ALGORITHMS FOR BALL HULLS AND BALL INTERSECTIONS IN NORMED PLANES

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Abstract. Extending results of Hershberger and Suri for the Euclidean plane, we show that ball hulls and ball intersections of sets of \( n \) points in normed planes can be constructed in \( O(n \log n) \) time. In addition, we confirm that the 2-center problem with constrained circles for arbitrary normed planes can be solved in \( O(n^2) \) time. Some ideas about the geometric structure of the ball hull in a normed plane are also presented.

1 Introduction

The ball hull and the ball intersection of a given point set \( K \) are common notions in Banach space theory; see, e.g., [5], [15], and [19]. They denote intersections of congruent balls with a fixed radius which, in the first case, contain \( K \) and, in the second one, have their centers in \( K \). For \( d = 2 \) and \( K \) a finite set, the boundary structure of \( \text{bi}(K, \lambda) \) consists of circular arcs of radius \( \lambda \) with centers belonging to \( K \). Extending previous results (see [15] and [16]), we describe the boundary structure of \( \text{bh}(K, \lambda) \). Continuing algorithmical investigations of Hershberger and Suri (for the Euclidean subcase; see [12]), we present algorithmical approaches to ball hulls and ball intersections of finite point sets \( K \) in arbitrary normed planes. Note that, although presenting only planar results, we stay with the notions of ball hull and ball intersection (instead of circular hull and circular intersection), since they are common in this form. In other situations, we replace “ball” and “sphere” by disc and circle, respectively. More precisely, we show that if \( K \) consists of \( n \) points, then the ball hull and the ball intersection of \( K \) can be constructed in \( O(n \log n) \) time. We also discuss a further geometric question. The 2-center problem asks for two closed discs to cover \( K \), and, if the centers of the discs are constrained to be points of \( K \), we discuss the so-called discrete 2-center problem or the 2-center problem with constrained circles (see [1], [2], [3], [10], [11], [13], and [18]). Nontrivial subquadratic algorithms for the Euclidean plane were obtained in [7], [8], and [18]. And other subcases of general normed planes, for instance \( L_1 \) and \( L_\infty \), have been studied in [6], [12], and [14]. Again generalizing results from [12], we show that the 2-center problem with centers in \( K \) and fixed radii can be solved in \( O(n^2) \) time, also if we extend it to arbitrary normed planes.

Let \( M^d = (\mathbb{R}^d, \| \cdot \|) \) be a \( d \)-dimensional normed (or Minkowski) space. As well-known, the unit ball \( B \) of \( M^d \) is a compact, convex set with non-empty interior (i.e., a convex body) centered at the origin \( o \). The boundary of a closed set \( A \) is denoted by \( \partial A \).
and \( \partial B \) is the unit sphere of \( \mathbb{M}^d \). Any homothetical copy \( x + \lambda B \) of \( B \) is called the ball with center \( x \in \mathbb{R}^d \) and radius \( \lambda > 0 \) and denoted by \( B(x, \lambda) \); its boundary is the sphere \( S(x, \lambda) \).

We use the usual abbreviation \( \text{conv} \) for convex hull, and the line segment connecting the different points \( p \) and \( q \) is denoted by \( \overrightarrow{pq} \), its affine hull is the line \( \langle p, q \rangle \). The vector \( p - q \) is denoted by \( -\overrightarrow{qp} \).

Let \( p \) and \( q \) be two points of the circle \( S(x, \lambda) \) in \( \mathbb{M}^2 \). The minimal circular arc of \( B(x, \lambda) \) meeting \( p \) and \( q \) is the connected piece of \( S(x, \lambda) \) with endpoints \( p \) and \( q \) which lies in the half-plane bounded by the line \( \langle p, q \rangle \) and does not contain the center \( x \). If \( p \) and \( q \) are opposite in \( S(x, \lambda) \), then both half-circles with endpoints \( p \) and \( q \) are minimal circular arcs of \( S(x, \lambda) \) meeting \( p \) and \( q \). We denote a minimal circular arc meeting \( p \) and \( q \) by \( \hat{pq} \).

