Weyl Type Theorems for Restrictions of Bounded Linear Operators

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Abstract: In this paper we give sufficient conditions for which Weyl’s theorems for a bounded linear operator $T$, acting on a Banach space $X$, can be reduced to the study of Weyl’s theorems for some restriction of $T$.

Key words: Weyl’s theorem, a-Weyl’s theorem, semi-Fredholm operator, pole of the resolvent.

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1. Introduction

Throughout this paper $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space $X$. For $T \in L(X)$, we denote by $N(T)$ the null space of $T$ and by $R(T) = T(X)$ the range of $T$. We denote by $\alpha(T) := \dim N(T)$ the nullity of $T$ and by $\beta(T) := \text{codim } R(T) = \dim X/R(T)$ the defect of $T$. Other two classical quantities in operator theory are the ascent $p = p(T)$ of an operator $T$, defined as the smallest non-negative integer $p$ such that $N(T^p) = N(T^{p+1})$ (if such an integer does not exist, we put $p(T) = \infty$), and the descent $q = q(T)$, defined as the smallest non-negative integer $q$ such that $R(T^q) = R(T^{q+1})$ (if such an integer does not exist, we put $q(T) = \infty$). It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Furthermore, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if $\lambda$ is a pole of the resolvent, see [12, Proposition 50.2]. An operator $T \in L(X)$ is said to be Fredholm (respectively, upper semi-Fredholm, lower semi-Fredholm), if $\alpha(T)$, $\beta(T)$ are both finite (respectively, $R(T)$ closed and $\alpha(T) < \infty$, $\beta(T) < \infty$). $T \in L(X)$ is said to be semi-Fredholm if $T$ is either an upper semi-Fredholm or a lower semi-Fredholm operator. If $T$ is semi-Fredholm the index of $T$ defined by $\text{ind } T := \alpha(T) - \beta(T)$. Other two
important classes of operators in Fredholm theory are the classes of semi-Browder operators. These classes are defined as follows, $T \in L(X)$ is said to be Browder (resp. upper semi-Browder, lower semi-Browder) if $T$ is a Fredholm (respectively, upper semi-Fredholm, lower semi-Fredholm) and both $p(T)$, $q(T)$ are finite (respectively, $p(T) < \infty$, $q(T) < \infty$). A bounded operator $T \in L(X)$ is said to be upper semi-Weyl (respectively, lower semi-Weyl) if $T$ is upper Fredholm operator (respectively, lower semi-Fredholm) and index $\text{ind} \, T \leq 0$ (respectively, $\text{ind} \, T \geq 0$). $T \in L(X)$ is said to be Weyl if $T$ is both upper and lower semi-Weyl, i.e. $T$ is a Fredholm operator having index 0. The Browder spectrum and the Weyl spectrum are defined, respectively, by

$$\sigma_b(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder} \},$$

and

$$\sigma_w(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl} \}.$$ 

Since every Browder operator is Weyl then $\sigma_w(T) \subseteq \sigma_b(T)$. Analogously, The upper semi-Browder spectrum and the upper semi-Weyl spectrum are defined by

$$\sigma_{ub}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder} \},$$

and

$$\sigma_{uw}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Weyl} \}.$$ 

In the sequel we need the following basic result:

**LEMMA 1.1.** If $T \in L(X)$ and $p = p(T) < \infty$, then the following statements are equivalent:

(i) There exists $n \geq p + 1$ such that $T^n(X)$ is closed;

(ii) $T^n(X)$ is closed for all $n \geq p$.

**Proof.** Define $c_i'(T) := \dim (N(T^i)/N(T^{i+1}))$. Clearly, $p = p(T) < \infty$ entails that $c_i'(T) = 0$ for all $i \geq p$, so $k_i(T) := c_i(T) - c_{i+1}(T) = 0$ for all $i \geq p$. The equivalence easily follows from [13, Lemma 12]. □

