0 : TRV_\Phi\Big(\frac{f}{t},[\mathbf{a},\mathbf{b}]\Big) \le 1\Big\},$$

defines a seminorm on $BRV^n_\Phi([\mathbf{a},\mathbf{b}];\mathcal{M})$, and therefore

$$\|f\| := \|f\|_{BRV^n_\Phi([\mathbf{a},\mathbf{b}];\mathcal{M})} := d(f(\mathbf{a}),0) + p_\Lambda(f) \tag{3.2}$$

defines a norm on $BRV^n_\Phi([\mathbf{a},\mathbf{b}];\mathcal{M})$.

LEMMA 3.6. *Let* $f \in BRV^n_\Phi([\mathbf{a},\mathbf{b}];\mathcal{M})$,

(i) *If* $\|f\| \neq 0$ *then* $TRV_\Phi(f/\|f\|,[\mathbf{a},\mathbf{b}]) \leq 1$;

(ii) *if* $0 \neq \|f\| \leq 1$ *then* $TRV_\Phi(f,[\mathbf{a},\mathbf{b}]) \leq \|f\|$.

Proof. (i) From definition 3.2 $p_\Lambda(f) \leq \|f\|$.

If $p_\Lambda(f) < \|f\|$ then there is $\xi \in \Lambda$ such that $p_\Lambda(f) < \xi \leq \|f\|$ and $TRV_\Phi(\frac{f}{\xi},[\mathbf{a},\mathbf{b}]) \leq 1$. So, since Λ is absorbent, $\frac{f}{\|f\|} \in \Lambda$.

If $p_\Lambda(f) = \|f\|$, then there is a sequence $t_n \in \Lambda$ such that

$$t_n \to \|f\| \quad \text{and} \quad TRV_\Phi\left(\frac{f}{t_n},[\mathbf{a},\mathbf{b}]\right) \leq 1.$$

It follows, by the continuity of the functional $TRV_\Phi(\cdot,[\mathbf{a},\mathbf{b}])$, that

$$TRV_\Phi\Big(\frac{f}{\|f\|},[\mathbf{a},\mathbf{b}]\Big) \leq 1.$$

(ii) follows from (i)and the convexity of $TRV_\Phi(\cdot,[\mathbf{a},\mathbf{b}])$. ∎

Remark 3.7. It follows from Lemma 3.6 (i) that if $p_\Lambda(f) \neq 0$ and $t > \|f\|$ then $t \in \Lambda$.

4. COMPOSITION OPERATOR IN THE SPACE $BRV^n_\Phi([\mathbf{a},\mathbf{b}];\mathcal{M})$

In this section we state and prove the main results of this paper concerning the action of a superposition operator between spaces of functions of bounded n-dimensional Φ-variation. For the sake of clarity of exposition, we will denote the norm of $BRV^n_\Phi([\mathbf{a},\mathbf{b}];\mathcal{M})$ by $\|\cdot\|_{(\Phi,\mathcal{M})}$.

THEOREM 4.1. *Suppose that* $[\mathbf{a},\mathbf{b}] \subseteq \mathbb{R}^n$ *is an* n*-dimensional closed interval, and that* Φ, Ψ *are* φ*-functions. Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be linear metric spaces,* $\mathcal{C} \subseteq \mathcal{M}$ *a convex and closed set with non empty interior and let* $h : [\mathbf{a},\mathbf{b}]\times\mathcal{C} \to \mathcal{N}$ *be a continuous function. If the Nemytskij operator* H, *generated by the function* h, *applies the set* $\mathrm{K} = \{f \in BRV^n_\Phi([\mathbf{a},\mathbf{b}];\mathcal{M}) : f([\mathbf{a},\mathbf{b}]) \subset \mathcal{C}\}$ *into* $BRV^n_\Psi([\mathbf{a},\mathbf{b}];\mathcal{N})$ *and satisfies the inequality*

