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$\mathfrak{aff}(1|1)$-trivial deformations of $\mathfrak{aff}(2|1)$-modules of weighted densities on the superspace $\mathbb{R}^{1|2}$

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Abstract: Over the (1|2)-dimensional real superspace, we study $\mathfrak{aff}(1|1)$-trivial deformations of the action of the affine Lie superalgebra $\mathfrak{aff}(2|1)$ on the direct sum of the superspaces of weighted densities. We compute the necessary and sufficient integrability conditions of a given infinitesimal deformation of this action and we prove that any formal deformation is equivalent to its infinitisemal part.

Key words: relative cohomology, trivial deformation, Lie superalgebra, symbol.

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1. Introduction

Let $\mathrm{Vect}(\mathbb{R})$ be the Lie algebra of polynomial vector fields on the real space. Consider the 1-parameter deformation of $\mathrm{Vect}(\mathbb{R})$-action on $C^\infty(\mathbb{R})$

$$\mathcal{L}^{\lambda}_{X\frac{d}{dx}}(f) = Xf' + \lambda X' f,$$

where $X,\, f \in C^\infty(\mathbb{R})$ and $X' := \frac{dX}{dx}$. Denote by $\mathcal{F}_\lambda$ the $\mathrm{Vect}(\mathbb{R})$-module structure on $C^\infty(\mathbb{R})$ defined by $\mathcal{L}^\lambda$ for a fixed λ. Geometrically, $\mathcal{F}_\lambda = \{ f dx^\lambda : f \in C^\infty(\mathbb{R})\}$ is the space of polynomial weighted densities of weight $\lambda \in \mathbb{R}$. The space $\mathcal{F}_\lambda$ coincides with the space of vector fields, functions and differential 1-forms for $\lambda = -1$, 0 and 1, respectively.

The superspace $\mathrm{D}_{\lambda,\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$ the linear differential operators with the natural $\mathrm{Vect}(\mathbb{R})$-action denoted $\mathcal{L}^{\lambda,\mu}_X(A) = \mathcal{L}^\mu_X \circ A - A \circ \mathcal{L}^\lambda_X$. Each module $\mathrm{D}_{\lambda,\mu}$ has a natural filtration by the order of differential operators; the graded module $\mathcal{S}_{\lambda,\mu} := \mathrm{gr}\, \mathrm{D}_{\lambda,\mu}$ is the space of symbols. The quotient-module $\mathrm{D}^k_{\lambda,\mu} / \mathrm{D}^{k-1}_{\lambda,\mu}$ is isomorphic to the module of weighted densities $\mathcal{F}_{\mu-\lambda-k}$; the

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isomorphism is defined by the principal symbol map $\sigma_{\rm pr}$ defined by

$$A=\sum_{i=0}^{k}a_i(x)\left(\frac{\partial}{\partial x}\right)^i\longmapsto\sigma_{\rm pr}(A)=a_k(x)(dx)^{\mu-\lambda-k},$$

(see, e.g.,[15]). Therefore, as a Vect($\mathbb{R}$)-module, the space $\mathcal{S}_{\lambda,\mu}$ depends on the difference $\beta=\mu-\lambda$, so that $\mathcal{S}_{\lambda,\mu}$ be written as $\mathcal{S}_\beta$, and

$$\mathcal{S}_\beta=\bigoplus_{k=0}^{\infty}\mathcal{F}_{\beta-k}$$

as Vect($\mathbb{R}$)-modules. The space of symbols of order $\leq n$ is

$$\mathcal{S}_\beta^n:=\bigoplus_{k=0}^{n}\mathcal{F}_{\beta-k}.$$

The space $\mathrm{D}_{\lambda,\mu}$ cannot be isomorphic as a Vect($\mathbb{R}$)-module to the space of symbols, but is a deformation of this space in the sense of Richardson and Neijenhuis [19].

Deformation theory plays a crucial role in all branches of physics. In physics the mathematical theory of deformations has been proved to be a powerful tool in modeling physical reality. The concepts symmetry and deformations are considered to be two fundamental guiding principles for developing the physical theory further. The notion of deformation was applied to Lie algebras by Nijenhuis and Richardson [19, 18]. This theory is developed by Ovsienko and by other authors [3, 9, 19].

We consider the superspace $\mathbb{R}^{1|2}$ endowed with its standard contact structure defined by the 1-form α_2, and $\mathcal{K}(2)$ of contact vector fields on $\mathbb{R}^{1|2}$. We introduce the $\mathcal{K}(2)$-module $\mathfrak{F}_\lambda^2$ of λ-densities on $\mathbb{R}^{1|2}$ and the $\mathcal{K}(2)$-module of linear differential operators, $\mathfrak{D}_{\lambda,\mu}^2:=\mathrm{Hom}_{\rm diff}(\mathfrak{F}_\lambda^2,\mathfrak{F}_\mu^2)$, which are super analogs of the spaces $\mathcal{F}_\lambda$ and $\mathrm{D}_{\lambda,\mu}$, respectively. The Lie superalgebra $\mathfrak{aff}(2|1)$, a super analog of $\mathfrak{aff}(1)$, is a subalgebra of $\mathcal{K}(2)$. We classify the $\mathfrak{aff}(1|1)$-trivial deformations of the structure of the $\mathfrak{aff}(2|1)$-module

$$\mathfrak{S}_{\mu-\lambda}^2=\bigoplus_{k=0}^{\infty}\mathfrak{F}_{\mu-\lambda-\frac{k}{2}}^2,$$

which is super analog of the space $\mathcal{S}_\beta$. We prove that any formal deformation is equivalent to its infinitesimal part and we give an example of deformation with one parameter.