Given a set \( K \) of points in \( \mathbb{M}^2 \) and \( \lambda > 0 \), the \( \lambda \)-ball hull \( \text{bh}(K, \lambda) \) of \( K \) is defined as the intersection of all balls of radius \( \lambda \) that contain \( K \):

\[
\text{bh}(K, \lambda) = \bigcap_{K \subseteq B(x, \lambda)} B(x, \lambda).
\]

The \( \lambda \)-ball intersection \( \text{bi}(K, \lambda) \) of \( K \) is the intersection of all balls of radius \( \lambda \) whose centers are from \( K \):

\[
\text{bi}(K, \lambda) = \bigcap_{x \in K} B(x, \lambda).
\]

Of course, these notions make only sense if \( \text{bi}(K, \lambda) \neq \emptyset \) and \( \text{bh}(K, \lambda) \neq \emptyset \). It is clear that \( \text{bh}(K, \lambda) \neq \emptyset \) if and only if \( \lambda \geq \lambda_K \), where \( \lambda_K \) is the smallest number such that \( K \) is contained in a translate of \( \lambda_K B \). Such a translate is called a minimal enclosing ball (or circumball) of \( K \), and \( \lambda_K \) is said to be the minimal enclosing radius (or circumradius or Chebyshev radius) of \( K \). In the Euclidean subcase the minimal enclosing ball of a bounded set is always unique, but this is not true for all norms. It is easy to check that

\[
\{ x \in \mathbb{M}^d : x \text{ is the center of a minimal enclosing disc of } K \} = \text{bi}(K, \lambda_K),
\]

yielding that \( \text{bi}(K, \lambda) \neq \emptyset \) if and only if \( \lambda \geq \lambda_K \). The set of centers of minimal enclosing balls of \( K \) is called the Chebyshev set of \( K \). Note that, in contrast to the Euclidean situation, in general normed spaces the Chebyshev set of a bounded set does not necessarily belong to the convex hull of this set (see [15] for some examples).

For a bounded compact set \( K \) in \( \mathbb{M}^d \) denote by \( \text{diam}(K) := \max\{ \|x - y\| : x, y \in K \} \) the diameter of \( K \).

In what follows, when we speak about the \( \lambda \)-ball intersection or \( \lambda \)-ball hull of a set \( K \), we always mean that \( \lambda \geq \lambda_K \). It is easy to check that

\[
\lambda_K \leq \text{diam}(K) \leq 2\lambda_K;
\]

see also [5].

The strong relationship between the ball hull and the ball intersection of a set \( K \) is proved in [16] for strictly convex normed planes or \( \lambda \geq \text{diam}(K) \). Our Theorem 2 completes the proof for a general normed plane.
The notions of ball hull and ball intersection are both used for solving some versions of the 2-center problem in the Euclidean plane (see [11] and [12]).

Our paper is organized as follows: Section 2 contains some results studying the ball hull structure in a normed plane. Section 3 presents an algorithm for the ball intersection which takes $O(n \log n)$ time and an algorithm for the 2-center problem with constrained circles which takes $O(n^2)$ time, both for general normed planes. Section 4 shows an algorithm for the ball hull, taking $O(n \log n)$ time for general normed planes and based on the results of Section 3. All the run times are relative to the cost of elementary operations, like computing sphere intersections.

2 The ball hull structure

In this section, we study the structure of $bh(K, \lambda)$ for a general normed plane. We start with the following result by Grünbaum [9] and Banasiak [4] (see also [17, §3.3]).

**Lemma 1.** Let $M^2$ be a normed plane. Let $C \subset M^2$ be a compact, convex disc whose boundary is the closed curve $\gamma$; $v$ be a vector in $M^2$; $C' = C + v$ be a translate of $C$ with boundary $\gamma'$. Then $\gamma \cap \gamma'$ is the union of two segments, each of which may degenerate to a point or to the empty set.

Suppose that this intersection consists of two connected non-empty components $A_1, A_2$. Then the two lines parallel to the line of translation and supporting $C \cap C'$ intersect $C \cap C'$ exactly in $A_1$ and $A_2$.