$$||H(f_1) - H(f_2)||_{(\Psi,\mathcal{N})} \leq \gamma\left(||f_1 - f_2||_{(\Phi,\mathcal{M})}\right) \quad f_1,\ f_2 \in \mathrm{K}, \tag{4.1}$$

for some function $\gamma : [0,\infty) \to [0,\infty)$*, then there are functions* $A : [\mathbf{a},\mathbf{b}] \to \mathcal{L}(\mathcal{M},\mathcal{N})$ *and* $B \in \mathcal{N}^{[\mathbf{a},\mathbf{b}]}$ *such that*

$$h(\mathbf{x},u) = A(\mathbf{x})u + B(\mathbf{x}), \qquad \mathbf{x} \in [\mathbf{a},\mathbf{b}], \quad u \in \mathcal{C}.$$

If $0 \in \mathcal{C}$*, then* $B \in BRV^n_\Psi([\mathbf{a},\mathbf{b}];\mathcal{N})$*.*

Proof. We will show that h satisfies the Jensen equation in the second variable.

Indeed, let $\mathbf{t}_1 = \left(t_1^{(i)}\right)_{i=1}^n$ and $\mathbf{t}_2 = \left(t_2^{(i)}\right)_{i=1}^n \in [\mathbf{a},\mathbf{b}]$, suppose further that $\mathbf{t}_1 \leq \mathbf{t}_2$, and define the functions

$$\eta_i(t) := \begin{cases} 0 & \text{if } a_i \leq t \leq t_1^{(i)} \\ \dfrac{t - t_1^{(i)}}{t_2^{(i)} - t_1^{(i)}} & \text{if } t_1^{(i)} \leq t \leq t_2^{(i)} \\ 1 & \text{if } t_2^{(i)} \leq t \leq b_i. \end{cases}$$

Next, consider $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{C}$, $\mathbf{y}_1 \neq \mathbf{y}_2$ and define

$$f_j(\mathbf{x}) := \frac{1}{2}\Big[\prod_{i=1}^n \eta_i(x_i)(\mathbf{y}_1 - \mathbf{y}_2) + \mathbf{y}_j + \mathbf{y}_2\Big], \tag{4.2}$$

for $j = 1,2$, where $\mathbf{x} := (x_1, x_2, ..., x_n)$.

Notice that

$$\begin{aligned} f_1(\mathbf{x}) - f_2(\mathbf{x}) &= \frac{1}{2}\Big[\prod_{i=1}^n \eta(x_i)(\mathbf{y}_1 - \mathbf{y}_2) + \mathbf{y}_1 + \mathbf{y}_2 - \prod_{i=1}^n \eta(x_i)(\mathbf{y}_1 - \mathbf{y}_2) - \mathbf{y}_2 - \mathbf{y}_2\Big] \\ &= \frac{\mathbf{y}_1 - \mathbf{y}_2}{2}. \end{aligned}$$

Hence $f_1 - f_2$ has zero Φ-variation and

$$\begin{aligned} \|f_1 - f_2\|_{(\Phi,\mathcal{M})} &= d((f_1 - f_2)(\mathbf{a}), 0) + p_\varphi(f_1 - f_2) \\ &= d((f_1 - f_2)(\mathbf{a}), 0) = d\Big(\frac{\mathbf{y}_1 - \mathbf{y}_2}{2}, 0\Big) \\ &= d\Big(\frac{\mathbf{y}_1}{2}, \frac{\mathbf{y}_2}{2}\Big) > 0. \end{aligned}$$

Notice further that

- If $\mathbf{x} = \mathbf{t}_\alpha$ where $\alpha_i = 2$ for $i = 1, 2, ..., n$ then

$$\prod_{i=1}^{n} \eta(t_{\alpha_i}^{(i)}) = \prod_{i=1}^{n} \frac{t_{\alpha_i}^{(i)} - t_1^{(i)}}{t_2^{(i)} - t_1^{(i)}} = 1.$$