2. Definitions and notations

We briefly give in this section the basics definitions of geometrical objects on $\mathbb{R}^{1|2}$ that will be needed for our purpose, for more details, see [7, 11, 6, 16, 15, 17].

2.1. The Lie superalgebra of contact vector fields on $\mathbb{R}^{1|2}$.

Let $\mathbb{R}^{1|2}$ be the superspace with coordinates $(x,\ \theta_1,\ \theta_2)$, where θ_1 and θ_2 are odd indeterminates: $\theta_i\theta_j = -\theta_j\theta_i$. We consider the superspace $C^\infty(\mathbb{R}^{1|2})$ of polynomial functions. Any element of $C^\infty(\mathbb{R}^{1|2})$ has the form

$$F = f_0 + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2,$$

where $f_0, f_1, f_2, f_{12} \in C^\infty(\mathbb{R})$. Even elements in $C^\infty(\mathbb{R}^{1|2})$ are the functions $F(x,\theta) = f_0(x) + f_{12}(x)\theta_1\theta_2$, the functions $F(x,\theta) = f_1(x)\theta_1 + f_2(x)\theta_2$ are odd elements. We denote by $|F|$ the parity of a function F. Let $\mathrm{Vect}(\mathbb{R}^{1|2})$ be the space of polynomial vector fields on $\mathbb{R}^{1|2}$:

$$\mathrm{Vect}(\mathbb{R}^{1|2}) = \left\{F_0\partial_x + F_1\partial_1 + F_2\partial_2\ :\ F_i \in C^\infty(\mathbb{R}^{1|2})\right\},$$

where ∂_i and ∂_x stand for $\frac{\partial}{\partial\theta_i}$ and $\frac{\partial}{\partial x}$. The superbracket of two vector fields is bilinear and defined for two homogeneous vector fields by

$$[X, Y] = X \circ Y - (-1)^{|X||Y|} Y \circ X.$$

The supespace $\mathbb{R}^{1|2}$ is equipped with the contact structure given by the 1-form

$$\alpha_2 = dx + \theta_1 d\theta_1 + \theta_2 d\theta_2.$$

This contact structure is equivalently defined by the kernel of α_2, spanned by the odd vector fields

$$\overline{\eta}_i = \partial_i - \theta_i\partial_x.$$

We consider the superspace $\mathcal{K}(2)$ of contact vector fields on $\mathbb{R}^{1|2}$. That is,

$$\mathcal{K}(2) = \left\{X \in \mathrm{Vect}(\mathbb{R}^{1|2})\ :\ \exists\ F \in C^\infty(\mathbb{R}^{1|2}) \text{ such that } \mathfrak{L}_X(\alpha_2) = F\alpha_2\right\},$$

where $\mathfrak{L}_X$ is the Lie derivative of a vector field, acting on the space of functions, forms, vector fields, ...

Any contact vector field on $\mathbb{R}^{1|2}$ can be expressed as

$$X_F = F\partial_x - \frac{1}{2}(-1)^{|F|}\sum_{i=1}^{2}\overline{\eta}_i(F)\overline{\eta}_i\,, \qquad \text{where } F \in C^\infty(\mathbb{R}^{1|2}).$$

Of course, $\mathcal{K}(2)$ is a subalgebra of $\mathrm{Vect}(\mathbb{R}^{1|2})$, and $\mathcal{K}(2)$ acts on $C^\infty(\mathbb{R}^{1|2})$ through

$$\mathfrak{L}_{X_F}(G) = FG' - \frac{1}{2}(-1)^{|F|}\sum_{i=1}^{2}\overline{\eta}_i(F)\cdot\overline{\eta}_i(G), \tag{2.1}$$

where $G \in C^\infty(\mathbb{R}^{1|2})$.

The contact bracket is defined by $[X_F,\, X_G] = X_{\{F,G\}}$. The space $C^\infty(\mathbb{R}^{1|2})$ is thus equipped with a Lie superalgebra structure isomorphic to $\mathcal{K}(2)$. The explicit formula can be easily calculated:

$$\{F,G\} = FG' - F'G - \frac{1}{2}(-1)^{|F|}\sum_{i=1}^{2}\overline{\eta}_i(F)\cdot\overline{\eta}_i(G). \tag{2.2}$$

2.2. THE SUPERALGEBRA $\mathfrak{aff}(2|1)$. Recall that the Lie algebra $\mathfrak{aff}(1)$ can be realized as a subalgebra of $\mathrm{Vect}(\mathbb{R})$:

$$\mathfrak{aff}(1) = \mathrm{Span}\left(X_1,\, X_x\right),$$

and the affine Lie superalgebra $\mathfrak{aff}(1|1)$ is realized as a subalgebra of $\mathcal{K}(1)$:

$$\mathfrak{aff}(1|1) = \mathrm{Span}(X_1,\, X_x,\, X_\theta).$$

The space $\mathfrak{aff}(1|1)_0$ is isomorphic to $\mathfrak{aff}(1)$, while

$$(\mathfrak{aff}(1|1))_{\bar{1}} = \mathrm{Span}(X_\theta).$$

Similarly, the affine Lie superalgebra $\mathfrak{aff}(2|1)$ can be realized as a subalgebra of $\mathcal{K}(2)$:

$$\mathfrak{aff}(2|1) = \mathrm{Span}(X_1,\, X_x,\, X_{\theta_1},\, X_{\theta_2},\, X_{\theta_1\theta_2}),$$

where

$$(\mathfrak{aff}(2|1))_{\bar{0}} = \mathrm{Span}(X_1,\, X_x,\, X_{\theta_1\theta_2}),$$
$$(\mathfrak{aff}(2|1))_{\bar{1}} = \mathrm{Span}(X_{\theta_1},\, X_{\theta_2}).$$

We easily see that $\mathfrak{aff}(1|1)$ is subalgebra of the Lie superalgebra $\mathfrak{aff}(2|1)$.