Now, we introduce an important property in local spectral theory. The localized version of this property has been introduced by Finch [11], and in the framework of Fredholm theory this property has been characterized in several ways, see [1, Chapter 3]. A bounded operator $T \in L(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated, SVEP at
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$\lambda_0$, if for every open disc $\mathbb{D}_{\lambda_0} \subseteq \mathbb{C}$ centered at $\lambda_0$ the only analytic function $f : \mathbb{D}_{\lambda_0} \rightarrow X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{D}_{\lambda_0},$$

is the function $f \equiv 0$ on $\mathbb{D}_{\lambda_0}$. The operator $T$ is said to have SVEP if $T$ has the SVEP at every point $\lambda \in \mathbb{C}$. Evidently, $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic functions it is easily seen that $T$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, $T$ has SVEP at every isolated point of the spectrum. Note that (see [1, Theorem 3.8])

$$p(\lambda I - T) < \infty \quad \Rightarrow \quad T \text{ has SVEP at } \lambda, \quad (1.1)$$

and dually

$$q(\lambda I - T) < \infty \quad \Rightarrow \quad T^* \text{ has SVEP at } \lambda. \quad (1.2)$$

Recall that $T \in L(X)$ is said to be bounded below if $T$ is injective and has closed range. Denote by $\sigma_{ap}(T)$ the classical approximate point spectrum defined by

$$\sigma_{ap}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below} \}.$$

Note that if $\sigma_s(T)$ denotes the surjectivity spectrum

$$\sigma_s(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not onto} \},$$

then $\sigma_{ap}(T) = \sigma_s(T^*)$ and $\sigma_s(T) = \sigma_{ap}(T^*)$.

It is easily seen from definition of localized SVEP that

$$\lambda \notin \text{acc } \sigma_{ap}(T) \quad \Rightarrow \quad T \text{ has SVEP at } \lambda, \quad (1.3)$$

where acc $K$ means the set of all accumulation points of $K \subseteq \mathbb{C}$, and if $T^*$ denotes the dual of $T$, then

$$\lambda \notin \text{acc } \sigma_s(T) \quad \Rightarrow \quad T \text{ has SVEP at } \lambda. \quad (1.4)$$

Remark 1.2. The implications (1.1), (1.2), (1.3) and (1.4) are actually equivalences whenever $T \in L(X)$ is semi-Fredholm (see [1, Chapter 3]).
Denote by iso $K$ the set of all isolated points of $K \subseteq \mathbb{C}$. Let $T \in L(X)$, define
\[
\pi_{00}(T) = \{ \lambda \in \sigma(T) : 0 < \alpha(\lambda I - T) < \infty \},
\]
\[
\pi_{00}^a(T) = \{ \lambda \in \text{iso} \sigma_{ap}(T) : 0 < \alpha(\lambda I - T) < \infty \}.
\]
Clearly, for every $T \in L(X)$ we have $\pi_{00}(T) \subseteq \pi_{00}^a(T)$.

Let $T \in L(X)$ be a bounded operator. Following Coburn [8], $T$ is said to satisfy Weyl’s theorem, in symbol (W), if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. According to Rakočević [15], $T$ is said to satisfy $a$-Weyl’s theorem, in symbol (aW), if $\sigma_{ap}(T) \setminus \sigma_{uw}(T) = \pi_{00}^a(T)$.

Note that
\[
a\text{-Weyl’s theorem} \quad \Rightarrow \quad \text{Weyl’s theorem},
\]
see for instance [1, Chapter 3]. The converse of these implication in general does not hold.

Weyl type theorems have been recently studied by several authors ([2], [3], [5], [6], [8], [9], [10], [15] and [16]). In these papers several results are obtained, by considering an operator $T \in L(X)$ in the whole space $X$. In this paper we give sufficient conditions for which Weyl type theorems holds for $T$, if and only if there exists $n \in \mathbb{N}$ such that the range $R(T^n)$ of $T^n$ is closed and Weyl type theorems holds for $T_n$, where $T_n$ denote the restriction of $T$ on the subspace $R(T^n) \subseteq X$.

2. Preliminaries

In this section we establish several lemmas that will be used throughout the paper. We begin examining some algebraic relations between $T$ and $T_n$, $T_n$ viewed as a operator from the space $R(T^n)$ in to itself.