- If $\mathbf{x} = \mathbf{t}_\alpha$ with $\alpha_i \neq 2$ for some $1 \leq i \leq n$ then

$$\prod_{i=1}^{n} \eta(t_{\alpha_i}^{(i)}) = \prod_{i=1}^{n} \frac{t_{\alpha_i}^{(i)} - t_1^{(i)}}{t_2^{(i)} - t_1^{(i)}} = 0.$$

Thus, by (4.2)

- If $\alpha_i = 2$ for $i = 1, 2, ..., n$ then

$$f_1(\mathbf{t}_\alpha) := \frac{1}{2}\Big[\prod_{i=1}^{n} \eta(t_{\alpha_i}^{(i)})(\mathbf{y}_1 - \mathbf{y}_2) + \mathbf{y}_1 + \mathbf{y}_2\Big] = \mathbf{y}_1,$$

  and

$$f_2(\mathbf{t}_\alpha) := \frac{1}{2}\Big[\prod_{i=1}^{n} \eta(t_{\alpha_i}^{(i)})(\mathbf{y}_1 - \mathbf{y}_2) + \mathbf{y}_2 + \mathbf{y}_2\Big] = \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}.$$

- If $\alpha_k \neq 2$ for some $1 \leq k \leq n$ then

$$f_1(\mathbf{t}_\alpha) := \frac{1}{2}[\mathbf{y}_1 + \mathbf{y}_2] = \frac{\mathbf{y}_1 + \mathbf{y}_2}{2},$$
$$f_2(\mathbf{t}_\alpha) := \mathbf{y}_2.$$

Thus, by the definition of $\mathbf{H}$, we have

$$\mathbf{H}f_1(\mathbf{t}_2) = h(\mathbf{t}_2, f_1(\mathbf{t}_2)) = h(\mathbf{t}_2, \mathbf{y}_1)$$
$$\mathbf{H}f_2(\mathbf{t}_2) = h(\mathbf{t}_2, f_2(\mathbf{t}_2)) = h\Big(\mathbf{t}_2, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\Big)$$
$$\mathbf{H}f_1(\mathbf{t}_1) = h(\mathbf{t}_1, f_1(\mathbf{t}_1)) = h\Big(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\Big)$$
$$\mathbf{H}f_2(\mathbf{t}_1) = h(\mathbf{t}_1, f_2(\mathbf{t}_1)) = h(\mathbf{t}_1, \mathbf{y}_2),$$

and, if θ is a non-zero multi-index different from $\mathbf{1}$

$$\mathbf{H}f_1(\theta\,\mathbf{t}_1 + (1-\theta)\mathbf{t}_2) = h\Big(\theta\,\mathbf{t}_1 + (1-\theta)\mathbf{t}_2, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\Big)$$
$$\mathbf{H}f_2(\theta\,\mathbf{t}_1 + (1-\theta)\mathbf{t}_2) = h(\theta\,\mathbf{t}_1 + (1-\theta)\mathbf{t}_2, \mathbf{y}_2).$$

On the other hand, for $f_1, f_2 \in \mathrm{K}$ we have

$$||\mathbf{H}(f_1) - \mathbf{H}(f_2)||_{(\Psi,\mathcal{N})} \leq \gamma\left(||f_1 - f_2||_{(\Phi,\mathcal{M})}\right),$$

thus

$$p_{\Psi}(\mathbf{H}(f_1) - \mathbf{H}(f_2)) \leq ||\mathbf{H}(f_1) - \mathbf{H}(f_2)||_{(\Psi,\mathcal{N})} \leq \gamma\left(||f_1 - f_2||_{(\Phi,\mathcal{M})}\right).$$