2.3. THE SPACE OF WEIGHTED DENSITIES ON $\mathbb{R}^{1|2}$. We introduce a one-parameter family of modules over the Lie superalgebra $\mathcal{K}(2)$. As vector spaces all these modules are isomorphic to $C^\infty(\mathbb{R}^{1|2})$, but not as $\mathcal{K}(2)$-modules.

For every contact vector field X_F, define a one-parameter family of first-order differential operators on $C^\infty(\mathbb{R}^{1|2})$:

$$\mathfrak{L}^\lambda_{X_F} = X_F + \lambda F', \qquad \lambda \in \mathbb{R}. \tag{2.3}$$

We easily check that

$$[\mathfrak{L}^\lambda_{X_F}, \mathfrak{L}^\lambda_{X_G}] = \mathfrak{L}^\lambda_{X_{\{F,G\}}}. \tag{2.4}$$

We thus obtain a one-parameter family of $\mathcal{K}(2)$-modules on $C^\infty(\mathbb{R}^{1|2})$ that we denote $\mathfrak{F}^2_\lambda$, the space of all weighted densities on $C^\infty(\mathbb{R}^{1|2})$ of weight λ with respect to α_2:

$$\mathfrak{F}^2_\lambda = \left\{ F\alpha_2^\lambda \,:\, F \in C^\infty(\mathbb{R}^{1|2}) \right\}. \tag{2.5}$$

In particular, we have $\mathfrak{F}^0_\lambda = \mathcal{F}_\lambda$. Obviously the adjoint $\mathcal{K}(2)$-module is isomorphic to the space of weighted densities on $\mathbb{R}^{1|2}$ of weight -1.

2.4. DIFFERENTIAL OPERATORS ON WEIGHTED DENSITIES. A differential operator on $\mathbb{R}^{1|2}$ is an operator on $C^\infty(\mathbb{R}^{1|2})$ of the form:

$$A = \sum_{j=0}^{m} a_j \partial_x^j + \sum_{i=1}^{2}\sum_{k=0}^{n_i} b_{k,i}\partial_x^k \partial_i + \sum_{\ell=0}^{n} c_\ell \partial_x^\ell \partial_1\partial_2, \tag{2.6}$$

where $a_j,\ b_{k,i},\ c_\ell \in C^\infty(\mathbb{R}^{1|2})$. Any differential operator defines a linear mapping $F\alpha_2^\lambda \mapsto (AF)\alpha_2^\mu$ from $\mathfrak{F}^2_\lambda$ to $\mathfrak{F}^2_\mu$ for any $\lambda,\ \mu \in \mathbb{K}$; thus, the space of differential operators becomes a family of $\mathfrak{osp}(2|2)$-modules $\mathfrak{D}^2_{\lambda,\mu}$ for the natural action:

$$X_F \cdot A = \mathfrak{L}^\mu_{X_F} \circ A - (-1)^{|A||F|} A \circ \mathfrak{L}^\lambda_{X_F}. \tag{2.7}$$

PROPOSITION 2.1. *Every differential operator $A \in \mathfrak{D}^2_{\lambda,\mu}$ can be expressed in the form*

$$A(F\alpha_2^\lambda) = \sum_{\ell,m} a_{\ell,m}(x,\theta)\overline{\eta}_1^\ell \overline{\eta}_2^m (F)\alpha_2^\mu, \tag{2.8}$$

where $a_{\ell,m}(x,\theta)$ are arbitrary functions.

Proof. Since $-\overline{\eta}_i^2 = \partial_x$, and $\partial_i = \overline{\eta}_i - \theta_i\overline{\eta}_i^2$, every differential operator A given by (2.6) is a polynomial expression in $\overline{\eta}_1$ and $\overline{\eta}_2$. ∎

Proposition 2.2. *As a* $\mathfrak{aff}(1|1)$*-module, we have*

$$\mathfrak{D}^2_{\lambda,\mu} \simeq \mathfrak{D}^1_{\lambda,\mu} \oplus \mathfrak{D}^1_{\lambda+\frac{1}{2},\mu+\frac{1}{2}} \oplus \Pi\left(\mathfrak{D}^1_{\lambda,\mu+\frac{1}{2}} \oplus \mathfrak{D}^1_{\lambda+\frac{1}{2},\mu}\right). \tag{2.9}$$

Proof. Any element $F \in C^\infty(\mathbb{R}^{1|2})$ can be uniquely written as follows: $F = F_1 + F_2\theta_2$, where $\partial_2 F_1 = \partial_2 F_2 = 0$. Therefore, for any $X_H \in \mathfrak{aff}(1|1)$, we easily chek that

$$\mathfrak{L}^\lambda_{X_H}(F) = \mathfrak{L}^\lambda_{X_H}(F_1) + \mathfrak{L}^{\lambda+\frac{1}{2}}_{X_H}(F_2)\theta_2.$$