**Lemma 2.1.** Let $T \in L(X)$ and $T_n$, $n \in \mathbb{N}$, be the restriction of the operator $T$ on the subspace $R(T^n) = T^n(X)$. Then, for all $\lambda \neq 0$, we have:

(i) $N((\lambda I - T_n)^m) = N((\lambda I - T)^m)$, for any $m$;

(ii) $R((\lambda I - T_n)^m) = R((\lambda I - T)^m) \cap R(T^n)$, for any $m$;

(iii) $\alpha(\lambda I - T_n) = \alpha(\lambda I - T)$;

(iv) $p(\lambda I - T_n) = p(\lambda I - T)$;

(v) $\beta(\lambda I - T_n) = \beta(\lambda I - T)$.
Proof. (i) For $m = 0$,

$$N((\lambda I - T_n)^m) = N((\lambda I - T)^m)$$

holds trivially. Let $x \in N((\lambda I - T)^m)$, $m \geq 1$, then

$$0 = (\lambda I - T)^m x = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} (-1)^k \lambda^{m-k} T^k x$$

$$= \lambda^m x + \sum_{k=1}^{m} \frac{m!}{k!(m-k)!} (-1)^k \lambda^{m-k} T^k x.$$ 

Thus $0 = \lambda^m x + h(T)x$, where

$$h(T) = \sum_{k=1}^{m} \frac{m!}{k!(m-k)!} (-1)^k \lambda^{m-k} T^k.$$ 

Hence $-\lambda^m x = h(T)x$, and since $\lambda \neq 0$, then $x = -\lambda^{-m} h(T)x$. From this equality, it follows that

$$(-\lambda^{-m} h(T))^2 x = -\lambda^{-m} h(T)(-\lambda^{-m} h(T)x) = -\lambda^{-m} h(T)x = x.$$ 

Consequently $x = (-\lambda^{-m} h(T))^2 x$. By repeating successively the same argument, we obtain that $x = (-\lambda^{-m} h(T))^j x$, for all $j \in \mathbb{N}$. But since $-\lambda^{-m} h(T)x \in R(T)$, then $(-\lambda^{-m} h(T))^j x \in R(T^j)$, for all $j \in \mathbb{N}$. Therefore $x = (-\lambda^{-m} h(T))^n x \in R(T^n)$, and since $R(T^n)$ is $T$-invariant subspace, we conclude that

$$0 = (\lambda I - T)^m x = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} (-1)^k \lambda^{m-k} T^k x$$

$$= \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} (-1)^k \lambda^{m-k} (T_n)^k x = (\lambda I - T_n)^m x.$$ 

So $x \in N((\lambda I - T_n)^m)$, and we get the inclusion

$$N((\lambda I - T)^m) \subseteq N((\lambda I - T_n)^m).$$

On the other hand, since $T_n$ is the restriction of $T$ on $R(T^n)$, and $R(T^n)$ is invariant under $T$, it then follows the inclusion

$$N((\lambda I - T_n)^m) \subseteq N((\lambda I - T)^m).$$
From which, we obtain that $N((\lambda I - T_n)^m) = N((\lambda I - T)^m)$.

(ii) Since $T_n$ is the restriction of $T$ on $R(T^n)$, and $R(T^n)$ is invariant under $T$, then

$$R((\lambda I - T_n)^m) \subseteq R((\lambda I - T)^m) \cap R(T^n).$$

Now, we show the inclusion $R((\lambda I - T)^m) \cap R(T^n) \subseteq R((\lambda I - T_n)^m)$. For this, it will suffice to show that for $m \in \mathbb{N}$, the implication

$$(\lambda I - T)^m x \in R(T^n) \quad \Rightarrow \quad x \in R(T^n),$$

holds. For $m = 1$. Let $y \in R(\lambda I - T) \cap R(T^n)$, then there exists $x \in X$ such that $\lambda x - Tx = (\lambda I - T)x = y \in R(T^n)$, so $\lambda^2 x - \lambda Tx = \lambda y \in R(T^n)$. But since $\lambda Tx - T^2 x = Ty \in R(T^n)$, because $\lambda x - Tx = y$ and $R(T^n)$ is invariant under $T$, we have that $\lambda^2 x - \lambda Tx, \lambda Tx - T^2 x \in R(T^n)$. Then

$$\lambda^2 x - T^2 x = \lambda^2 x - \lambda Tx + \lambda Tx - T^2 x \in R(T^n).$$

Thus $\lambda^2 x - T^2 x \in R(T^n)$. Hence $\lambda^3 x - \lambda T^2 x = \lambda(\lambda^2 x - T^2 x) \in R(T^n)$, and since $\lambda T^2 x - T^3 x = T^2 y \in R(T^n)$, we have that $\lambda^3 x - \lambda T^2 x, \lambda T^2 x - T^3 x \in R(T^n)$. From which,

$$\lambda^3 x - T^3 x = \lambda^3 x - \lambda T^2 x + \lambda T^2 x - T^3 x \in R(T^n).$$