Hence, by Remark 3.7 we have

$$\begin{aligned}
\rho^n_\Phi\Big(\frac{\mathbf{H}f_1 - \mathbf{H}f_2}{\gamma(\|f_1 - f_2\|_{(\Phi,\mathcal{M})})}, [\mathbf{a}, \mathbf{b}]\Big) &\leq TRV_\Phi\Big(\frac{\mathbf{H}f_1 - \mathbf{H}f_2}{\gamma(\|f_1 - f_2\|_{(\Phi,\mathcal{M})})}, [\mathbf{a}, \mathbf{b}]\Big) \\
&\leq TRV_\Phi\Big(\frac{\mathbf{H}f_1 - \mathbf{H}f_2}{\|\mathbf{H}f_1 - \mathbf{H}f_2\|_{(\Psi,\mathcal{N})}} \frac{\|\mathbf{H}f_1 - \mathbf{H}f_2\|_{\Psi,\mathcal{N})}}{\gamma(\|f_1 - f_2\|_{(\Phi,\mathcal{M})})}, [\mathbf{a}, \mathbf{b}]\Big) \\
&\leq \frac{\|\mathbf{H}f_1 - \mathbf{H}f_2\|_{(\Psi,\mathcal{N})}}{\gamma(\|f_1 - f_2\|_{(\Phi,\mathcal{M})})} TRV_\Phi\Big(\frac{\mathbf{H}f_1 - \mathbf{H}f_2}{\|\mathbf{H}f_1 - \mathbf{H}f_2\|_{(\Psi,\mathcal{N})}}, [\mathbf{a}, \mathbf{b}]\Big) \leq 1.
\end{aligned} \tag{4.3}$$

Thus

$$\begin{aligned}
1 &\geq \rho^n_\Phi\Big(\frac{\mathbf{H}(f_1) - \mathbf{H}(f_2)}{\gamma(\|f_1 - f_2\|_{(\Phi,\mathcal{M})})}, [\mathbf{a}, \mathbf{b}]\Big) \\
&\geq \Phi\left(\frac{\Delta_n\Big(\frac{\mathbf{H}(f_1) - \mathbf{H}(f_2), [\mathbf{t}_1, \mathbf{t}_2]}{\gamma(\|f_1 - f_2\|_{(\Phi,\mathcal{M})})}\Big)}{\mathrm{Vol}\,[\mathbf{t}_1, \mathbf{t}_2]}\right) \mathrm{Vol}\,[\mathbf{t}_1, \mathbf{t}_2],
\end{aligned}$$

which implies

$$\Phi^{-1}\left(\frac{1}{\mathrm{Vol}\,[\mathbf{t}_1, \mathbf{t}_2]}\right) \mathrm{Vol}\,[\mathbf{t}_1, \mathbf{t}_2] \;\geq\; \Delta_n\left(\frac{\mathbf{H}(f_1) - \mathbf{H}(f_2), [\mathbf{t}_1, \mathbf{t}_2]}{\gamma(\|f_1 - f_2\|_{(\Phi,\mathcal{M})})}\right)$$

and

$$\begin{aligned}
&\Delta_n\left(\mathbf{H}(f_1) - \mathbf{H}(f_2), [\mathbf{t}_1, \mathbf{t}_2]\right) \\
&\qquad \leq \Phi^{-1}\left(\frac{1}{\mathrm{Vol}\,[\mathbf{t}_1, \mathbf{t}_2]}\right) \mathrm{Vol}\,[\mathbf{t}_1, \mathbf{t}_2] \gamma(\|f_1 - f_2\|_{(\Phi,\mathcal{M})}).
\end{aligned} \tag{4.4}$$

