Automorphisms of classical geometries in the sense of Klein

Navarro, A.* Navarro, J.†

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Abstract

In this note, we compute the group of automorphisms of Projective, Affine and Euclidean Geometries in the sense of Klein.

As an application, we give a simple construction of the outer automorphism of S_6.

1 Introduction

Let $\mathbb{P}_n$ be the set of 1-dimensional subspaces of a $n+1$-dimensional vector space E over a (commutative) field k. This standard definition does not capture the "structure" of the projective space although it does point out its automorphisms: they are projectivizations of semilinear automorphisms of E, also known as *Staudt projectivities.*

A different approach (v. gr. [1]), defines the projective space as a lattice (the lattice of all linear subvarieties) satisfying certain axioms. Then, the so named Fundamental Theorem of Projective Geometry ([1], Thm 2.26) states that, when $n > 1$, collineations of $\mathbb{P}_n$ (i.e., bijections preserving alignment, which are the automorphisms of this lattice structure) are precisely Staudt projectivities.

In this note we are concerned with geometries in the sense of Klein: a *geometry* is a pair (X, G) where X is a set and G is a subgroup of the group $\mathrm{Biy}(X)$ of all bijections of X. In Klein's view, Projective Geometry is the pair $(\mathbb{P}_n, \mathrm{PGl}_n)$, where PGl_n is the group of projectivizations of k-linear automorphisms of the vector space E (see Example 2.2). The main result of this note is a computation, analogous to the aforementioned theorem for collineations, but in the realm of Klein geometries:

Theorem 3.4 *The group of automorphisms of the Projective Geometry* $(\mathbb{P}_n, \mathrm{PGl}_n)$ *is the group of Staudt projectivities, for any* $n \geq 1$.

Let us remark this statement covers the case $n = 1$ of the projective line, which usually requires a separated treatment ([1], [5]).

*ICMAT, Madrid, Spain

†Corresponding author. Department of Mathematics, University of Extremadura, Badajoz, Spain. *Email address:* navarrogarmendia@unex.es

Also, we compute in Theorems 3.10 and 3.14 the group of automorphisms for other classical geometries – namely Affine and Euclidean Geometries.

Finally, as an application, we use Theorem 3.4 to give a description of the outer automorphism of S_6, which is simple in comparison with other constructions existing in the literature (see [2, 3] or [4]).

2 Preliminaries

Let us firstly introduce the main definitions and examples that will be used later on:

Definition 2.1. A *geometry* in the sense of Klein is a pair (X, G) where X is a set and G is a subgroup of the group $\mathrm{Biy}(X)$ of all bijections $X \to X$.

The *concepts* of a geometry are those notions invariant with respect to the action of the structural group G.

Example 2.2. The examples that will appear in this note are:

- The *Projective Geometry* $(\mathbb{P}_n, \mathrm{PGl}_n)$, where $\mathbb{P}_n$ is the set of 1–dimensional vector subspaces of a k–vector space E of dimension $n+1$ and PGl_n is the group projectivizations of k–linear automorphisms of E with the obvious action on $\mathbb{P}_n$.
- The *Affine Geometry* $(\mathbb{A}_n, \mathrm{Aff}_n)$, where $\mathbb{A}_n$ is the complement of an hyperplane H in $\mathbb{P}_n$, called *hyperplane at infinity*, and Aff_n is the subgroup of PGl_n consisting of all projectivities $\varphi\colon \mathbb{P}_n \to \mathbb{P}_n$ preserving the hyperplane at infinity (i.e., such that $\varphi(H) = H$), with the action by restriction on $\mathbb{A}_n$.
- The *Euclidean Geometry* $(\mathbb{A}_n(\mathbb{R}), \mathrm{Mot}_n)$, where $\mathbb{A}_n(\mathbb{R})$ denotes the affine space over the real numbers, endowed with a positive definite, non-singular metric g on its vector space of directions, and Mot_n stands for the group of *motions*; i.e., the group of those affinities $\varphi\colon \mathbb{A}_n(\mathbb{R}) \to \mathbb{A}_n(\mathbb{R})$ whose tangent linear map $\vec{\varphi}$ satisfy $\vec{\varphi}_* g = g$.

Definition 2.3. An *isomorphism* between two geometries (X, G) and (X', G') is a bijection $f\colon X \to X'$ such that the map

$$\phi\colon \mathrm{Biy}(X) \to \mathrm{Biy}(X') \qquad , \qquad \phi(g) := \varphi \circ g \circ \varphi^{-1}$$

preserves the structural groups, $\phi(G) = G'$; that is to say, such that $\phi G \phi^{-1} = G'$.