That is, $\lambda^3 x - T^3 x \in R(T^n)$. Now, suppose that $\lambda^j x - T^j x \in R(T^n)$, for some $j \in \mathbb{N}$. From this, $\lambda^{j+1} x - \lambda T^j x = \lambda(\lambda^j x - T^j x) \in R(T^n)$, and $\lambda T^j x - T^{j+1} x = T^j y \in R(T^n)$, thus $\lambda^{j+1} x - \lambda T^j x, \lambda T^j x - T^{j+1} x \in R(T^n)$. From which,

$$\lambda^{j+1} x - T^{j+1} x = \lambda^{j+1} x - \lambda T^j x + \lambda T^j x - T^{j+1} x \in R(T^n).$$

Consequently, by mathematical induction, we obtain that $\lambda^j x - T^j x \in R(T^n)$ for all $j \in \mathbb{N}$. In particular, $\lambda^n x - T^n x \in R(T^n)$, and since $\lambda \neq 0$, then

$$x = \lambda^{-n}((\lambda^n x - T^n x) + T^n x) \in R(T^n).$$

By the above reasoning, we conclude that, for $m = 1$, the implication

$$(\lambda I - T)x \in R(T^n) \quad \Rightarrow \quad x \in R(T^n)$$

holds.

Now, suppose that for $m \geq 1$,

$$(\lambda I - T)^m x \in R(T^n) \quad \Rightarrow \quad x \in R(T^n).$$
If \((\lambda I - T)^{m+1} x \in R(T^n)\), then \((\lambda I - T)((\lambda I - T)^{m} x) \in R(T^n)\). From the proof of case \(m = 1\), we conclude that \((\lambda I - T)^{m} x \in R(T^n)\). Therefore by inductive hypothesis, \(x \in R(T^n)\). Then, by mathematical induction, we conclude that for all \(m \in \mathbb{N}\)

\[(\lambda I - T)^{m} x \in R(T^n) \implies x \in R(T^n)\]

holds.

Finally, if \(y \in R((\lambda I - T)^{m} \cap R(T^n)\) there exists \(x \in X\) such that \((\lambda I - T)^{m} x = y \in R(T^n)\), then \((\lambda I - T)^{m} x \in R(T^n)\). As the above proof, we conclude that \(x \in R(T^n)\). Thus

\[y = (\lambda I - T)^{m} x = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \lambda^{m-k} T^{k} x\]

\[= \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \lambda^{m-k} (T^{n})^{k} x = (\lambda I - T^{n})^{m} x,\]

then \(y \in R((\lambda I - T^{n})^{m})\). This shows that,

\[R((\lambda I - T)^{m}) \cap R(T^n) \subseteq R((\lambda I - T^{n})^{m})\, .\]

Consequently, \(R((\lambda I - T^{n})^{m}) = R((\lambda I - T)^{m}) \cap R(T^n)\).

(iii) and (iv), it follows immediately from the equality

\[N((\lambda I - T^{n})^{m}) = N((\lambda I - T)^{m}) \quad \text{for all } m \in \mathbb{N} .\]

(v) Observe that \(R(\lambda I - T^{n})\) is a subspace of \(R(T^n)\). Let \(M\) be a subspace of \(R(T^n)\) such that \(R(T^n) = R(\lambda I - T^{n}) \oplus M\). Since \(R(\lambda I - T^{n}) = R(\lambda I - T) \cap R(T^n)\), we have

\[R(\lambda I - T) \cap M = R(\lambda I - T) \cap R(T^n) \cap M = R(\lambda I - T^{n}) \cap M = 0 .\]

Thus \(R(\lambda I - T) \cap M = \{0\}\). Now, we show that \(X = R(\lambda I - T) + M\).