Making $\mathbf{t}_2 \to \mathbf{t}_1$ on the left hand side of (4.4) we get

$$\begin{aligned}
&\lim_{\mathbf{t}_2\to\mathbf{t}_1} d\Big(\sum_{\theta\leq 1}(-1)^{|\theta|}\,(\mathbf{H}(f_1)-\mathbf{H}(f_2))(\theta\,\mathbf{t}_1+(\mathbf{1}-\theta)\mathbf{t}_2),0\Big)\\
&= d\Big(h(\mathbf{t}_1,\mathbf{y}_1)-h(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2})\\
&\qquad+\lim_{\mathbf{t}_2\to\mathbf{t}_1}\sum_{\substack{\theta\leq\mathbf{1}\\ \theta\neq\mathbf{1}}}(-1)^{|\theta|}\,(\mathbf{H}(f_1)-\mathbf{H}(f_2))(\theta\,\mathbf{t}_1+(\mathbf{1}-\theta)\mathbf{t}_2),0\Big)\\
&= d\Big(h(\mathbf{t}_1,\mathbf{y}_1)-h(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2})\\
&\qquad+\lim_{\mathbf{t}_2\to\mathbf{t}_1}\sum_{\substack{\theta\leq\mathbf{1}\\ \theta\neq\mathbf{1}}}(-1)^{|\theta|}\,\big[h(\theta\,\mathbf{t}_1+(\mathbf{1}-\theta)\mathbf{t}_2,\frac{\mathbf{y}_1+\mathbf{y}_2}{2})\\
&\qquad-h(\theta\,\mathbf{t}_1+(\mathbf{1}-\theta)\mathbf{t}_2,\mathbf{y}_2)\big],0\Big)\\
&= d\Big(h(\mathbf{t}_1,\mathbf{y}_1)-h(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2})\\
&\qquad+\sum_{\substack{\theta\leq\mathbf{1}\\ \theta\neq\mathbf{1}}}(-1)^{|\theta|}\,\big[h(\theta\,\mathbf{t}_1+(\mathbf{1}-\theta)\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2})-h(\theta\,\mathbf{t}_1+(\mathbf{1}-\theta)\mathbf{t}_1,\mathbf{y}_2)\big],0\Big)\\
&= d\Big(h(\mathbf{t}_1,\mathbf{y}_1)-h(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}) \qquad\qquad (4.5)\\
&\qquad+\sum_{\substack{\theta\leq\mathbf{1}\\ \theta\neq\mathbf{1}}}(-1)^{|\theta|}\,\big[h(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2})-h(\mathbf{t}_1,\mathbf{y}_2)\big],0\Big).
\end{aligned}$$

Now, the number of n-tuples that contain k $1s$, with $k>0$, is equal to $\binom{n}{k}=\dfrac{n!}{(n-k)!k!}$, thus

$$\begin{aligned}
&\sum_{\substack{\theta\leq\mathbf{1}\\ \theta\neq\mathbf{0}}}(-1)^{|\theta|}\,\left[h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right)-h(\mathbf{t}_1,\mathbf{y}_2)\right]\\
&=\left[h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right)-h(\mathbf{t}_1,\mathbf{y}_2)\right]\sum_{k=1}^{n}(-1)^k\binom{n}{k}\\
&=\left[h\left(\mathbf{t}_1,\frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right)-h(\mathbf{t}_1,\mathbf{y}_2)\right]\left\{\sum_{k=0}^{n}(-1)^k\binom{n}{k}-\binom{n}{0}\right\}
\end{aligned}$$

$$= \left[h\left(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right) - h(\mathbf{t}_1,\mathbf{y}_2)\right]\left\{(-1+1)^n - \binom{n}{0}\right\}$$
$$= \left[h\left(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\right) - h(\mathbf{t}_1,\mathbf{y}_2)\right]\{-1\}.$$

