Exposed Polynomials of $\mathcal{P}(\mathbb{R}^2_{h(\frac{1}{2})})$

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Abstract: We show that every extreme polynomials of $\mathcal{P}(\mathbb{R}^2_{h(\frac{1}{2})})$ is exposed.

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1. Introduction

According to the Krein-Milman Theorem, every nonempty convex set in a Banach space is fully described by the set of its extreme points. Let $n \in \mathbb{N}$. We write $B_E$ for the closed unit ball of a real Banach space $E$ and the dual space of $E$ is denoted by $E'$. We recall that if $x \in B_E$ is said to be an extreme point of $B_E$ if $y, z \in B_E$ and $x = \lambda y + (1 - \lambda)z$ for some $0 < \lambda < 1$ implies that $x = y = z$. $x \in B_E$ is called an exposed point of $B_E$ if there is an $f \in E'$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of $B_E$ is an extreme point. We denote by $\text{ext}B_E$ and $\text{exp}B_E$ the sets of extreme and exposed points of $B_E$, respectively.

We denote by $L(n,E)$ the Banach space of all continuous $n$-linear forms on $E$ endowed with the norm $\|T\| = \sup_{\|x\| = 1} |T(x_1, \ldots, x_n)|$. A $n$-linear form $T$ is symmetric if $T(x_1, \ldots, x_n) = T(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every permutation $\sigma$ on $\{1, 2, \ldots, n\}$. We denote by $L_s(n,E)$ the Banach space of all continuous symmetric $n$-linear forms on $E$. A mapping $P : E \to \mathbb{R}$ is a continuous $n$-homogeneous polynomial if there exists a unique $T \in L_s(n,E)$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. In this case it is convenient to write $T = \hat{P}$. We denote by $\mathcal{P}(n,E)$ the Banach space of all continuous $n$-homogeneous polynomials from $E$ into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\| = 1} |P(x)|$. Note that the spaces $L(n,E)$,

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\(L_1(nE), \mathcal{P}_1(nE)\) are very different from a geometric point of view. In particular, for integral multilinear forms and integral polynomials one has ([2], [9], [42])

\[
\text{ext}B_{L_1(nE)} = \{ \phi_1 \phi_2 \cdots \phi_n : \phi_i \in \text{ext}B_{E^*} \},
\]

\[
\text{ext}B_{\mathcal{P}_1(nE)} = \{ \pm \phi^n : \phi \in E^*, \|\phi\| = 1 \},
\]

where \(L_1(nE)\) and \(\mathcal{P}_1(nE)\) are the spaces of integral \(n\)-linear forms and integral \(n\)-homogeneous polynomials on \(E\), respectively. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [10].

Let us say about the stories of the classification problems of \(\text{ext}B_X\) and \(\exp B_X\) if \(X = \mathcal{P}(nE)\). Choi et al. ([4], [5]) initiated the classification problems and classified \(\text{ext}B_X\) if \(X = \mathcal{P}(2l_2^n)\) for \(p = 1, 2\), where \(l_2^n = \mathbb{R}^2\) with the \(l_p\)-norm. B. Grecu [14] classified \(\text{ext}B_X\) if \(X = \mathcal{P}(2l_2^n)\) for \(1 < p < 2\) or \(2 < p < \infty\). Kim [18] classified \(\exp B_X\) if \(X = \mathcal{P}(2l_2^n)\) for \(1 \leq p \leq \infty\). Kim et al. [34] showed that every extreme \(2\)-homogeneous polynomials on a real separable Hilbert space is also exposed. Kim ([20], [26]) characterized \(\text{ext}B_X\) and \(\exp B_X\) for \(X = \mathcal{P}(2d_s(1,w)^2)\), where \(d_s(1,w)^2 = \mathbb{R}^2\) with the octagonal norm

\[
\|(x,y)\|_{d_s} = \max \left\{ |x|, |y|, \frac{|x| + |y|}{1+w} : 0 < w < 1 \right\}.
\]

He showed [26] that \(\text{ext}B_{\mathcal{P}(2d_s(1,w)^2)} \neq \exp B_{\mathcal{P}(2d_s(1,w)^2)}\). In [31], Kim classified \(\text{ext}B_X\) and using the classification of \(\text{ext}B_X\), Kim computed the polarization and unconditional constants of the space \(X\) if \(X = \mathcal{P}(2\mathbb{R}^2_{h(\frac{1}{2})})\), where \(\mathbb{R}^2_{h(w)}\) denotes the space \(\mathbb{R}^2\) endowed with the hexagonal norm

\[
\|(x,y)\|_{h(w)} := \max \{|y|, |x| + (1 - w)|y|\}.
\]

We refer to ([1]–[9], [11]–[43]) and references therein for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces.