Let \(\mu \in \mathbb{C}\) such that \(\mu I - T\) is invertible in \(L(X)\), then \((\mu I - T)^{j}\) is invertible in \(L(X)\), for all \(j \in \mathbb{N}\). In particular \((\mu I - T)^{m}\) is invertible in \(L(X)\), for all \(m \geq n\). Thus, if \(y \in X\) there exists \(x \in X\) such that \(y = (\mu I - T)^{m} x\). Thus,

\[y = (\mu I - T)^{m} x = \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} (-1)^{j} \mu^{m-j} T^{j} x\]

\[= \sum_{j=0}^{n-1} \frac{m!}{j!(m-j)!} (-1)^{j} \mu^{m-j} T^{j} x + \sum_{j=n}^{m} \frac{m!}{j!(m-j)!} (-1)^{j} \mu^{m-j} T^{j} x .\]
Since $R(T^j) \subseteq R(T^n)$, for $n \leq j \leq m$, then we can write $y = u + v$, where:

$$u = \sum_{j=0}^{n-1} \frac{m!}{j!(m-j)!}(-1)^j \mu^{m-j}T^j x \in X,$$

$$v = \sum_{j=n}^{m} \frac{m!}{j!(m-j)!}(-1)^j \mu^{m-j}T^j x \in R(T^n).$$

Now, from the above decomposition and for any $\lambda \neq 0$, we obtain a sequence $(y_k)_{k=0}^{\infty}$, where $y_k = \lambda^{-k-1}(\lambda I - T)T^ku$, for $k = 0, 1, \ldots$, such that

$$u = y_0 + y_1 + \cdots + y_{n-1} + \lambda^{-n}T^n u \in R(\lambda I - T) + R(T^n),$$

because $y_k = \lambda^{-k-1}(\lambda I - T)T^ku \in R(\lambda I - T)$ and $\lambda^{-n}T^n u \in R(T^n)$.

On the other hand,

$$v + \lambda^{-n}T^n u \in R(T^n) + R(T^n) = R(T^n) = R(\lambda I - T_n) + M.$$

Thus $v + \lambda^{-n}T^n u = z + m$, where $z \in R(\lambda I - T_n)$ and $m \in M$. From this, and since $R(\lambda I - T_n) \subseteq R(\lambda I - T)$, we obtain that

$$y = u + v = y_0 + y_1 + \cdots + y_{n-1} + \lambda^{-n}T^n u + v$$

$$= y_0 + y_1 + \cdots + y_{n-1} + z + m$$

$$= (y_0 + y_1 + \cdots + y_{n-1} + z) + m \in R(\lambda I - T) + M.$$

Therefore, we have that $X \subseteq R(\lambda I - T) + M$, consequently $X = R(\lambda I - T) + M$. But since $R(\lambda I - T) \cap M = \{0\}$, and hence it follows that $X = R(\lambda I - T) \oplus M$, which implies that

$$\beta(\lambda I - T) = \dim M = \beta(\lambda I - T_n).$$

This shows that $\beta(\lambda I - T) = \beta(\lambda I - T_n)$. 

The following result concerning the ranges of the powers of $\lambda I - T$, where $\lambda \in \mathbb{C}$ and $T \in L(X)$, plays an important role in this paper. In the proof of this corollary we use the notion of paraclosed (or paracomplete) subspace and the Neubauer Lemma (see [14]).

**Lemma 2.2.** If $R(T^n)$ is closed in $X$ and $R((\lambda I - T_n)^m)$ is closed in $R(T^n)$, then there exists $k \in \mathbb{N}$ such that $R((\lambda I - T)^k)$ is closed in $X$. 

Proof. Observe that for $\lambda = 0$,

$$R((0I - T_n)^m) = R((T_n)^m) = R(T^{m+n}) .$$

Then $R(T^{m+n})$ is a closed subspace of $R(T^n)$. Since $R(T^n)$ is closed, we have that $R((0I - T)^{m+n}) = R(T^{m+n})$ is closed. On the other hand, if $\lambda \neq 0$ and $R((\lambda I - T_n)^m)$ is a closed subspace of $R(T^n)$, since $R(T^n)$ is closed in $X$, we have that $R((\lambda I - T_n)^m)$ is closed in $X$. But, from the incise (ii) in Lemma 2.1,

$$R((\lambda I - T_n)^m) = R((\lambda I - T)^m) \cap R(T^n) .$$

Thus $R((\lambda I - T)^m) \cap R(T^n)$ is closed in $X$. Also, if $\lambda \neq 0$ the polynomials $(\lambda - z)^m$ and $z^n$ have no common divisors, so there exist two polynomials $u$ and $v$ such that $1 = (\lambda - z)^m u(z) + z^n v(z)$, for all $z \in \mathbb{C}$. Hence $I = (\lambda I - T)^m u(T) + T^n v(T)$ and so $R((\lambda I - T)^m) + R(T^n) = X$. Since both $R((\lambda I - T)^m)$ and $R(T^n)$ are paraclosed subspaces, and $R((\lambda I - T)^m) \cap R(T^n)$ and $R((\lambda I - T)^m) + R(T^n)$ are closed, using Neubauer Lemma [14, Proposition 2.1.2], we have that $R((\lambda I - T)^m)$ is closed. 