Thus, the following map is an $\mathfrak{aff}(1|1)$-isomorphism:

$$\begin{aligned} \Phi_\lambda : \mathfrak{F}^2_\lambda &\longrightarrow \mathfrak{F}^1_\lambda \oplus \Pi(\mathfrak{F}^1_{\lambda+\frac{1}{2}}) \\ F\alpha_2^\lambda &\longmapsto \left(F_1\alpha_1^\lambda, \Pi(F_2\alpha_1^{\lambda+\frac{1}{2}})\right). \end{aligned} \tag{2.10}$$

So, we deduce an $\mathfrak{aff}(1|1)$-isomorphism:

$$\begin{aligned} \Psi_{\lambda,\mu} : \mathfrak{D}^1_{\lambda,\mu} \oplus \mathfrak{D}^1_{\lambda+\frac{1}{2},\mu+\frac{1}{2}} \oplus \Pi\left(\mathfrak{D}^1_{\lambda,\mu+\frac{1}{2}} \oplus \mathfrak{D}^1_{\lambda+\frac{1}{2},\mu}\right) &\longrightarrow \mathfrak{D}^2_{\lambda,\mu} \\ A &\longmapsto \Phi_\mu^{-1} \circ A \circ \Phi_\lambda. \end{aligned} \tag{2.11}$$

We identify the $\mathfrak{aff}(1|1)$-modules the following isomorphisms:

$$\begin{array}{rclrcl} \Pi\left(\mathfrak{D}^1_{\lambda,\mu+\frac{1}{2}}\right) &\longrightarrow& \mathrm{Hom_{diff}}\left(\mathfrak{F}^1_\lambda, \Pi(\mathfrak{F}^1_{\mu+\frac{1}{2}})\right), & \Pi(A) &\longmapsto& \Pi\circ A, \\ \Pi\left(\mathfrak{D}^1_{\lambda+\frac{1}{2},\mu}\right) &\longrightarrow& \mathrm{Hom_{diff}}\left(\mathfrak{F}^1_{\lambda+\frac{1}{2}}, \Pi(\mathfrak{F}^1_\mu)\right), & \Pi(A) &\longmapsto& A\circ\Pi, \\ \mathfrak{D}^1_{\lambda+\frac{1}{2},\mu+\frac{1}{2}} &\longrightarrow& \mathrm{Hom_{diff}}\left(\Pi(\mathfrak{F}^1_{\lambda+\frac{1}{2}}), \Pi(\mathfrak{F}^1_{\mu+\frac{1}{2}})\right), & \Pi(A) &\longmapsto& \Pi\circ A\circ\Pi. \end{array}$$

■

3. $\mathfrak{aff}(1|1)$-trivial deformation of $\mathfrak{aff}(2|1)$-modules

Deformation theory of Lie algebra was first considered with one-parameter of deformation [13, 19, 12, 21, 4, 5]. Recently, deformations of Lie (super)algebras with multi-parameters were intensively studied (see, e.g., [1, 2, 3, 8, 20]).

3.1. Infinitesimal deformations and the first cohomology. Let $\rho_0 : \mathfrak{g} \to \mathrm{End}(V)$ be an action of a Lie superalgebra $\mathfrak{g}$ on a vector superspace V and let $\mathfrak{h}$ be a subagebra of $\mathfrak{g}$ (if $\mathfrak{h}$ is omitted it assumed to be $\{0\}$). When studying $\mathfrak{h}$-trivial deformations of the $\mathfrak{g}$-action ρ_0, one usually starts with *infinitesimal* deformations

$$\rho = \rho_0 + t\,\Upsilon, \tag{3.1}$$

where $\Upsilon : \mathfrak{g} \to \mathrm{End}(V)$ is a linear map vanishing on $\mathfrak{h}$ and t is a formal parameter with $p(t) = p(\Upsilon)$. The homomorphism condition

$$[\rho(x), \rho(y)] = \rho([x, y]), \tag{3.2}$$

where $x,\ y \in \mathfrak{g}$, is satisfied in order 1 in t if and only if Υ is a $\mathfrak{h}$-relative 1-cocycle. That is, the map Υ satisfies

$$(-1)^{|x||\Upsilon|}[\rho_0(x), \Upsilon(y)] - (-1)^{|y|(|x|+|\Upsilon|)}[\rho_0(y), \Upsilon(x)] - \Upsilon([x,\ y]) = 0.$$

Moreover, two $\mathfrak{h}$-trivial infinitesimal deformations $\rho = \rho_0 + t\,\Upsilon_1$, and $\rho = \rho_0 + t\,\Upsilon_2$, are equivalents if and only if $\Upsilon_1 - \Upsilon_2$ is $\mathfrak{h}$-relative coboundary:

$$(\Upsilon_1 - \Upsilon_2)(x) = (-1)^{|x||A|}[\rho_0(x), A] := \delta A(x),$$

where $A \in \mathrm{End}(V)^{\mathfrak{h}}$ and δ stands for differential of cochains on $\mathfrak{g}$ with values in $\mathrm{End}(V)$ (see, e.g., [14, 19]). So, the space $\mathrm{H}^1(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V))$ determines and classifies infinitesimal deformations up to equivalence. If

$$\dim \mathrm{H}^1(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V)) = m,$$

then choose 1-cocycles $\Upsilon_1, \dots, \Upsilon_m$ representing a basis of $\mathrm{H}^1(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V))$ and consider the infinitesimal deformation

$$\rho = \rho_0 + \sum_{i=1}^{m} t_i\,\Upsilon_i, \tag{3.3}$$

where $t_1, \dots, t_m$ are independent parameters with $|t_i| = |\Upsilon_i|$.