An *automorphism* of a geometry (X, G) is an isomorphism of geometries $f\colon (X, G) \to (X, G)$. With the composition of maps, the automorphisms of a geometry (X, G) are a group.

In other words, the group of automorphisms of a geometry (X, G) is the normalizer of G inside the group $\mathrm{Biy}(X)$ of all bijections of X.

3 Automorphisms of classical geometries

3.1 Projective Geometry

Let $\mathbb{P}_n$ be the set of 1-dimensional subspaces of a $(n+1)$-dimensional k-vector space E and let:

$$\pi\colon E-\{0\} \longrightarrow \mathbb{P}_n \quad , \quad e \longmapsto \langle e\rangle \ ,$$

be the projectivization map. Recall that linear subvarieties of $\mathbb{P}_n$ (lines, planes,...) are defined as the projectivization of linear subspaces of E.

Definition 3.1. Three points $p_1, p_2, p_3 \in \mathbb{P}_n$ are *collinear* if there exists a line L passing through them.

A bijection $\varphi\colon \mathbb{P}_n \to \mathbb{P}_n$ is a *collineation* if it transforms lines into lines; that is, it maps collinear points into collinear points.

Definition 3.2. A *semilinear automorphism* $f\colon E \to E$ is a bijection such that there exists an automorphism of fields $h\colon k \to k$ satisfying

$$f(\lambda e + \mu v) = h(\lambda) f(e) + h(\mu) f(v) \quad , \quad \forall\ e, v \in E, \quad \lambda, \mu \in k \ .$$

A bijection $\Phi\colon \mathbb{P}_n \to \mathbb{P}_n$ is a *Staudt projectivity* if it is the projectivization of a semilinear automorphism.

Observe that elements in PGl_n are precisely the Staudt projectivities with associated automorphism of fields $h = \mathrm{Id}$.

For what follows, it will be useful to characterize collineations in terms of projectivities; to this end, let us consider the following sets:

$$\mathsf{P}_{p_1,p_2}(p_3) := \{\, p \in \mathbb{P}_n - \{p_3\} \colon\ \exists\ \varphi \in \mathrm{PGl}_n\ ,\ \varphi(p_1) = p_1,\ \varphi(p_2) = p_2,\ \varphi(p_3) = p \,\}\ .$$

Lemma 3.3. *Three different points $p_1, p_2, p_3 \in \mathbb{P}_n$ are collinear if and only if*

$$\mathsf{P}_{p_1,p_2}(p_3) = \mathsf{P}_{p_1,p_3}(p_2) = \mathsf{P}_{p_2,p_3}(p_1)\ .$$

Proof: If p_1, p_2, p_3 are collinear, then $\mathsf{P}_{p_1,p_2}(p_3)$ equals the complement of p_1, p_2, p_3 in the line passing through them.

Conversely, if p_1, p_2, p_3 are not collinear, the set $\mathsf{P}_{p_1,p_2}(p_3)$ is the complement of the line L joining p_1 and p_2 (apart from p_3). Then, any point $q \in L$, $q \neq p_1, p_2$ satisfies that $q \notin \mathsf{P}_{p_1,p_2}(p_3)$ and $q \in \mathsf{P}_{p_1,p_3}(p_2) \cap \mathsf{P}_{p_2,p_3}(p_1)$.

□

Theorem 3.4. *The group of automorphisms of the Projective Geometry* $(\mathbb{P}_n, \mathrm{PGl}_n)$ *is the group of Staudt projectivities, for any $n \geq 1$.*

That is to say, the group of Staudt projectivities is the normalizer of the group PGl_n *in the group of all bijections of* $\mathbb{P}_n$*.*

Proof: Let $\Phi\colon \mathbb{P}_n \to \mathbb{P}_n$ be a Staudt projectivity, with associated automorphism of fields $h\colon k \to k$.

If $\varphi \in \mathrm{PGl}_n$ then $\Phi\varphi\Phi^{-1}$ is a Staudt projectivity, with associated automorphism $h \circ \mathrm{Id} \circ h^{-1} = \mathrm{Id}$, so that $\Phi \mathrm{PGl}_n \Phi^{-1} \subseteq \mathrm{PGl}_n$. Since Φ^{-1} is also a Staudt projectivity, it also holds $\Phi^{-1}\mathrm{PGl}_n\Phi \subseteq \mathrm{PGl}_n$ and the reverse inclusion follows. We conclude that Φ belongs to the normalizer of PGl_n in the group $\mathrm{Biy}(\mathbb{P}_n)$.