Recall that for an operator $T \in L(X)$, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when $\lambda$ is a pole of the resolvent of $T$ (see [12, Proposition 50.2]).

**Lemma 2.3.** If 0 is not a pole of the resolvent of $T \in L(X)$ and $R(T^n)$ is closed, then $\pi_{00}(T) \subseteq \pi_{00}(T_n)$.

**Proof.** By Lemma 2.1, $\sigma(T_n) \setminus \{0\} = \sigma(T) \setminus \{0\}$. Also, $0 \notin \sigma(T)$ implies $T$ bijective, thus $T = T_n$. Hence $\sigma(T_n) \subseteq \sigma(T)$. Moreover, $\sigma(T) \subseteq \sigma(T_n)$. Since, if $\lambda \in \sigma(T)$, then $\sigma(T) \cap \mathbb{D}_\lambda = \{\lambda\}$ for some open disc $\mathbb{D}_\lambda \subseteq \mathbb{C}$ centered at $\lambda$. Thus,

$$\sigma(T_n) \cap \mathbb{D}_\lambda \subseteq \sigma(T) \cap \mathbb{D}_\lambda = \{\lambda\} .$$

Consequently $\sigma(T_n) \cap \mathbb{D}_\lambda = \{\lambda\}$ or $\sigma(T_n) \cap \mathbb{D}_\lambda = \emptyset$. If $\sigma(T_n) \cap \mathbb{D}_\lambda = \emptyset$, then $\lambda \notin \sigma(T_n)$, so that $p(\lambda I - T_n) = \beta(\lambda I - T_n) = 0$. For the case $\lambda \neq 0$, from Lemma 2.1, $p(\lambda I - T) = 0$ and $\beta(\lambda I - T) = 0$, then $\lambda \notin \sigma(T)$ a contradiction. In the case where $\lambda = 0$, $p(T_n) = q(T_n) = 0$ implies, by [7, Lemma 2 and Lemma 3] and [12, Proposition 38.6], that $0 < p(T) = q(T) < \infty$, which is impossible, because 0 is not a pole of the resolvent of $T$. Consequently, $\sigma(T_n) \cap \mathbb{D}_\lambda = \{\lambda\}$, so we have that $\lambda \in \sigma(T_n)$.

Now, the following argument shows that $\pi_{00}(T) \subseteq \pi_{00}(T_n)$. If $\lambda \in \pi_{00}(T)$, we have that $\lambda \in \sigma(T_n)$, because $\lambda \in \sigma(T)$. On the other hand, for
\[ \lambda \neq 0, \text{ Lemma 2.1 implies that } \alpha(\lambda I - T) = \alpha(\lambda I - T_n), \text{ so } 0 < \alpha(\lambda I - T_n) < \infty. \]

For \( \lambda = 0 \), we claim that \( \alpha(T_n) > 0 \). If \( \alpha(T_n) = 0 \), we have that \( p(T_n) = 0 \).

By [7, Lemma 2], \( p(T) < \infty \). Moreover [7, Remark 1],

\[
p(T) = \inf\{ k \in \mathbb{N} : T_k \text{ is injective} \} \leq n.
\]

Thus, by Lemma 1.1, \( T_n \) is bounded below, because \( T_n \) is injective and \( R(T_n) = R(T^{n+1}) \) is closed, so \( T_n \) is semi-Fredholm. Also \( (T_n)^* \) has SVEP at 0, because \( 0 \in \text{ iso } \sigma(T_n) \), then \( q(T_n) < \infty \) ([1, Chapter 3]), which implies that \( q(T) < \infty \) ([7, Lemma 3]). Hence \( 0 < p(T) = q(T) < \infty \), a contradiction, since 0 is not a pole of the resolvent of \( T \). Thus \( 0 < \alpha(T_n) = \alpha(0I - T_n) \).