Since we are interested in the $\mathfrak{aff}(1|1)$-trivial deformations of the $\mathfrak{aff}(2|1)$-module structure on the space

$$\mathfrak{S}^{2,m}_{\beta} = \bigoplus_{k=0}^{2m} \mathfrak{F}^2_{\beta-\frac{k}{2}}, \qquad \text{where } m \in \frac{1}{2}\mathbb{N}. \tag{3.4}$$

The first differential cohomology spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{aff}(2|1), \mathfrak{aff}(1|1), \mathfrak{D}^2_{\lambda,\mu})$ was computed in [10]. The result is as follows:

$$\dim(\mathrm{H}^1(\mathfrak{aff}(2|1), \mathfrak{aff}(1|1), \mathfrak{D}^2_{\lambda,\mu})) = \begin{cases} 1 & \text{if } \mu = \lambda, \\ 2 & \text{if } \mu - \lambda = k,\ k \in \{1, 2, \dots\}, \\ 0 & \text{otherwise.} \end{cases}$$

The following 1-cocycles span the corresponding cohomology spaces:

$$\Upsilon_{\lambda,\lambda}(X_G) = \overline{\eta}_1\overline{\eta}_2(G),$$
$$\Gamma^1_{\lambda,\lambda+k}(X_G) = \overline{\eta}_1\overline{\eta}_2(G)\partial_x^k,$$
$$\Gamma^2_{\lambda,\lambda+k}(X_G) = \overline{\eta}_1\overline{\eta}_2(G)\overline{\eta}_1\overline{\eta}_2\partial_x^{k-1}.$$

We consider the space $\mathrm{H}^1(\mathfrak{aff}(2|1), \mathfrak{aff}(1|1), \mathrm{End}(\mathfrak{S}^{2,m}_\beta))$ spanned by the classes $\Upsilon^i_{\lambda,\lambda}$ and $\Gamma^i_{\lambda,\lambda+k}$, $i = 1, 2$, where $k \in \{1, \dots, [m]\}$, $[m]$ denoting the integer part of m, and $2(\beta - \lambda) \in \{2k, \dots, 2m\}$ for a generic β. Any infinitesimal $\mathfrak{aff}(1|1)$-trivial deformation of the $\mathfrak{aff}(2|1)$-module structure on $\mathfrak{S}^{2,m}_\beta$ is then of the form

$$\widetilde{\mathfrak{L}}_{X_F} = \mathfrak{L}_{X_F} + \mathfrak{L}^{(1)}_{X_F}, \tag{3.5}$$

where $\mathfrak{L}_{X_F}$ is the Lie derivate of $\mathfrak{S}^{2,m}_\beta$ along the vector X_F defined by (2.3), and

$$\mathfrak{L}^{(1)}_{X_F} = \sum_\lambda t_{\lambda,\lambda}\Upsilon_{\lambda,\lambda}(X_F) + \sum_\lambda\sum_{k=1}^{[m]}\sum_{i=1}^{2} t^i_{\lambda,\lambda+k}\Gamma^i_{\lambda,\lambda+k}(X_F), \tag{3.6}$$

where $t_{\lambda,\lambda}$ and $t^i_{\lambda,\lambda+k}$ are independent parameters with $|t_{\lambda,\lambda}| = |\Upsilon_{\lambda,\lambda}|$ and $|t^i_{\lambda,\lambda+k}| = |\Gamma^i_{\lambda,\lambda+k}|$.

3.2. Integrability conditions and deformations over supercommutative algebras. Consider the superalgebra with unity $\mathbb{C}[[t_1, \dots, t_m]]$ and consider the problem of integrability of infinitesimal deformations. Starting with the infinitesimal deformation (3.3), we look for a formal series

$$\rho = \rho_0 + \sum_{i=1}^{m} t_i\,\Upsilon_i + \sum_{i,j} t_i t_j\,\rho^{(2)}_{ij} + \cdots\,, \tag{3.7}$$

where the higher order terms $\rho^{(2)}_{ij}$, $\rho^{(3)}_{ijk}$, … are linear maps from $\mathfrak{g}$ to $\mathrm{End}(V)$ with $|\rho^{(2)}_{ij}| = |t_i t_j|$, $|\rho^{(3)}_{ijk}| = |t_i t_j t_k|$, … such that the map

$$\rho : \mathfrak{g} \to \mathbb{C}[[t_1, \dots, t_m]] \otimes \mathrm{End}(V) \tag{3.8}$$

satisfies the homomorphism condition (3.2).

Quite often the above problem has no solution. Following [13] and [2], we will impose extra algebraic relations on the parameters $t_1, \dots, t_m$. Let $\mathcal{R}$ be an ideal in $\mathbb{C}[[t_1, \dots, t_m]]$ generated by some set of relations, and we can speak about deformations with base $\mathcal{A} = \mathbb{C}[[t_1, \dots, t_m]]/\mathcal{R}$, (for details, see [13]). The map (3.8) sends $\mathfrak{g}$ to $\mathcal{A} \otimes \mathrm{End}(V)$.