Now, let us prove that any automorphism φ of $(\mathbb{P}_n, \mathrm{PGl}_n)$ is a Staudt projectivity.

$n \geq 2$. As we already mentioned, in this case Staudt projectivities are precisely collineations ([1], Thm 2.26), and hence it is enough to check that φ is a collineation. Since φ preserves elements in PGl_n by hypothesis, it easily follows that

$$\varphi(\mathsf{P}_{p_1,p_2}(p_3)) = \mathsf{P}_{\varphi(p_1),\varphi(p_2)}(\varphi(p_3)) ,$$

so Lemma 3.3 allows to deduce that $\varphi(p_1), \varphi(p_2), \varphi(p_3)$ are collinear whenever so are p_1, p_2, p_3.

$n = 1$. Let us fix a projective reference (p_0, p_∞, p_1) in $\mathbb{P}_1$, and write $p'_0 := \varphi(p_0),\ p'_1 := \varphi(p_1),\ p'_\infty := \varphi(p_\infty)$.

In the affine line $\mathbb{A}_1 = \mathbb{P}_1 - \{p_\infty\}$, consider the origin p_0 and the unit point p_1, thus inducing a bijection $\mathbb{A}_1 \simeq k$. Let p_λ denote the point corresponding to $\lambda \in k$ (and analogously in the affine line $\mathbb{A}'_1 = \mathbb{P}_1 - \{p'_\infty\}$).

The composition $k \simeq \mathbb{A}_1 \xrightarrow{\varphi} \mathbb{A}'_1 \simeq k$ defines a bijection $h\colon k \to k$ that is an automorphism of the field k (Lemma 3.5) and such that $\varphi(p_\lambda) = p'_{h(\lambda)}$.

Therefore, if (e_0, e_1) is a basis of E normalized to the reference (p_0, p_∞, p_1) and (e'_0, e'_1) is a basis normalized to the reference (p'_0, p'_∞, p'_1), then φ coincides with the projectivization of the semilinear map $f(\lambda e_0 + \mu e_1) = h(\lambda)\, e'_0 + h(\mu)\, e'_1$.

□

Lemma 3.5. *The map $h\colon k \to k$ defined in the proof above is an automorphism of fields.*

Proof: By definition, $h(0) = 0$ and $h(1) = 1$, so we only have to show that h is compatible with additions and products.

Let τ_μ and σ_μ denote the translation by $\mu \in k$ in $\mathbb{A}_1$ and the homothety with center at p_0 and ratio μ, respectively (and analogously, τ'_μ and σ'_μ in $\mathbb{A}'_1$).

Let us first check the equality $\varphi\tau_\mu\varphi^{-1} = \tau'_{h(\mu)}$: since p_∞ is the only point that is fixed by τ_μ, its image p'_∞ is the unique fixed point of $\varphi\tau_\mu\varphi^{-1}$; hence this composition is a translation in $\mathbb{A}'_1$ (it is an homography by hypothesis), and it suffices to see that it transforms p'_0 into $p'_{h(\mu)}$:

$$\varphi\tau_\mu\varphi^{-1}(p'_0) = \varphi\tau_\mu(p_0) = \varphi(p_\mu) = p'_{h(\mu)}.$$

Using this equality, $\varphi\tau_\mu = \tau'_{h(\mu)}\varphi$, and applying it to p_λ:

$$\varphi\tau_\mu(p_\lambda) = \varphi(p_{\lambda+\mu}) = p'_{h(\lambda+\mu)}$$
$$\tau'_{h(\mu)}\varphi(p_\lambda) = \tau'_{h(\mu)}(p'_{h(\lambda)}) = p'_{h(\lambda)+h(\mu)}$$

it follows that $h(\lambda + \mu) = h(\lambda) + h(\mu)$.