Choose a point $p_i$ from each component $A_i$ and let $c_i = p_i - v$ and $c'_i = p_i + v$ for $i = 1, 2$. Let $\gamma_1$ be the part of $\gamma$ on the same side of the line $\langle p_1, p_2 \rangle$ as $c_1$ and $c_2$; let $\gamma_2$ be the part of $\gamma'$ on the side of $\langle p_1, p_2 \rangle$ opposite to $c_1$ and $c_2$; and similarly for $\gamma'_1, \gamma'_1$, and $\gamma'_2$.

Then $\gamma_2 \subseteq \text{conv}(\gamma'_1)$ and $\gamma'_2 \subseteq \text{conv}(\gamma_1)$.

**Lemma 2.** Let $M^2$ be a normed plane. For every pair of points $p$ and $q$ whose distance is less than or equal to $2\lambda$, there exist two minimal circular arcs of radius $\lambda$ meeting them (eventually only one, if they degenerate to the same segment) which belong to every disc of radius $\lambda$ containing $p$ and $q$. These two arcs (if they are really two) are situated in different half planes bounded by the line $\langle p, q \rangle$. The centers of the discs defining these two minimal arcs are situated in the extreme points of the connected components $S(p, \lambda) \cap S(q, \lambda)$.

**Proof.** If either $\lambda \geq \|p - q\|$ or $M^2$ is strictly convex, the statement is proved in Lemma 3 and Lemma 4 in [16].

Let $M^2$ be a normed plane which is not strictly convex. Let us consider a disc of radius $\lambda$ such that $p$ and $q$ belong to its boundary. Without loss of generality, we can assume that the center of this disc is the origin $o$ of the plane.

If $S(p, \lambda) \cap S(q, \lambda)$ are two different non-empty, connected components, by Lemma 1, $S(o, \lambda)$ contains every minimal arc whose center belongs to the component where $o$ is not situated.
If $S(p, \lambda) \cap S(q, \lambda)$ are two segments parallel to $\overrightarrow{pq}$ and supporting $S(p, \lambda) \cap S(q, \lambda)$, we consider an extreme point $a$ of the segment where $o$ is situated; namely, $a$ and $o$ are in the same half-plane defined by $\langle p, q \rangle$. Then the vector $\overrightarrow{o a}$ is parallel to $\overrightarrow{pq}$, and, without loss of generality, we can assume that $\overrightarrow{o a} = \alpha \overrightarrow{pq}$ with $\alpha > 0$. The point $a + \alpha \overrightarrow{pq}$ belongs to the line $\langle p, q \rangle$, and it is situated at distance $\lambda$ from $a$. Therefore, the segment meeting $a + \alpha \overrightarrow{pq}$, $p$ and $q$ belongs to $S(a, \lambda)$, and the segment $\overrightarrow{pq}$ is a minimal arc meeting $p$ and $q$ on the circle $S(a, \lambda)$. By convexity, this segment belongs to the disc $B(o, \lambda)$.

If $S(p, \lambda) \cap S(q, \lambda)$ has only one connected component, then it is a segment, and the points $o$ and $p + q$ belong to it. Let $a$ and $b$ be the extreme points of this segment such that $a, o, p + q$ and $b$ is the sequence along this segment-component. The line $\langle p, q \rangle$ separates the points $a$ and $o$ from $p + q$ and $b$. Again by Lemma 1, the minimal arcs meeting $p$ and $q$ defined by $S(a, \lambda)$ and $S(b, \lambda)$ are contained in the disc $B(o, \lambda)$.

Due to all facts shown above, if $p, q \in S(o, \lambda)$, then the disc $B(o, \lambda)$ contains the minimal arcs defined by the circles whose centers are the extreme points of the connected components of $S(p, \lambda) \cap S(q, \lambda)$. These minimal arcs could degenerate to only one segment.