Setting

$$\varphi_t = \rho - \rho_0,\ \rho^{(1)} = \sum t_i\, \Upsilon_i,\ \rho^{(2)} = \sum t_i t_j\, \rho_{ij}^{(2)}, \dots,$$

we can rewrite the relation (3.2) in the following way:

$$[\varphi_t(x), \rho_0(y)] + [\rho_0(x), \varphi_t(y)] - \varphi_t([x,y]) + \sum_{i,j>0} [\rho^{(i)}(x), \rho^{(j)}(y)] = 0. \tag{3.9}$$

The first three terms are $(\delta\varphi_t)(x,y)$. For arbitrary linear maps $\gamma_1,\ \gamma_2 : \mathfrak{g} \to \mathrm{End}(V)$, consider the standard *cup-product*: $[\![\gamma_1, \gamma_2]\!] : \mathfrak{g}\otimes\mathfrak{g} \to \mathrm{End}(V)$ defined by:

$$\begin{aligned}[\![\gamma_1, \gamma_2]\!](x,y) = (-1)^{|\gamma_2|(|\gamma_1|+|x|)}&[\gamma_1(x), \gamma_2(y)] \\ &+ (-1)^{|\gamma_1||x|}[\gamma_2(x), \gamma_1(y)].\end{aligned} \tag{3.10}$$

The relation (3.9) becomes now equivalent to

$$\delta\varphi_t + \frac{1}{2}[\![\varphi_t, \varphi_t]\!] = 0. \tag{3.11}$$

Expanding (3.11) in power series in $t_1, \dots, t_m$, we obtain the following equation for $\rho^{(k)}$:

$$\delta\rho^{(k)} + \frac{1}{2}\sum_{i+j=k} [\![\rho^{(i)}, \rho^{(j)}]\!] = 0. \tag{3.12}$$

The first non-trivial relation $\delta\rho^{(2)} + \frac{1}{2}[\![\rho^{(1)}, \rho^{(1)}]\!] = 0$ gives the first obstruction to integration of an infinitesimal deformation. Thus, considering the coefficient of $t_i\, t_j$, we get

$$\delta\rho_{ij}^{(2)} + \frac{1}{2}[\![\Upsilon_i, \Upsilon_j]\!] = 0. \tag{3.13}$$

It is easy to check that for any two 1-cocycles γ_1 and $\gamma_2 \in Z^1(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V))$, the bilinear map $[\![\gamma_1, \gamma_2]\!]$ is a $\mathfrak{h}$-relative 2-cocycle. The relation (3.13) is precisely the condition for this cocycle to be a coboundary. Moreover, if one of the cocycles γ_1 or γ_2 is a $\mathfrak{h}$-relative coboundary, then $[\![\gamma_1, \gamma_2]\!]$ is a $\mathfrak{h}$-relative 2-coboundary. Therefore, we naturally deduce that the operation (3.10) defines a bilinear map:

$$\mathrm{H}^1(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V)) \otimes \mathrm{H}^1(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V)) \longrightarrow \mathrm{H}^2(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V)). \tag{3.14}$$

All the obstructions lie in $\mathrm{H}^2(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V))$ and they are in the image of $\mathrm{H}^1(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V))$ under the cup-product.

3.3. Equivalence. Two deformations ρ and ρ' of a $\mathfrak{g}$-module V over $\mathcal{A}$ are said to be *equivalent* (see [13]) if there exists an inner automorphism Ψ of the associative superalgebra $\mathcal{A}\otimes\mathrm{End}(V)$ such that

$$\Psi\circ\rho=\rho' \qquad \text{and} \qquad \Psi(\mathbb{I})=\mathbb{I},$$

where $\mathbb{I}$ is the unity of the superalgebra $\mathcal{A}\otimes\mathrm{End}(V)$.

The following notion of miniversal deformation is fundamental. It assigns to a $\mathfrak{g}$-module V a canonical commutative associative algebra $\mathcal{A}$ and a canonical deformation over A. A deformation (3.7) over $\mathcal{A}$ is said to be *miniversal* if

(i) for any other deformation ρ' with base (local) $\mathcal{A}'$, there exists a homomorphism $\psi:\mathcal{A}'\to\mathcal{A}$ satisfying $\psi(1)=1$, such that

$$\rho=(\psi\otimes\mathrm{Id})\circ\rho';$$

(ii) under notation of (i), if ρ is infinitesimal, then ψ is unique.

If ρ satisfies only the condition (i), then it is called versal. This definition does not depend on the choice 1-cocycles $\Upsilon_1,\dots,\Upsilon_m$ representing a basis of $\mathrm{H}^1(\mathfrak{g},\mathfrak{h};\mathrm{End}(V))$.

The miniversal deformation corresponds to the smallest ideal $\mathcal{R}$. We refer to [13] for a construction of miniversal deformations of Lie algebras and to [2] for miniversal deformations of $\mathfrak{g}$-modules. Superization of these results is immediate by the Sign Rule.

3.4. Integrability conditions. In this subsection, we obtain the integrability conditions for the infinitesimal deformation (3.5).

Theorem 3.1. (i) *The following conditions are necessary and sufficient for integrability of the infinitesimal deformation* (3.5)*:*

$$\left.\begin{array}{l} t^1_{\lambda,\lambda+k}t_{\lambda,\lambda}-t_{\lambda+k,\lambda+k}\,t^1_{\lambda,\lambda+k}=0\\ t^2_{\lambda,\lambda+k}t_{\lambda,\lambda}-t_{\lambda+k,\lambda+k}\,t^2_{\lambda,\lambda+k}=0\end{array}\right\} \quad \text{for } 2(\beta-\lambda)\in\{2k,\dots,2m\}. \tag{3.15}$$

(ii) *Any formal $\mathfrak{aff}(1|1)$-trivial deformation of the $\mathfrak{aff}(2|1)$-module $\mathfrak{S}^{2,m}_{\beta}$ is equivalent to a deformation of order* 1*, that is, to a deformation given by* (3.5).

