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On the Completion of (LF)-Spaces

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Abstract. Once the existence of metrizable (LF)-spaces was discovered, the problem whether the completion of an (LF)-space is or is not an (LF)-space was answered in the negative, because no (LF)-space can be a Fréchet space. However, some (non-metrizable) (LF)-spaces are complete, e.g. the classical Köthe's strict (LF)-spaces. In this paper we will carry out a thorough study of the completeness of (LF)-spaces stressing upon the stable completion properties of (LB)-spaces. A basic tool for handling this problem is an Open Mapping Theorem for completions of (LF)-spaces, which is also proved in the present paper.

1. Introduction and Preliminary Results. In [8], Saxon and Narayanaswami partition the class of all the (LF)-spaces into three mutually disjoint, non-empty classes, denoted $(\mathrm{LF})_i$-spaces or (LF)-spaces of type i ($i = 1, 2, 3$) (see definitions below). $(\mathrm{LF})_1$-spaces include the well known class of strict (LF)-spaces, while $(\mathrm{LF})_3$-spaces are those (LF)-spaces which are metrizable (the first example of a metrizable (LF)-space is due to A. Grothendieck [2]; these spaces were later on studied in [5], [7] and [8]).

Certain completeness properties of (LF)-spaces depend upon the type of the space. For example, no $(\mathrm{LF})_3$-space is complete, because no (LF)-space can be Fréchet, while many $(\mathrm{LF})_1$-spaces are complete, for example all strict (LF)-spaces. We will prove that (LB)-spaces exhibit a fair behaviour with respect to the completion, while the remaining (LF)-spaces are, in general, not stable for complete hulls.

For the general theory of (LF)-spaces, we follow throughout [3], § 19. We recall that a Hausdorff locally convex space (E, τ) is an

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(LF)-space if there exists a strictly increasing sequence $\{(E_n, \tau_n)\}$ of Fréchet spaces, called a defining sequence for E, such that $E = \bigcup E_n$ and $\tau_{n+1}|_{E_n} \leqslant \tau_n$ for every $n \in \mathbb{N}$ and τ is the finest Hausdorff locally convex topology on E such that $\tau|_{E_n} \leqslant \tau_n$ for all $n \in \mathbb{N}$. We write $(E, \tau) = \lim (E_n, \tau_n)$. If each space E_n is a Banach space, the (LF)-space E is called an (LB)-space. If $\tau_{n+1}|_{E_n} = \tau_n$ for every $n \in \mathbb{N}$, the (LF)- or (LB)-space is said to be strict.

1.1. *Definition.* An (LF)-space E is said to be of type i or an $(\mathrm{LF})_i$-space ($i = 1, 2, 3$), if it satisfies the following condition (i):

(1) E has a defining sequence $\{E_n\}$ such that no E_n is dense in E.

(2) E is not metrizable and it has a defining sequence $\{E_n\}$ such that some E_n is dense in E.

(3) E is metrizable.

The disjointness of the three classes is readily seen: Since any two defining sequences for an (LF)-space are equivalent we have $(\mathrm{LF})_1 \cap (\mathrm{LF})_2 = \emptyset$; $(\mathrm{LF})_2 \cap (\mathrm{LF})_3 = \emptyset$ is obvious; $(\mathrm{LF})_1 \cap (\mathrm{LF})_3 = \emptyset$ follows from the fact that every metrizable, barrelled space is Baire like (cf. [6] Corollary 2.5.).

In the sequel, φ will denote a countable-dimensional linear space endowed with the finest locally convex topology.

2. Open Mapping Theorem for (LF)-Spaces. In this Section we will make use of the classical Pták's Open Mapping Theorem to deduce a similar theorem for completions of (LF)-spaces. The following lemma is obvious.

2.1. Lemma. *If (E_1, Γ_1) and (E_2, Γ_2) are Fréchet spaces continuously included in a Hausdorff topological space, then the locally convex topology Γ on $E_1 \cap E_2$ with neighborhood basis $\{U \cap V; U \in \Gamma_1, V \in \Gamma_2\}$ is Fréchet.*

2.2. Theorem. *Let Ψ be an* (LF) *topology on the completion $(\tilde{E}, \tilde{\tau})$ of the* (LF)*-space (E, τ), such that the identity map $I: (\tilde{E}, \Psi) \to (\tilde{E}, \tilde{\tau})$ is continuous. Then I is a topological isomorphism.*

Proof. Let us assume that $(E, \tau) = \lim (E_n, \tau_n)$ and $(\tilde{E}, \Psi) = \lim (F_j, \Psi_j)$ for Fréchet spaces E_n, F_j $(n, j \in \mathbb{N})$. Fix $n \in \mathbb{N}$. Since $E_n \subset \tilde{E}$, $E_n = \bigcup_j (E_n \cap F_j)$ and, therefore, there exists $j \in \mathbb{N}$ such that $H =$