Let $p, q \in B(o, \lambda)$ but not both belonging to $S(o, \lambda)$. Let $\gamma_a$ be a minimal arc meeting $p$ and $q$ defined by an extreme point $a$ as above. By Lemma 1, there exists a connected component of $S(a, \lambda)$ which is completely contained in $B(o, \lambda)$, and its complementary connected component on $S(a, \lambda)$ has no point in the interior of $B(o, \lambda)$. Since $p, q \in \gamma_a$, we conclude that $\gamma_a$ belongs to the first connected component and is completely contained in $B(o, \lambda)$.

Equipped with Lemma 2, we obtain the following

**Theorem 1.** Let $K = \{p_1, p_2, \ldots, p_n\}$ be a finite set in a normed plane $\mathbb{R}^2$, and let $\lambda \geq \lambda_K$. Then

$$bh(K, \lambda) = \bigcap_{K \subseteq B(x_s, \lambda)} B(x_s, \lambda) = \text{conv} \left( \bigcup_{i,j=1}^n \overrightarrow{p_ip_j} \right),$$

where $x_s$ are some extreme points of the components $S(p_1, \lambda) \cap S(p_j, \lambda)$, and $\overrightarrow{p_ip_j}$ are minimal arcs meeting points of $K$ and whose centers are these extreme points $x_s$.

**Proof.** A constructive proof of the statement is presented in [15] and [16], where either $\lambda \geq \text{diam}(K)$ or $\mathbb{R}^2$ is strictly convex. This proof describes itself an algorithm for $bh(K, \lambda)$, and it is possible to apply it for the general case. Doing that, finally we obtain some minimal arcs $\overrightarrow{p_1p_2}, \overrightarrow{p_2p_3}, \ldots, \overrightarrow{p_kp_1}$ whose centers are some points $x_1, x_2, \ldots, x_k$, respectively. The union of the arcs is the boundary of $\bigcap_{K \subseteq B(x_s, \lambda)} B(x_s, \lambda)$, and one realizes that the described process forces every center $x_s$ to be an extreme point of a connected component from $S(p_i, \lambda) \cap S(p_{i+1}, \lambda)$. By Lemma 2, the related minimal arc with center in $x_s$ is contained in every ball of radius $\lambda$ which contains $K$, and consequently we obtain our statement.

**Theorem 2.** Let $K = \{p_1, p_2, \ldots, p_n\}$ be a finite set in a normed plane $\mathbb{R}^2$ and $\lambda \geq \lambda_K$. Every arc of the boundary of $bh(K, \lambda)$ has a vertex of $bh(K, \lambda)$ as center. Moreover, every vertex of $bh(K, \lambda)$ is the center of an arc belonging to the boundary of $bh(K, \lambda)$.
Proof. For the cases that $M^2$ is strictly convex or $\lambda \geq \text{diam}(K)$, this statement is proved in [16]. After the constructive process in Theorem 1, we obtain that the union of the arcs $\widehat{p_1p_2}, \widehat{p_2p_3}, ..., \widehat{p_kp_1}$ is the boundary of $bh(K, \lambda)$, and the centers of these arcs are the points $x_1, x_2, ..., x_k$. Let $x_{k+1} := x_1$ and let us consider the arc $\widehat{x_ix_{i+1}}$ in $S(p_i, \lambda)$ meeting (clockwise) $x_i$ and $x_{i+1}$. Since the constructive process assures that every disc of radius $\lambda$ whose center belongs to $\widehat{x_ix_{i+1}}$ contains $K$, then $\widehat{x_ix_{i+1}} \subset bi(K, \lambda)$. We conclude that the union of the arcs $\widehat{x_ix_{i+1}}$ is the boundary of $bi(K, \lambda)$, and $x_1, x_2, ..., x_k$ are the vertices. Besides, the centers $p_i$ of the arcs $\widehat{x_ix_{i+1}}$ are the vertices of $bh(K, \lambda)$.