In a similar way, $\varphi\sigma_\mu\varphi^{-1}$ can be proved to be the homothety $\sigma'_{h(\mu)}$: since p_0 and p_∞ are fixed points of σ_μ, both p'_0 and p'_∞ are fixed points of $\varphi\sigma_\mu\varphi^{-1}$, and hence this composition is a homothety in $\mathbb{A}'_1$ with center at p'_0 that satisfies:

$$\varphi\sigma_\mu\varphi^{-1}(p'_1) = \varphi\sigma_\mu(p_1) = \varphi(p_\mu) = p'_{h(\mu)} \ .$$

We conclude $h(\lambda\mu) = h(\lambda)h(\mu)$ by using the equality $\varphi\sigma_\mu = \sigma'_{h(\mu)}\varphi$:

$$\varphi\sigma_\mu(p_\lambda) = \varphi(p_{\lambda\mu}) = p'_{h(\lambda\mu)}$$
$$\sigma'_{h(\mu)}\varphi(p_\lambda) = \sigma'_{h(\mu)}(p'_{h(\lambda)}) = p'_{h(\lambda)h(\mu)} \ .$$

□

The outer automorphism of S_6

As an application of Theorem 3.4, let us construct an automorphism of S_6 which is not inner; i.e., which does not coincide with conjugation by an element of S_6.

Let k be the field with 5 elements. The projective line over k has 6 elements, so that there are $6 \cdot 5 \cdot 4$ homographies and PGl_1 is a subgroup of S_6 of index 6. Since the identity is the unique automorphism of the field $k = \mathbb{Z}/5\mathbb{Z}$, Theorem 3.4 states that PGl_1 coincides with its normalizer in S_6; hence it has 6 conjugated subgroups $H_1 = \mathrm{PGl}_1, H_2, \ldots, H_6$.

Any permutation $\tau \in S_6$ of the projective line defines, by conjugation, a permutation $F(\tau)$ of this set $\{H_1, \ldots, H_6\}$. Thus, we obtain an automorphism

$$S_6 = \mathrm{Biy}(\mathbb{P}_1) \xrightarrow{\ F\ } \mathrm{Perm}(\{H_1, \ldots, H_6\}) = S_6$$

such that $F(\mathrm{PGl}_1)$ is contained in the stabilizer of H_1; i.e., in the subgroup of all permutations fixing the element H_1 (in fact they coincide, since both subgroups have index 6).

As the image of a stabilizer under an inner automorphism is the stabilizer of another point, we conclude that F cannot be inner, since no point of the projective line is fixed by the group of all homographies PGl_1.

3.2 Affine Geometry

Let $\mathbb{A}_n$ be the complementary of an hyperplane H on $\mathbb{P}_n$, and let Aff_n be the group of affinites; i.e., the group of projectivities $\varphi\colon \mathbb{P}_n \to \mathbb{P}_n$ such that $\varphi(H) = H$.

Linear subvarieties of $\mathbb{A}_n$ (affine lines, affine planes,...) are defined as the restriction of affine subvarieties on $\mathbb{P}_n$.

Lemma 3.6. *Let $\varphi\colon \mathbb{A}_n \to \mathbb{A}_n$ be a collineation of an affine space of dimension $n > 1$ over a field $k \neq \mathbb{F}_2$.*

If $\Pi \subset \mathbb{A}_n$ is an affine plane, then $\varphi(\Pi)$ is contained in some affine plane.

Proof: Let L_1, L_2 be two lines in Π intersecting at a point z. Their images $\varphi(L_1)$ and $\varphi(L_2)$ are lines with one point in common, so both lie in some affine plane Π'.

For any point $p \in \Pi - (L_1 \cup L_2)$, let $L_p \subset \Pi$ be any line passing through p, not parallel neither to L_1 nor to L_2, and such that $z \notin L_p$. As L_p intersects both L_1 and L_2, it follows that $\varphi(p) \in \varphi(L_p) \subset \Pi'$.

□

If the base field has 2 elements, then there not exist 3 *different* affine collinear points; in that case all bijections of $\mathbb{A}_n$ are in fact collineations.

Definition 3.7. A Staudt projectivity $\Phi\colon \mathbb{P}_n \to \mathbb{P}_n$ is called a *Staudt affinity* if it preserves the hyperplane at infinity; i.e., if $\Phi(H) = H$.

Lemma 3.8. *Let $\varphi\colon \mathbb{A}_n \to \mathbb{A}_n$ be a collineation of an affine space of dimension $n > 1$ over a field $k \neq \mathbb{F}_2$.*

There exists a unique Staudt affinity $\Phi\colon \mathbb{P}_n \to \mathbb{P}_n$ such that $\varphi = \Phi_{|\mathbb{A}_n}$.