$= E_n \cap F_j$ is a dense and barrelled subspace of (E_n, τ_n) (let us note that E_n, being Fréchet, is a (db)-space, see [4]). By hypothesis, E_n and F_j are continuously included in the (Hausdorff) space $(\tilde{E}, \tilde{\tau})$. By means of the Lemma 2.1., H is endowed with a Fréchet topology Γ. Since $\Gamma \geqslant \tau_n|_H$, Pták's Open Mapping Theorem applied to the identity map $(H, \Gamma) \to (H, \tau_n|_H)$ yields $\Gamma = \tau_n|_H$. We deduce that H is closed in (E_n, τ_n), that is $H = E_n \subset F_j$. We also deduce that the inclusion map $(E_n, \tau_n) \hookrightarrow (F_j, \Psi_j)$ is continuous and, thus, $(E_n, \tau_n) \hookrightarrow (\tilde{E}, \Psi)$ is continuous as well. Since $n \in \mathbb{N}$ was arbitrarily fixed, we conclude that the canonical immersion $(E, \tau) \hookrightarrow (\tilde{E}, \Psi)$ is continuous. Extending by continuity to the completions, we prove the continuity of the identity $(\tilde{E}, \tilde{\tau}) \to (\tilde{E}, \Psi)$.

3. Completion of (LF)-Spaces. While it is true that the completion of an $(LF)_3$-space is never an (LF)-space (because it is a Fréchet space), the completion of an $(LF)_1$-space or an $(LF)_2$-space can be an (LF)-space. As a matter of fact, any strict (LF)-space is a complete $(LF)_1$-space ([3], 19.5.3.) and the $(LB)_2$-spaces $l_{p^-} = \lim l_{p_n}$ $(1 < p_n < p_{n+1} < p$, $\lim p_n = p)$ defined by Saxon and Narayanaswami in [7] are complete as well. However, we will show in the next examples that the completion of an $(LF)_i$-space, $i = 1, 2$, is not, in general, an (LF)-space.

3.1. *Example.* Let $E = \lim E_n$ be a complete $(LF)_2$-space. For each $n \in \mathbb{N}$ we define $F_n = E_n \times E_n \times \ldots$ endowed with the (Fréchet) product topology denoted by Ψ_n. The (LF)-space $(F, \Psi) = \lim (F_n, \Psi_n)$ is a dense, proper subspace of $E \times E \times \ldots$ because of its "diagonal" construction. Using the fact that E is of type 2 it is easy to prove that Ψ is the relative topology induced in F by the product topology of $E \times E \times \ldots$ We first observe that (F, Ψ) is not metrizable, because E is not metrizable. Since E is of type 2, some F_n is dense in $E \times E \times \ldots$ and hence also in F. So (F, Ψ) is an $(LF)_2$-space and its completion is $E \times E \times \ldots$ which is not an (LF)-space because no infinite product of (LF)-spaces is an (LF)-space.

3.2. *Example* (P. Pérez-Carreras) Let F be the non-complete $(LF)_2$-space described in Example 3.1. Then obviously $F \times \varphi$ is an $(LF)_1$-space. The completion $\tilde{F} \times \varphi$ of $F \times \varphi$ is not an (LF)-space because it has a quotient $\tilde{F}$ which is neither an (LF)-space nor a Fréchet space.

We will now study separately the completion of $(\mathrm{LB})_1$-spaces and $(\mathrm{LB})_2$-spaces (note that (LB)-spaces of type 3 do not exist because no (LB)-space is metrizable). The following lemma will be needed:

3.3. Lemma. *Let E be a normed space with closed unit ball B and let F be a Hausdorff complete locally convex space such that E is continuously included in F. If $\tilde{B}$ is the closure of B in F, the linear span $F_0 = \mathrm{sp}(\tilde{B})$ is a Banach space under the norm topology for which $\tilde{B}$ is the closed unit ball.*

Proof. Since F is complete and the canonical injection $E \hookrightarrow F$ is continuous, $\tilde{B}$ is a complete bounded disk in F. Therefore the norm gauge of $\tilde{B}$ defines on F_0 a Banach topology ([3] 18.4.4.) with closed unit ball equal to $\tilde{B}$.

If $\tilde{E}$ is the completion of E and $H \subset E$, we will denote in the sequel by $\bar{H}$ and $\tilde{H}$ the closures of H in E and in $\tilde{E}$ respectively.