3 An algorithm for the ball intersection

Hershberger and Suri (see [12], Section 6.1, page 459) describe an algorithm for computing $bi(K, \lambda)$ in $O(n \log n)$ time for the Euclidean subcase. They use this algorithm as a subroutine to solve the 2-center problem with centers at points of $K$ (called the 2-center problem with constrained circles) in $O(n^2)$ time. In the present section, we rewrite the algorithm described in [12] for arbitrary normed planes, and we use it to solve the 2-center problem with constrained circles in normed planes. The run time is relative to the cost of elementary operations, like computing the intersection of two spheres.

Let us fix an Euclidean orthonormal system of reference in the plane with basis $\{v_1, v_2\}$. The points of a finite set in this plane can be ordered by their $x$-coordinates with respect to this basis, using the $y$-coordinate order for breaking the ties.

We consider the two lines parallel to the vector $v_2$ and supporting $bi(K, \lambda)$, and the supporting sets from $\partial bi(K, \lambda)$, forming the intersections of $bi(K, \lambda)$ and these supporting lines. Let us choose two points, one from each of the two supporting sets. The line meeting these two points separates $\partial bi(K, \lambda)$ in two components, called upper chain and lower chain of $\partial bi(K, \lambda)$.

We say that an arc $a_1$ is on the left with respect to the other arc $a_2$ if the leftmost point of $a_1$ has an $x$-coordinate smaller than the $x$-coordinate of the leftmost point of $a_2$, breaking the ties similarly as with the point order.

Every arc of $bi(K, \lambda)$ has a center belonging to $K$. It is possible that some points of $K$ are not centers of arcs of $bi(K, \lambda)$. The arcs of the upper (lower) chain can be ordered by this left-to-right order induced by their leftmost points.

Lemma 3. Let $M^2$ be a normed plane. With the above conditions, if $K$ is a finite set in $M^2$, then the left-to-right order of the arcs along the upper (lower) chain of $bi(K, \lambda)$ is just the reverse of the left-to-right order of the centers of these arcs.

Proof. The upper and the lower chain cases are similar, and it is sufficient to prove the upper chain case.

Let us consider a set $K$ of two points. Let $p^0$ be the fixed leftmost point on $bi(K, \lambda)$ with respect to the system of reference. Namely, $p^0$ is the fixed point from $\partial bi(K, \lambda)$ created by a supporting line parallel to $v_2$ with the smallest first coordinate. The upper chain of
∂bi(K, λ) going clockwise is the part of ∂bi(K, λ) from p^0 to the boundary point fixed on the other parallel supporting line. The lower chain is the other part of ∂bi(K, λ).

Let o and x^1 be the center of the first and the second arc, respectively, from left to right (in the arc order sense) over the upper chain of ∂bi(K, λ). If the line parallel to v_2 is parallel to the line passing through o and x^1, then the upper chain of the boundary of ∂bi(K, λ) is a single circular arc, and there is nothing to prove. Otherwise, o and x^1 will be separated by the line passing through the midpoint of o and x^1 and parallel to v_2 (see Figure 1). By Lemma 1 and the convexity of bi(K, λ), x^1 will be situated on the left part of this line and, therefore, it is also on the left part of o.

If K is a set of n points, again p^0 is the leftmost point fixed on bi(K, λ) with respect to the system of reference; p^1, p^2, ..., p^{m-1} are the following vertices on the upper chain of ∂bi(K, λ), clockwise; p^m is the rightmost point fixed on bi(K, λ); o is the center of the arc p^0p^1; and x^1, x^2, ..., x^m are the centers of the left-to-right ordered arcs p^1p^2, p^2p^3, ..., p^{m-1}p^m, respectively.

We have proved the statement for a set K containing two points. But if the set K has n points and a connected piece of S(x^i, λ) ∩ S(x^{i+1}, λ) belongs to the upper chain of ∂bi(K, λ), then this piece also belongs to the upper chain of bi({x^i, x^{i+1}}, λ), and their common arcs are located in the same arc order. Therefore, we can repeatedly apply the statement proved for two points to the pairs (x^i, x^{i+1}) and justify that the centers x^1, x^2, ..., x^m are ordered conversely to the sequence of the arcs p^1p^2, p^2p^3, ..., p^{m-1}p^m.