Proof: We first define Φ on the points at infinity: if $p \in H$, let $\Phi(p)$ be the point at infinity of $\varphi(L_p)$, where L_p is any line passing through p. If L'_p is another line parallel to L_p, Lemma 3.6 shows that $\varphi(L_p)$ and $\varphi(L'_p)$ are contained in some affine plane; since they do not intersect, they are parallel and both have the same point at infinity $\Phi(p)$.

Then, it is enough to check that Φ is a collineation on the hyperplane at infinity (for collineations correspond with Staudt projectivities, [1] Thm 2.26): if three points p_1, p_2, p_3 of the infinity are collinear, they lie in the direction of some affine plane Π. By Lemma 3.6, $\varphi(\Pi)$ is contained in some affine plane Π', so that $\Phi(p_1), \Phi(p_2), \Phi(p_3) \in H$ are in the direction of Π' and are thus collinear.

□

Let us now characterize collinear points in terms of affinities. Consider:

$$\mathsf{A}_{p_1,p_2}(p_3) := \{p \in \mathbb{A}_n - \{p_3\} \colon\ \exists\, \varphi \in \mathrm{Aff}_n \ \ \varphi(p_1) = p_1,\ \varphi(p_2) = p_2,\ \varphi(p_3) = p\ \}\ .$$

Since any affinity fixing p_1 and p_2 also fixes the point at infinity of the line passing through them, it has to be the identity on such line. Hence,

Lemma 3.9. *Three different points $p_1, p_2, p_3 \in \mathbb{A}_n$ are collinear if and only if*

$$\mathsf{A}_{p_1,p_2}(p_3) = \mathsf{A}_{p_1,p_3}(p_2) = \mathsf{A}_{p_2,p_3}(p_1) = \emptyset\ .$$

Theorem 3.10. *The group of automorphisms of the Affine Geometry $(\mathbb{A}_n, \mathrm{Aff}_n)$ over a field $k \neq \mathbb{F}_2$ is the group of Staudt affinities, for any $n \geq 1$.*

Proof: On the one hand, it is trivial to check that Staudt affinities are indeed automorphisms, arguing as in the proof of Theorem 3.4.

On the other hand, let us prove that any automorphism of $(\mathbb{A}_n, \mathrm{Aff}_n)$ is indeed a Staudt affinity:

$n \geq 2$. If $\varphi \in \mathrm{Biy}(\mathbb{A}_n)$ is in the normalizer of Aff_n, it preserves affinities and so $\varphi(\mathsf{A}_{p_1,p_2}(p_3)) = \mathsf{A}_{\varphi(p_1),\varphi(p_2)}(\varphi(p_3))$. By Lemma 3.9, it follows that φ is a collineation of $\mathbb{A}_n$, and hence it coincides with the restriction to $\mathbb{A}_n$ of a unique Staudt affinity (Lemma 3.8).

$n = 1$. Let $\varphi \in \mathrm{Biy}(\mathbb{A}_1)$ be in the normalizer of Aff_1, and fix a projective reference (p_0, p_∞, p_1) such that $\mathbb{A}_1 = \mathbb{P}_1 - \{p_\infty\}$. Via the induced bijection $\mathbb{A}_1 \simeq k$, let p_λ denote the point corresponding to $\lambda \in k$. Consider $p'_0 = \varphi(p_0)$ and $p'_1 = \varphi(p_1)$ as another origin and unit point, and let p'_λ denote the point with coordinate $\lambda \in k$ via the corresponding bijection $\mathbb{A}_1 \simeq k$.

The composition $k \simeq \mathbb{A}_1 \xrightarrow{\varphi} \mathbb{A}_1 \simeq k$ defines a bijection $h\colon k \to k$ such that

$$\varphi(p_\lambda) = p'_{h(\lambda)}\ .$$

A similar proof to that of Lemma 3.5 shows that h is an automorphism of the field k.

Then, it is easy to check that the bijection $\Phi\colon \mathbb{P}_1 \to \mathbb{P}_1$ defined as $\Phi|_{\mathbb{A}_1} := \varphi$, $\Phi(p_\infty) := p_\infty$, is a Staudt afinity (with associated automorphism h).

□

3.3 Euclidean Geometry

Let $\mathbb{A}_n(\mathbb{R})$ be the affine space over the real numbers, endowed with a positive definite, non-singular metric g on its vector space of directions. Let Mot_n denote the group of motions; i.e., those affinities $\varphi\colon \mathbb{A}_n(\mathbb{R}) \to \mathbb{A}_n(\mathbb{R})$ whose tangent linear map $\vec{\varphi}$ preserves the metric g.

