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A note on the paper "On Invariant Submanifolds of LP-Sasakian Manifolds"

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Abstract: The object of the present paper is to find a counterexample to prove that every totally geodesic submanifold of an LP-Sasakian manifold need not be an invariant submanifold.

Key words: LP-Sasakian manifold, invariant submanifolds, totally geodesic.

AMS *Subject Class.* (2010): 53C15, 53C40.

1. Introduction

K. Matsumoto [5] introduced the idea of Lorentzian almost para-contact manifolds. After that, many geometers studied different structures on these manifolds. Study of Lorentzian para-Sasakian manifolds has become a topic of increasing research interest after the works in ([6], [7], [8]). Matsumoto, Mihai and Rosca cited an example of a five-dimensional Lorentzian para-Sasakian manifold in [7].

Submanifold theory is an active field of research due to its important applications in Mathematical Physics and some other applied parts of science. The notion of invariant submanifold is used to discuss properties of non-linear autonomous system [12]. Also, the notion of geodesics plays an important role in the theory of relativity [7]. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. That is the reason why totally geodesic submanifolds are so important in submanifold theory. U.C. De et al. ([3], [2], [4]) studied the geometry of submanifolds of LP-Sasakian manifolds. In [2], it is proved that a submanifold of an LP-Sasakian manifold is invariant if and only if $B(X, \xi) = 0$, where B is the second fundamental form of the submanifold. In [9], Özgür and Murathan obtained some necessary and sufficient conditions under which an invariant submanifold of an LP-Sasakian manifold becomes totally geodesic.

In [11] the authors proved that every odd dimensional totally geodesic submanifold of an LP-Sasakian manifold is invariant. In the present paper we nullify this result with the help of an example of a 3-dimensional totally geodesic submanifold of an LP-Sasakian manifold which is not an invariant submanifold. It can be observed that the vector Z in the proof of [11, Theorem 3.3] is not an arbitrary one. And hence, $g(Z,Z)=0$ in the proof does not imply any contradiction.

2. Example

In this section we construct an example of a totally geodesic three-dimensional submanifold of a five-dimensional LP-Sasakian manifold and show that this submanifold is not an invariant submanifold.

Let us consider the 5-dimensional manifold

$$\tilde{M}=\left\{(x,y,z,u,v)\in\mathbb{R}^5 : (x,y,z,u,v)\neq(0,0,0,0,0)\right\},$$

where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. The vector fields

$$e_1=-2\frac{\partial}{\partial x}+2y\frac{\partial}{\partial z}, \qquad e_2=\frac{\partial}{\partial y}, \qquad e_3=\frac{\partial}{\partial z},$$
$$e_4=-2\frac{\partial}{\partial u}+2v\frac{\partial}{\partial z}, \qquad e_5=\frac{\partial}{\partial v}$$

are linearly independent at each point of M. Let g be the metric defined by

$$g(e_i,e_i)=1 \quad \text{for } i\neq 3, \qquad g(e_3,e_3)=-1,$$
$$g(e_i,e_j)=0 \quad \text{for } i\neq j.$$

Here i and j runs from 1 to 5. Let η be the 1-form defined by $\eta(Z)=g(Z,e_3)$, for any vector field Z tangent to $\tilde{M}$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1)=e_2$, $\phi(e_2)=e_1$, $\phi(e_3)=0$, $\phi(e_4)=e_5$, $\phi(e_5)=e_4$. Then, using the linearity of ϕ and g we have

$$\eta(e_3)=-1, \qquad \phi^2Z=Z+\eta(Z)e_3$$

for any vector field Z tangent to $\tilde{M}$. Thus for $e_3=\xi$, $\tilde{M}(\phi,\xi,\eta,g)$ defines an almost para-contact metric manifold. Let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$ with respect to the metric g. Then we have

$$[e_1,e_2]=-2e_3, \qquad [e_1,e_3]=0, \qquad [e_1,e_4]=0,$$
$$[e_1,e_5]=0, \qquad [e_2,e_3]=0, \qquad [e_2,e_4]=0,$$
$$[e_2,e_5]=0, \qquad [e_3,e_4]=0, \qquad [e_4,e_5]=-2e_3.$$