The super-commutative algebra defined by relations (3.15) corresponds to the miniversal deformation of the Lie derivative $\mathfrak{L}_X$. Note that the super-commutative algebra defined in Theorem 3.1 is infinite-dimensional.

The proof of Theorem 3.1 consists in two steps. First, we compute explicitly the obstructions for existence of the second-order term, this will prove that relations (3.15) are necessary. Second we show that under relations (3.15) the highest-order terms of the deformation can be chosen identically zero, so that relations (3.15) are indeed sufficient.

Proof. Assume that the infinitesimal deformation (3.5) can be integrated to a formal deformation

$$\widetilde{\mathfrak{L}}_X = \mathfrak{L}_X + \mathfrak{L}_X^{(1)} + \mathfrak{L}_X^{(2)} + \cdots ,$$

where $\mathfrak{L}_X^{(1)}$ is given by (3.6) and $\mathfrak{L}_X^{(2)}$ is a quadratic polynomial in t with coefficients in $\mathfrak{S}_\beta^{2,m}$. Considering the homomorphism condition, we compute the second order term $\mathfrak{L}^{(2)}$ which is a solution of the Maurer-Cartan equation:

$$\partial(\mathfrak{L}^{(2)}) = -\frac{1}{2}[\![\mathfrak{L}^{(1)}, \mathfrak{L}^{(1)}]\!]. \tag{3.16}$$

For arbitrary λ, the right hand side of (3.16) yields the following $\mathfrak{aff}(1|1)$-relative 2-cocycles:

$$\begin{aligned} B_{\lambda,\lambda} &= [\![\Upsilon_{\lambda,\lambda}, \Upsilon_{\lambda,\lambda}]\!] : \mathfrak{aff}(2|1) \otimes \mathfrak{aff}(2|1) \longrightarrow \mathfrak{D}^2_{\lambda,\lambda}, \\ \widetilde{B}^i_{\lambda,\lambda+k} &= [\![\Gamma^i_{\lambda,\lambda+k}, \Upsilon_{\lambda,\lambda}]\!] : \mathfrak{aff}(2|1) \otimes \mathfrak{aff}(2|1) \longrightarrow \mathfrak{D}^2_{\lambda,\lambda+k}, \quad i \in \{1,2\}, \\ \overline{B}^i_{\lambda,\lambda+k} &= [\![\Upsilon_{\lambda+k,\lambda+k}, \Gamma^i_{\lambda,\lambda+k}]\!] : \mathfrak{aff}(2|1) \otimes \mathfrak{aff}(2|1) \longrightarrow \mathfrak{D}^2_{\lambda,\lambda+k}, \quad i \in \{1,2\}. \end{aligned}$$

By a straightforward computation, we check that

$$B_{\lambda,\lambda} = 0, \qquad \widetilde{B}^1_{\lambda,\lambda+k} = -\overline{B}^1_{\lambda,\lambda+k} \qquad \text{and} \qquad \widetilde{B}^2_{\lambda,\lambda+k} = -\overline{B}^2_{\lambda,\lambda+k},$$

with

$$\begin{aligned} \widetilde{B}^1_{\lambda,\lambda+k}(X_F, X_G) &= \big(\overline{\eta}_1\overline{\eta}_2(F)G' - F'\overline{\eta}_1\overline{\eta}_2(G)\big)\,\partial_x^k, \\ \widetilde{B}^2_{\lambda,\lambda+k}(X_F, X_G) &= \big(\overline{\eta}_1\overline{\eta}_2(F)G' - F'\overline{\eta}_1\overline{\eta}_2(G)\big)\,\overline{\eta}_1\overline{\eta}_2\partial_x^{k-1}. \end{aligned}$$

We will need the following:

Proposition 3.2. *Each of the bilinear map*

$$\widetilde{B}^1_{\lambda,\lambda+k},\ \widetilde{B}^2_{\lambda,\lambda+k}$$

define generically nontrivial cohomology class. Moreover, these cohomology classes are linearly independent.

Proof. Each map $\widetilde{B}^i_{\lambda,\lambda+k}, i = 1,2$, is a $\mathfrak{aff}(1|1)$-relative 2-cocycle on $\mathfrak{aff}(2|1)$ since it is the Kolmogorov-Alexander product of two $\mathfrak{aff}(1|1)$-relative 1-cocycles. Assume that, for some differential 1-cochain $b_{\lambda,\lambda+k}$ on $\mathfrak{aff}(2|1)$ with coefficients in $\mathfrak{D}^2_{\lambda,\lambda+k}$, we have $\widetilde{B}^i_{\lambda,\lambda+k} = \partial(b_{\lambda,\lambda+k})$. The general form of such a cochain is

$$b_{\lambda,\lambda+k}(X_F) = \sum a_{\ell_1\ell_2m_1m_2}(x,\theta_1,\theta_2)\eta_1^{\ell_1}\eta_2^{\ell_2}(F)\eta_1^{m_1}\eta_2^{m_2},$$

where the coefficients $a_{\ell_1\ell_2m_1m_2}(x,\theta_1,\theta_2)$ are arbitrary functions.