Hence, substituting this last identity in (4.5) we get

$$\lim_{\mathbf{t}_2\to\mathbf{t}_1} d\Big(\sum_{\theta\le\mathbf{1}}(-1)^{|\theta|}\,(\mathbf{H}(f_1)-\mathbf{H}(f_2))(\theta\,\mathbf{t}_1+(\mathbf{1}-\theta)\mathbf{t}_2),0\Big)$$
$$= d\Big(h(\mathbf{t}_1,\mathbf{y}_1) - h(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2})$$
$$+ \sum_{\substack{\theta\le\mathbf{1}\\ \theta\neq 0}}(-1)^{|\theta|}\,[h(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}) - h(\mathbf{t}_1,\mathbf{y}_2)],0\Big)$$
$$= d\Big(h(\mathbf{t}_1,\mathbf{y}_1) - h(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}) - h(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}) + h(\mathbf{t}_1,\mathbf{y}_2),0\Big). \tag{4.6}$$

On the other hand, the limit as $\mathbf{t}_2 \to \mathbf{t}_1$ on the right side of (4.4) is zero, therefore

$$d\left(h(\mathbf{t}_1,\mathbf{y}_1) - h\Big(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\Big) - h\Big(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\Big) + h(\mathbf{t}_1,\mathbf{y}_2),0\right) = 0$$

or equivalently

$$\frac{h(\mathbf{t}_1,\mathbf{y}_1)+h(\mathbf{t}_1,\mathbf{y}_2)}{2} = h\Big(\mathbf{t}_1, \frac{\mathbf{y}_1+\mathbf{y}_2}{2}\Big).$$

Thus $h(\mathbf{t}_1,\cdot)$ is solution for the Jensen equation in $\mathcal{C}$ for $\mathbf{t}_1 \in [\mathbf{a},\mathbf{b}]$.

Adapting the classical standard argument (cf. Kuczma [9], see also [12]) we conclude that there exist $A(\mathbf{t}_1) \in \mathcal{L}(\mathcal{M},\mathcal{N})$ and $B \in \mathcal{N}^{[\mathbf{a},\mathbf{b}]}$ such that

$$h(\mathbf{t}_1,\mathbf{y}) = A(\mathbf{t}_1)\mathbf{y} + B(\mathbf{t}_1) \qquad \mathbf{y}\in\mathcal{C}. \tag{4.7}$$

Finally, notice that if $0 \in \mathcal{C}$, then taking $y = 0$ in (4.7), we have $h(\mathbf{t},\mathbf{0}) = B(\mathbf{t})$, for $\mathbf{t} \in [\mathbf{a},\mathbf{b}]$, which implies that $B \in BRV^n_\Psi([\mathbf{a},\mathbf{b}];\mathcal{N})$. ∎

Notice that condition (4.1) is a generalization of the classical Lipschitz condition; indeed, that is the case if, in particular, the function γ is an increasing linear function.

In [12] J. Matkowski gives the following definition

Definition 4.2. Let Y and Z be two metric (or normed) spaces. We say that the map $H : Y \to Z$ is uniformly bounded if, for all $t > 0$ there exists a real number $\gamma(t)$ such that for all non empty set $B \subset Y$:

$$\operatorname{diam} B \leq t \implies \operatorname{diam} H(B) \leq \gamma(t). \tag{4.8}$$

Corollary 4.3. *Suppose that* $[\mathbf{a}, \mathbf{b}] \subseteq \mathbb{R}^n$ *is an* n*-dimensional closed interval, and that* Φ, Ψ *are* φ*-functions. Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be linear metric spaces,* $\mathcal{C} \subseteq \mathcal{M}$ *a convex and closed set with non empty interior and let* $h : [\mathbf{a}, \mathbf{b}] \times \mathcal{C} \to \mathcal{N}$ *be a continuous function. If the Nemytskij operator* H*, generated by the function* h*, applies the set* $\mathrm{K} = \{f \in BRV^n_\Phi([\mathbf{a}, \mathbf{b}]; \mathcal{M}) : f([\mathbf{a}, \mathbf{b}]) \subset \mathcal{C})\}$ *into* $BRV^n_\Psi([\mathbf{a}, \mathbf{b}]; \mathcal{N})$ *and is uniformly bounded then there are functions* $A : [\mathbf{a}, \mathbf{b}] \to \mathcal{L}(\mathcal{M}, \mathcal{N})$ *and* $B \in \mathcal{N}^{[\mathbf{a}, \mathbf{b}]}$ *such that*

$$h(\mathbf{x}, u) = A(\mathbf{x})u + B(\mathbf{x}), \qquad \mathbf{x} \in [\mathbf{a}, \mathbf{b}], \quad u \in \mathcal{C}.$$