Finally, since \( N(T_n) \subseteq N(T) \) and \( \alpha(T) < \infty \) it then follows the equality \( \alpha(T_n) = \alpha(0I - T_n) < \infty \). Thus, \( 0 \in \text{ iso } \sigma(T_n) \) and \( 0 < \alpha(0I - T_n) < \infty \). Consequently \( \lambda \in \pi_{00}(T_n) \), for each \( \lambda \in \pi_{00}(T) \), so we have the inclusion \( \pi_{00}(T) \subseteq \pi_{00}(T_n) \).

The result of Lemma 2.3 may be extended as follows.

**Lemma 2.4.** If \( 0 \) is not a pole of the resolvent of \( T \in L(X) \) and \( R(T^n) \) is closed, then \( \pi_{00}^a(T) \subseteq \pi_{00}^a(T_n) \).

**Proof.** If \( \lambda \notin \sigma_{ap}(T) \), then \( \lambda I - T \) is injective and \( R(\lambda I - T) \) is closed. Now, here we consider the two different cases \( \lambda \neq 0 \) and \( \lambda = 0 \). If \( \lambda \neq 0 \), by Lemma 2.1, \( N(\lambda I - T_n) = N(\lambda I - T) \) and \( R(\lambda I - T_n) = R(\lambda I - T) \cap R(T^n) \) is closed. Hence \( \lambda I - T_n \) is bounded below, and so \( \lambda \notin \sigma_{ap}(T_n) \). In the other case, \( -T \) bounded below implies that \( 0 = p(T) = p(T_n) \) and \( R(T) \) is closed. Thus \( T_n \) is injective and, by Lemma 1.1, \( R(T_n) = R(T^{n+1}) \) is closed. From this we obtain that \( T_n \) is bounded below. Consequently, \( \sigma_{ap}(T_n) \subseteq \sigma_{ap}(T) \).

Similarly, as in the proof of Lemma 2.3 and taking into account Lemma 2.2, we can prove that \( \sigma_{ap}(T) \subseteq \sigma_{ap}(T_n) \).

Finally, to show \( \pi_{00}^a(T) \subseteq \pi_{00}^a(T_n) \). Observe that, if \( \lambda \in \pi_{00}^a(T) \) then \( \lambda \in \sigma_{ap}(T) \) and \( 0 < \alpha(\lambda I - T) < \infty \). Thus \( \lambda \in \sigma_{ap}(T_n) \). For \( \lambda \neq 0 \), by Lemma 2.1, \( \alpha(\lambda I - T_n) = \alpha(\lambda I - T) \), and so \( 0 < \alpha(\lambda I - T_n) < \infty \). In the case \( \lambda = 0 \), \( p(T_n) = 0 \) and \( R(T^n) \) is closed. Similarly to the case \( p(T_n) = 0 \) and \( R(T^n) \) closed in the proof of Lemma 2.3, one shows that \( 0 < \alpha(0I - T_n) < \infty \). Consequently \( \pi_{00}^a(T) \subseteq \pi_{00}^a(T_n) \).

### 3. Weyl’s theorems and restrictions

In this section we give conditions for which Weyl’s theorem (resp. a-Weyl’s theorem) for an operator \( T \in L(X) \) is equivalent to Weyl’s theorem (resp. a-
Weyl’s theorem) for certain restriction $T_n$ of $T$.

It is well known that if $\lambda$ is a pole of the resolvent of $T$, then $\lambda$ is an isolated point of the spectrum $\sigma(T)$. Thus, the following result is an immediate consequence of Lemma 2.1 and Lemma 2.3.

**Theorem 3.1.** Suppose that $0$ is not an isolated point of $\sigma(T)$. Then $T$ satisfies (W) if and only if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_n$ satisfies (W).