Taking $e_3 = \xi$ and using Koszul's formula for g, it can be easily calculated that

$$\begin{array}{lll}
\tilde{\nabla}_{e_1} e_5 = 0\,, & \tilde{\nabla}_{e_1} e_4 = 0\,, & \tilde{\nabla}_{e_1} e_3 = e_2\,, \\
\tilde{\nabla}_{e_1} e_2 = -e_3\,, & \tilde{\nabla}_{e_1} e_1 = 0\,, & \tilde{\nabla}_{e_2} e_5 = 0\,, \\
\tilde{\nabla}_{e_2} e_4 = 0\,, & \tilde{\nabla}_{e_2} e_3 = e_1\,, & \tilde{\nabla} e_2 e_2 = 0\,,
\end{array}$$

$$\begin{array}{lll}
\tilde{\nabla}_{e_2} e_1 = e_3\,, & \tilde{\nabla}_{e_3} e_5 = e_4\,, & \tilde{\nabla}_{e_3} e_4 = e_5\,, \\
\tilde{\nabla}_{e_3} e_3 = 0\,, & \tilde{\nabla}_{e_3} e_2 = e_1\,, & \tilde{\nabla}_{e_3} e_1 = e_2\,, \\
\tilde{\nabla}_{e_4} e_5 = -e_3\,, & \tilde{\nabla}_{e_4} e_4 = 0\,, & \tilde{\nabla} e_4 e_3 = e_5\,,
\end{array}$$

$$\begin{array}{lll}
\tilde{\nabla}_{e_4} e_2 = 0\,, & \tilde{\nabla}_{e_4} e_1 = 0\,, & \tilde{\nabla}_{e_5} e_5 = 0\,, \\
\tilde{\nabla}_{e_5} e_4 = e_3\,, & \tilde{\nabla}_{e_5} e_3 = e_4\,, & \tilde{\nabla}_{e_5} e_2 = 0\,, \\
 & \tilde{\nabla}_{e_5} e_1 = 0\,. &
\end{array}$$

From the above calculations, we see that the manifold under consideration satisfies $\eta(\xi) = -1$ and $\nabla_X \xi = \phi X$. Hence, it is an LP-Sasakian manifold.

Let $M = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = -2\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}\,, \qquad e_2 = \frac{\partial}{\partial y}\,, \qquad e_5 = \frac{\partial}{\partial v}$$

are linearly independent at each point of M.

Let the immersion f from M to $\tilde{M}$ be defined as $f(x, y, z) = (x, y, 0, 0, z)$.

Then, e_1, e_2 and e_5 form a basis of the tangent space of M and $T^{\perp}M$, the normal space of M in $\tilde{M}$ is spanned by the vectors e_4 and e_3.

Let g be the induced metric defined by

$$g(e_1, e_5) = g(e_2, e_5) = g(e_1, e_2) = 0\,,$$
$$g(e_1, e_1) = g(e_2, e_2) = g(e_5, e_5) = 1\,.$$

Let ∇ be the Levi-Civita connection on M with respect to the metric g. Then we have

$$[e_1, e_2] = -2e_3\,, \qquad [e_1, e_5] = 0\,, \qquad [e_2, e_5] = 0\,.$$

Using Koszul's formula for the metric g, it can be easily calculated that

$$\begin{array}{lll}
\nabla_{e_1} e_5 = 0\,, & \nabla_{e_1} e_2 = -e_3\,, & \nabla_{e_1} e_1 = 0\,, \\
\nabla_{e_2} e_5 = 0\,, & \nabla_{e_2} e_2 = 0\,, & \nabla_{e_2} e_1 = e_3\,, \\
\nabla_{e_5} e_5 = 0\,, & \nabla_{e_5} e_2 = 0\,, & \nabla e_5 e_1 = 0\,.
\end{array}$$

From the values of $\tilde{\nabla}_{e_i}e_j$ and $\nabla_{e_i}e_j$ calculated before and from the relation $B(e_i, e_j) = \tilde{\nabla}_{e_i}e_j - \nabla_{e_i}e_j$, we see that $B(U, V) = 0$, for all $U, V \in TM$. Hence, the submanifold is totally geodesic.

But since $\phi(e_5) = e_4$, we see that the submanifold is not an invariant submanifold.

The above arguments tell us that the submanifold M under consideration contradicts Theorem 3.3 which is the main result of [11].

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