After sorting the points of K by the x-coordinate, it is easier and cheaper to build bi(K, λ), because starting with the leftmost arc and its center, one only has to consider the centers at the left side to find the following arc at the right one. Therefore, the upper (lower) chain of bi(K, λ) can be constructed in O(n) time, as Hershberger and Suri describe.
Theorem 3. Let $\mathbb{M}^2$ be a normed plane. If $K$ is a set of $n$ points and $\lambda \geq \lambda_K$, then the set $\text{bi}(K, \lambda)$ can be constructed via an algorithm taking $O(n \log n)$ time.

Proof. Sorting the points of $K$ from left to right takes $O(n \log n)$ time. After the points are ordered, constructing $\text{bi}(K, \lambda)$ takes $O(n)$ time. Therefore, the total cost is $O(n \log n)$ time.

As a consequence, we obtain the following algorithm.

Corollary 1. Let $\mathbb{M}^2$ be a normed plane. If $K$ is a set of $n$ points and $r \geq 1$, the 2-center problem with constrained circles centered in the points of $K$ can be solved in $O(n^2)$ time.

Proof. The algorithm presented in [12] works correctly for a normed plane using Theorem 3 for the ball intersection: sorting the points of $K$ from left to right in $O(n \log n)$ time, according to the $x$-coordinate; for each point $p \in K$, determining the set $U$ of ordered points whose distance from $p$ is greater than the radius $r$; obtaining $\text{bi}(U, 1)$, which takes $O(n)$ time; testing if $\text{bi}(U, 1)$ contains some point of $K$ in $O(n)$ time, marching through $K$ from left to right, maintaining the two arcs of $\partial\text{bi}(U, 1)$ that overlap the $x$-coordinate of the current point. Therefore, the total time is $O(n \log n) + n \cdot O(n) = O(n^2)$.

4 An algorithm for the ball hull

In order to design an algorithm for $\text{bh}(K, \lambda)$, we can first construct $\text{bi}(K, \lambda)$ as in Section 3 in $O(n \log n)$ time, and after that we use Theorem 2 for building $\text{bh}(K, \lambda)$. We develop this idea in the present section. Again we note that the run time is relative to the cost of elementary operations, like intersection of spheres.

Theorem 4. Let $K = \{p_1, p_2, \ldots, p_n\}$ be a finite set in a normed plane $\mathbb{M}^2$ and $\lambda \geq \lambda_K$. Let $K'$ and $K''$ denote the set of vertices of $\text{bi}(K, \lambda)$ and the set of vertices of $\text{bh}(K, \lambda)$, respectively. Then

$$\text{bh}(K, \lambda) = \text{bi}(K', \lambda),$$

$$\text{bi}(K, \lambda) = \text{bi}(K'', \lambda).$$

Furthermore, the left-to-right order of the arcs along the upper (lower) chain of $\text{bh}(K, \lambda)$ is just the reverse of the left-to-right order of the centers of these arcs (which belong to $K'$).

Proof. From Theorem 2 one can deduce (1) and (2). Using Lemma 3, we have the last statement.

Therefore, having obtained $\text{bi}(K, \lambda)$, one can construct $\text{bh}(K, \lambda)$ in a normed plane by plotting the arcs with centers in the vertices of $\text{bi}(K, \lambda)$, describing finally the following algorithm:
1. Sorting the points of $K$ in $O(n \log n)$ time.

2. Building $\text{bi}(K, \lambda)$ in $O(n)$ time (Theorem 3).

3. Considering the set $K'$ of sorted vertices $\{x_1, ..., x_k\}$ of $\text{bi}(K, \lambda)$ obtained in (2).

4. Building $\text{bh}(K, \lambda) = \text{bi}(K', \lambda)$ (Theorem 4) in $O(n)$ time (Theorem 3).

**Theorem 5.** Let $M^2$ be a normed plane. If $K$ is a set of $n$ points and $\lambda \geq \lambda_K$, then the set $\text{bh}(K, \lambda)$ can be constructed via an algorithm taking $O(n \log n)$ time.

**References**