**Proof.** (Necessity) Assume that there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_n$ satisfies (W). Let $\lambda \in \pi_{00}(T)$, i.e. $\lambda \in \text{iso}(T)$ and $0 < \alpha(\lambda I - T) < \infty$. By hypothesis and Lemma 2.3, $0 \neq \lambda \in \pi_{00}(T_n) = \sigma(T_n) \setminus \sigma_w(T_n)$. Then $\alpha(\lambda I - T_n) = \beta(\lambda I - T_n) < \infty$ since $\lambda I - T_n$ is a Weyl operator, and so by Lemma 2.1

$$\alpha(\lambda I - T) = \alpha(\lambda I - T_n) = \beta(\lambda I - T_n) = \beta(\lambda I - T) < \infty.$$  

Furthermore, $\lambda \in \sigma(T)$ because $\lambda \in \sigma(T_n) \subseteq \sigma(T)$. Thus $\lambda I - T$ is Weyl, and hence $\lambda \in \sigma(T) \setminus \sigma_w(T)$. But since $\sigma(T) \setminus \sigma_w(T) \subseteq \pi_{00}(T)$, it then follows that $\pi_{00}(T) = \sigma(T) \setminus \sigma_w(T)$, which implies that $T$ satisfies (W).

(Sufficiency) Suppose that $T$ satisfies (W). Then for $n = 0$, $R(T^0) = X$ is closed and $T_0 = T$ satisfies (W).

In the same way as in Theorem 3.1, we have the following characterization of $a$-Weyl theorem for an operator throughout $a$-Weyl theorem for some restriction of the operator.

**Theorem 3.2.** Suppose that $0$ is not an isolated point of $\sigma(T)$. Then $T$ satisfies (aW) if and only if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_n$ satisfies (aW).

**Proof.** (Necessity) Suppose that there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_n$ satisfies (aW). Let $\lambda \in \pi_{00}^a(T)$, by hypothesis and Lemma 2.4, $\lambda \in \pi_{00}^a(T_n) = \sigma_{ap}(T_n) \setminus \sigma_{uw}(T_n)$. Thus $\lambda I - T_n$ is an upper semi-Fredholm operator, because $\lambda I - T_n$ is an upper semi-Weyl operator. Since $\lambda I - T_n$ is upper semi-Fredholm, it follows that $R((\lambda I - T_n)^m)$ is closed in $R(T_n)$ for all $m \in \mathbb{N}$, and so by Lemma 2.2, there exists $k \in \mathbb{N}$ such that $R((\lambda I - T)^k)$ is closed. But since $\alpha(\lambda I - T) < \infty$, then $\alpha((\lambda I - T)^k) < \infty$. That is, $(\lambda I - T)^k$ is an upper semi-Fredholm operator, which implies that $\lambda I - T$ is upper semi-Fredholm. Furthermore, $T$ has SVEP at $\lambda$ because $\lambda \in \text{iso}(\sigma_{ap}(T))$. Consequently, if
\( \lambda \in \pi_{00}^a(T) \) then \( \lambda I - T \) is upper semi-Fredholm and \( p(\lambda I - T) < \infty \). Hence \( \lambda I - T \) is upper semi-Weyl and \( \lambda \in \sigma_{ap}(T) \), thus \( \lambda \in \sigma_{ap}(T) \setminus \sigma_{uw}(T) \), and we obtain the inclusion \( \pi_{00}^a(T) \subseteq \sigma_{ap}(T) \setminus \sigma_{uw}(T) \). But since \( \sigma_{ap}(T) \setminus \sigma_{uw}(T) \subseteq \pi_{00}^a(T) \), it then follows that \( \pi_{00}^a(T) = \sigma_{ap}(T) \setminus \sigma_{uw}(T) \), which implies that \( T \) satisfies (aW).

(Sufficiency) If \( T \) satisfies (aW). Then for \( n = 0 \), trivially \( R(T^n) = X \) is closed and \( T_0 = T \) satisfies (aW). \( \square \)

Clearly, \( T \) has SVEP at every isolated point of \( \sigma(T) \). Thus, by Theorem 3.1 and Theorem 3.2, we have the following corollary.

**Corollary 3.3.** If \( T \) does not have SVEP at 0, then:

(i) there exists \( n \in \mathbb{N} \) such that \( R(T^n) \) is closed and \( T_n \) satisfies (W) if and only if \( T \) satisfies (W).

(ii) there exists \( n \in \mathbb{N} \) such that \( R(T^n) \) is closed and \( T_n \) satisfies (aW) if and only if \( T \) satisfies (aW).

**Remark 3.4.** There are more alternative ways to express Corollary 3.3. We may replace the assumption \( T \) does not have SVEP at 0 by: \( 0 \notin \partial \sigma(T) \), \( p(T) = \infty \) or \( q(T) = \infty \).

**References**