3.4. Theorem. *The completion of an* $(\mathrm{LB})_1$*-space is an* $(\mathrm{LB})_1$*-space.*

Proof. For each $n \in \mathbb{N}$, let (E_n, τ_n) be a Banach space with closed unit ball B_n, such that $(E, \tau) = \lim (E_n, \tau_n)$ is an $(\mathrm{LB})_1$-space. Let $(\tilde{E}, \tilde{\tau})$ be the completion of (E, τ). We can assume, without loss of generality, that $B_n \subset B_{n+1}$ for every $n \in \mathbb{N}$ (change, if necessary, B_n by a suitable multiple of B_n). By Lemma 3.3., $F_n = \mathrm{sp}(\tilde{B}_n)$ becomes a Banach space for a topology Ψ_n for which $\tilde{B}_n$ is the closed unit ball. Furthermore, for each $n \in \mathbb{N}$

$$\tilde{\tau}|_{F_n} \leqslant \Psi_n \,. \tag{1}$$

Indeed, if V is a closed 0-neighborhood in $\tilde{\tau}$, $V \cap E_n \in \tau_n$ and, consequently, $\lambda B_n \subset V \cap E_n$ for some $\lambda > 0$. Since $F_n \subset \tilde{E}_n$ we deduce $\lambda \tilde{B}_n \subset V \cap F_n$ and (1) is proved.

Let $F = \bigcup_n F_n$. If for some $m \in \mathbb{N}$ $F_m = F$, then $\tilde{E}_m \supset F_m \supset E$ and $\bar{E}_m = E \cap \tilde{E}_m = E$ contradicting that E is of type 1. Therefore, a strictly increasing subsequence $\{F_{n_j}\}_{j \in \mathbb{N}} \subset \{F_n\}_{n \in \mathbb{N}}$ can be chosen such that $F = \bigcup_j F_{n_j}$. We rename this subsequence as $\{F_n\}$ again and define the (LB)-space $(F, \Psi) = \lim (F_n, \Psi_n)$ which is (algebraically) a linear subspace of $\tilde{E}$. The absolutely convex hull $U = \Gamma_n B_n = \bigcup_n B_n$ is a neighborhood of 0 in (E, τ), and, in particular, is absorbing in E. By [1], Theorem 2, for every $\varepsilon > 0$

$$\tilde{U} \subset (1+\varepsilon) \bigcup_{n \in \mathbb{N}} \tilde{B}_n \subset (1+\varepsilon) \bigcup_{n \in \mathbb{N}} F_n = F \,. \tag{2}$$

Since $\tilde{U}$ is a $\tilde{\tau}$-neighborhood of 0 in $\tilde{E}$, we deduce that $\tilde{E} = \mathrm{sp}(\tilde{U}) \subset F$ and, hence, $\tilde{E} = F$.

By (1) the inclusion map $(F_n, \Psi_n) \hookrightarrow (\tilde{E}, \tilde{\tau})$ is continuous for every $n \in \mathbb{N}$. Thus, the identity map $(F, \Psi) \to (\tilde{E}, \tilde{\tau})$ is continuous and by Theorem 2.2. it is a topological isomorphism. We have, hence, proved that $(\tilde{E}, \tilde{\tau})$ is an (LB)-space. If for some $n \in \mathbb{N}$ $\tilde{F}_n = \tilde{E}$, one easily gets $\bar{E}_n = E$ which is impossible because E is of type 1. So $(\tilde{E}, \tilde{\tau})$ is of type 1.

3.5. Theorem. *The completion of an* $(\mathrm{LB})_2$*-space is either an* $(\mathrm{LB})_2$*-space or a Banach space.*

Proof. We keep all the notations of the Theorem 3.4. but that (E, τ) is now of type 2. If for some $m \in \mathbb{N}$ $F_m = F$, then the relation (2) yields now $\tilde{U} \subset F_m$. $\tilde{U}$ being a $\tilde{\tau}$-neighborhood of 0 in $\tilde{E}$, we deduce that $\tilde{E} = F_m$ and the Open Mapping Theorem applies to conclude $\tilde{\tau} = \Psi_m$, that is $(\tilde{E}, \tilde{\tau})$ is a Banach space. Otherwise, there exists a strictly increasing subsequence $\{F_{n_j}\}_{j \in \mathbb{N}} \subset \{F_n\}_{n \in \mathbb{N}}$ such that $F = \bigcup_j F_{n_j}$. We then proceed as in the proof of the Theorem 3.4. to conclude that $(\tilde{E}, \tilde{\tau})$ is an (LB)-space. Since some E_n is dense in E, some F_n is dense in $\tilde{E}$ and $(\tilde{E}, \tilde{\tau})$ is of type 2.

3.6. *Remark.* The Theorems 3.4. and 3.5. have nontrivial applications because non-complete $(\mathrm{LB})_1$- and $(\mathrm{LB})_2$-spaces do exist. Indeed, the (LB)-space E of [3] 31.6. is an example of a non-complete $(\mathrm{LB})_2$-space. The topological product of this space E and φ is an $(\mathrm{LB})_1$-space that is not complete, because the closed subspace E of $E \times \varphi$ is not complete. Since every strict (LF)-space is complete, $E \times \varphi$ supplies as well an example of a non-strict $(\mathrm{LB})_1$-space.

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