To complete the proof of the proposition we will need the following:

Lemma 3.3. *The condition $\widetilde{B}^i_{\lambda,\lambda+k} = \partial(b_{\lambda,\lambda+k})$ implies that the coefficients $a_{\ell_1\ell_2m_1m_2}(x,\theta_1,\theta_2)$ are functions of θ_i, not depending on x.*

Proof. The condition $\widetilde{B}^i_{\lambda,\lambda+k} = \partial(b_{\lambda,\lambda+k})$ reads

$$\begin{aligned}\widetilde{B}^i_{\lambda,\lambda+k}(X_F,X_G) = X_F\cdot b_{\lambda,\lambda+k}(X_G) - (-1)^{|G||F|}X_G\cdot b_{\lambda,\lambda+k}(X_F)\\ - b_{\lambda,\lambda+k}([X_F,\ X_G]).\end{aligned} \tag{3.17}$$

We choose a constant function $F = 1$, and prove that $X_1\cdot b_{\lambda,\lambda+k} = 0$. Indeed, we have

$$(X_1\cdot b_{\lambda,\lambda+k})(X_G) = X_1\cdot b_{\lambda,\lambda+k}(X_G) - b_{\lambda,\lambda+k}([X_1,\ X_G]).$$

Since $\ell_1+\ell_2 \geq 1$ in the expression of $b_{\lambda,\lambda+k}$, it follows that $b_{\lambda,\lambda+k}(X_1) = 0$, and thus the last equality gives

$$\begin{aligned}(X_1\cdot b_{\lambda,\lambda+k})(X_G) &= X_1\cdot b_{\lambda,\lambda+k}(X_G) - X_G\cdot b_{\lambda,\lambda+k}(X_1) - b_{\lambda,\lambda+k}([X_1,\ X_G])\\ &= \partial(b_{\lambda,\lambda+k})(X_1,X_G).\end{aligned}$$

By assumption, $\widetilde{B}^i_{\lambda,\lambda+k} = \partial(b_{\lambda,\lambda+k})$, from the explicit formula for $\widetilde{B}^i_{\lambda,\lambda+k}$, we obtain $\widetilde{B}^i_{\lambda,\lambda+k}(X_1,X_G) = 0$ for all $X_G \in \mathfrak{aff}(2|1)$. Therefore, $X_1\cdot b_{\lambda,\lambda+k} = 0$. Lemma 3.3 is proved. ∎

Now, by direct computation, we check that equation (3.17) has no solution. This is contradiction with the assumption, $\widetilde{B}^i_{\lambda,\lambda+k} = \partial(b_{\lambda,\lambda+k})$. Thus, the maps $\widetilde{B}^i_{\lambda,\lambda+k}$ are nontrivial $\mathfrak{aff}(1|1)$-relative 2-cocycles. Moreover, by a direct computation, we can check that, for some differential 1-cochain $b_{\lambda,\lambda+k}$ on $\mathfrak{aff}(2|1)$ with coefficients in $\mathfrak{D}^2_{\lambda,\lambda+k}$, the system

$$\left(\widetilde{B}^1_{\lambda,\lambda+k},\ \widetilde{B}^2_{\lambda,\lambda+k},\ \partial(b_{\lambda,\lambda+k})\right)$$

is linearly independent. Thus, the cohomology classes of $\widetilde{B}^i_{\lambda,\lambda+k}$ are linearly independent. This completes the proof of Proposition 3.2. ∎

Proposition 3.2 implies that equation (3.16) has solutions if and only if the quadratic polynomials given by (3.15) vanish simultaneously. We thus proved that conditions (3.15) are, indeed, necessary.

To prove that the conditions (3.15) are sufficient, we will find explicitly a deformation of $\mathfrak{L}_{X_F}$, whenever the conditions (3.15) are satisfied. The solution $\mathfrak{L}^{(2)}$ of (3.16) can be chosen identically zero. Choosing the hightest-order terms $\mathfrak{L}^{(m)}$ with $m \geq 3$, also identically zero, one obviously obtains a deformation which is of order 1 in t. Theorem 3.1, part (i) is proved.

The solution $\mathfrak{L}^{(2)}$ of (3.16) is defined up to a $\mathfrak{aff}(1|1)$-relative 1-cocycle and it has been shown in [13, 2] that different choices of solutions of the Maurer-Cartan equation correspond to equivalent deformations. Thus, we can always reduce $\mathfrak{L}^{(2)}$ to zero by equivalence. Then, by recurrence, the hightest-order terms $\mathfrak{L}^{(m)}$ satisfy the equation $\partial(\mathfrak{L}^{(m)}) = 0$ and can also be reduced to the identically zero map. This completes the proof of part (ii). ∎

EXAMPLE 3.1. For $m \in \frac{1}{2}\mathbb{N}$ and for arbitrary generic $\lambda \in \mathbb{R}$, the following example is a 1-parameter $\mathfrak{aff}(1|1)$-trivial deformation of the $\mathfrak{aff}(2|1)$-module $\mathfrak{S}^{2,m}_{\lambda+m}$

$$\widetilde{\mathfrak{L}}_{X_F} = \mathfrak{L}_{X_F} + t\sum_{\ell=0}^{2m}\sum_{k=1}^{[\frac{2m-\ell}{2}]}\sum_{i=1}^{2}\Gamma^i_{\lambda+\frac{\ell}{2},\lambda+\frac{\ell}{2}+k},$$

that is , we put $t^i_{\lambda+\frac{\ell}{2},\lambda+\frac{\ell}{2}+k} = t$ and $t_{\lambda,\lambda} = 0$.

Of course it is easy to give many other examples of true deformations with one parameter or with several parameters.

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