If $0 \in \mathcal{C}$*, then* $B \in BRV^n_\Psi([\mathbf{a}, \mathbf{b}]; \mathcal{N})$*.*

Proof. If $f_1, f_2 \in \mathrm{K}$ then $\operatorname{diam}(\{f_1, f_2\}) = ||f_1 - f_2||_\Phi$. Since H is uniformly bounded we have

$$\operatorname{diam} H(\{f_1, f_2\}) = ||H(\varphi) - H(\psi)||_\Psi \leq \gamma\left(||\varphi - \psi||_\Phi\right),$$

and the result readily follows from Theorem 4.1. ∎

Acknowledgements

The authors would like to express their gratitude to the referee of the first version of this paper who made a thorough revision of the manuscript and diligently made many valuable suggestions that help greatly to improve the presentation of the paper.

References

[1] C.R. Adams, J.A. Clarkson, Properties of functions $f(x, y)$ of bounded variation, *Transactions of the American Mathematical Society* 36 (1934), 711 – 736.

[2] J. Appell, P.P. Zabrejko, "Nonlinear Superposition Operator", Cambridge University Press, New York, 1990.

[3] M. Bracamonte, J. Ereú, J. Giménez, N. Merentes, Metric Semigroups-valued functions of bounded Riesz-Φ-variation in several variables, *Publicaciones en Ciencias y Tecnologia, Decanato de Ciencias y Tecnologia, UCLA, Venezuela* **7** (1) (2013), 37 – 50.

[4] V.V. Chistyakov, Functions of several variables of finite variation and superposition operators, in: “Real Analysis Exchange 26th Summer Symposium”, Lexington, VA, USA, (2002), 61 – 66.

[5] V.V. Chistyakov, Lipschitzian superposition operators between paces of functions of bounded generalized variation with weight, *Journal of Applied Analysis* **6** (2) (2000), 173 – 186.

[6] Chistyakov V.V., Y. Tretyachenko, Maps of several variables of finite total variation. I. Mixed differences and the total variation , *J. Math. Anal. Appl* **370** (2) (2010), 672 – 686.

[7] V. Clarkson, J.A., C.R. Adams, On definitions of bounded variation for functions of two variables, *Transactions of the American Mathematical Society 35* (1933), 824 – 854.

[8] T.H. Hildebrandt, “Introduction to the Theory of Integration”, Academic Press, New York and London, 1963.

[9] M. Kuzcma, “An Introduction to the Theory of Functional Equations and Inequalities”, Polish Scientific Editors and Silesian University, Warszawa, Kraków, Katowice, 1985.

[10] J. Matkowski, Uniformly continuous superposition operators in the spaces of differentiable functions and absolutely continuous functions, *Internat. Ser. Numer. Math.* **157** (2009), 155 – 166.

[11] J. Matkowski, Uniformly continuous superposition operators in the Banach space of Hölder functions, *J. Math. Anal. Appl.* **359** (1) (2009), 56 – 61.

[12] J. Matkowski, Uniformly bounded composition operators between general Lipschitz function normed spaces, *Topol. Methods Nonlinear Anal.* **38** (2) (2011), 395 – 406.

[13] F.A. Talalyan, A multidimensional analogue of a theorem of F. Riesz, *Sb. Math.* **186** (9) (1995), 1363 – 1374.

[14] G. Vitali, Sui gruppi di punti e sulle funzioni di variabili reali, *Atti Accad. Sci. Torino* **43** (1908), 75 – 92.