Lemma 3.11. *If an affinity $\varphi\colon \mathbb{A}_n(\mathbb{R}) \to \mathbb{A}_n(\mathbb{R})$ preserves motions (i.e., $\varphi \mathrm{Mot}_n \varphi^{-1} = \mathrm{Mot}_n$), then it maps perpendicular lines into perpendicular lines.*

Proof: Assume there exists perpendicular lines L_1, L_2 such that $\varphi(L_1)$ and $\varphi(L_2)$ are not perpendicular.

Let H_1 be the hyperplane perpendicular to L_2 passing through L_1. The symmetry σ with respect to H_1 is a motion such that $\varphi\sigma\varphi^{-1}$ is not a motion, for this composition (which is not the identity), fixes the hyperplane $\varphi(H_1)$ and preserves the oblique line $\varphi(L_2)$.

□

Definition 3.12. An affinity $\varphi\colon \mathbb{A}_n(\mathbb{R}) \to \mathbb{A}_n(\mathbb{R})$ is a *similarity* if there exists $\lambda \in \mathbb{R}$ such that $\vec{\varphi}_* g = \lambda^2 g$.

Observe that motions are a particular case of similarities, for which $\lambda = 1$.

The Fundamental Theorem of Euclidean Geometry ([1]) characterizes similarities as those affinities $\varphi\colon \mathbb{A}_n(\mathbb{R}) \to \mathbb{A}_n(\mathbb{R})$ that map perpendicular lines into perpendicular lines.

Analogously to what is made in previous sections, collinear points may be characterized in terms of motions, considering the sets:

$$\mathsf{M}_{p_1,p_2}(p_3) := \{p \in \mathbb{A}_n - \{p_3\}\colon\ \exists\, \varphi \in \mathrm{Mot}_n\ ,\ \varphi(p_1) = p_1,\ \varphi(p_2) = p_2,\ \varphi(p_3) = p\ \}\ .$$

Lemma 3.13. *Three different points $p_1, p_2, p_3 \in \mathbb{A}_n(\mathbb{R})$ are collinear if and only if*

$$\mathsf{M}_{p_1,p_2}(p_3) = \mathsf{M}_{p_1,p_3}(p_2) = \mathsf{M}_{p_2,p_3}(p_1) = \emptyset$$

Theorem 3.14. *The group of automorphisms of the Euclidean Geometry $(\mathbb{A}_n(\mathbb{R}), \mathrm{Mot}_n)$ is the group of similarities, for any $n > 1$.*

Proof: If $\varphi\colon \mathbb{A}_n(\mathbb{R}) \to \mathbb{A}_n(\mathbb{R})$ is a similarity, it is routine to check that, for any motion $\tau \in \mathrm{Mot}_n$, the composition $\varphi\tau\varphi^{-1}$ is also a motion.

On the other hand, if $\varphi \in \mathrm{Biy}(\mathbb{A}_n(\mathbb{R}))$ is a bijection that preserves motions, then $\varphi(\mathsf{M}_{p_1,p_2}(p_3)) = \mathsf{M}_{\varphi(p_1),\varphi(p_2)}(\varphi(p_3))$ and, using Lemma 3.13, it follows that φ is a collineation of $\mathbb{A}_n(\mathbb{R})$.

As the identity is the unique automorphism of the field $\mathbb{R}$, Lemma 3.8 then assures that φ is an affinity.

Finally, as φ is an affinity that preserves motions, it also preserves perpendicular lines (Lemma 3.11) and we can assure that φ is indeed a similarity.

□

The above theorem is false when $n = 1$: a counterexample is any bijection $\varphi\colon \mathbb{R} \to \mathbb{R}$ that respects the addition, v. gr., any $\mathbb{Q}$-linear automorphism φ of $\mathbb{R}$: if $\phi \in \mathrm{Mot}_1$, then $\phi(x) = \pm x + b$ and it follows that $(\varphi^{-1}\phi\varphi)(x) = \varphi^{-1}(\pm\varphi(x) + b) = \pm x + \varphi^{-1}(b)$, so that φ belongs to the normalizer of Mot_1.

However, it is not difficult to show that any *continuous* automorphism φ of the euclidean line $(\mathbb{A}_1(\mathbb{R}), \mathrm{Mot}_1)$ is a similarity: $\varphi(x) = ax + b$; that is, the group of similarities is the normalizer of Mot_1 in the group of all homeomorphisms of $\mathbb{A}_1(\mathbb{R})$.

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