We will denote by \(T((x_1,y_1), (x_2,y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1)\) and \(P(x,y) = ax^2 + by^2 + cxy\) a symmetric bilinear form and a \(2\)-homogeneous polynomial on a real Banach space of dimension 2, respectively. Recently, Kim [31] classified the extreme points of the unit ball of \(\mathcal{P}(2\mathbb{R}^2_{h(\frac{1}{2})})\) as follows:

\[
\text{ext}B_{\mathcal{P}(2\mathbb{R}^2_{h(\frac{1}{2})})} = \left\{ \pm y^2, \pm (x^2 + \frac{1}{4}y^2 \pm xy), \pm (x^2 + \frac{3}{4}y^2), \right. \\
\left. \pm [x^2 + \left(\frac{c^2}{4} - 1\right)y^2 \pm cxy], \right. \\
\left. \pm [cx^2 + \left(\frac{c^4 + \sqrt{1-c}}{4} - 1\right)y^2 \pm (c+2\sqrt{1-c})xy ] (0 \leq c \leq 1) \right\}.
\]
In this paper, we show that every extreme polynomials of \( P(2R^2 h(1/2)) \) is exposed.

2. Results

**Theorem 2.1.** ([31]) Let \( P(x, y) = ax^2 + by^2 + cxy \in P(2R^2 h(1/2)) \) with \( a \geq 0, c \geq 0 \) and \( a^2 + b^2 + c^2 \neq 0 \). Then:

Case 1: \( c < a \).

If \( a \leq 4b \), then
\[
\|P\| = \max \left\{ a, b, \left| \frac{1}{3}a + b \right| + \frac{1}{2}c, \frac{4ab-c^2}{4a}, \frac{4ab-c^2}{2c+a+4b}, \frac{4ab-c^2}{2c-a-4b} \right\}.
\]

If \( a > 4b \), then
\[
\|P\| = \max \left\{ a, b, \left| \frac{1}{4}a + b \right| + \frac{1}{2}c, \frac{|c^2-4ab|}{4a} \right\}.
\]

Case 2: \( c \geq a \).

If \( a \leq 4b \), then \( \|P\| = \max \left\{ a, b, \left| \frac{1}{3}a + b \right| + \frac{1}{2}c, \frac{c^2-4ab}{2c-a-4b} \right\} \).

If \( a > 4b \), then \( \|P\| = \max \left\{ a, b, \left| \frac{1}{4}a + b \right| + \frac{1}{2}c, c^2-4ab \right\} \).

**Theorem 2.2.** ([31])

\[
\text{ext} B_P(2R^2 h(1/2)) = \left\{ \pm y^2, \pm (x^2 + \frac{1}{4}y^2 \pm xy), \pm (x^2 + \frac{3}{4}y^2), \right. \]
\[
\left. \pm \left[ x^2 + \left( \frac{c^2}{4} - 1 \right)y^2 \pm cxy \right], \ 
\pm \left[ cx^2 + \left( \frac{c+4\sqrt{1-c}}{4} - 1 \right)y^2 \pm (c + 2\sqrt{1-c})xy \right] \left( 0 \leq c \leq 1 \right) \right\}.
\]

**Theorem 2.3.** Let \( f \in P(2R^2 h(1/2))^* \) with \( \alpha = f(x^2), \beta = f(y^2), \gamma = f(xy) \). Then
\[
\|f\| = \sup \left\{ |\beta|, |\alpha + \frac{1}{3}\beta| + |\gamma|, |\alpha + \frac{3}{7}\beta|, |\alpha + \left( \frac{c^2}{4} - 1 \right)\beta| + c|\gamma|, \right. \]
\[
\left. |ca + \left( \frac{c+4\sqrt{1-c}}{4} - 1 \right)\beta| + (c+2\sqrt{1-c})|\gamma| \left( 0 \leq c \leq 1 \right) \right\}.
\]

**Proof.** It follows from Theorem 2.2 and the fact that
\[
\|f\| = \sup \left\{ |f(P)| : P \in \text{ext} B_P(2R^2 h(1/2)) \right\}.
\]
Note that if $\|f\| = 1$, then $|\alpha| \leq 1$, $|\beta| \leq 1$, $|\gamma| \leq \frac{1}{2}$.

We are in a position to show the main result of this paper.

**Theorem 2.4.**

$$\exp B_{\mathcal{P}}(\mathbb{R}^2_{h(\frac{1}{2})}) = \text{ext} B_{\mathcal{P}}(\mathbb{R}^2_{h(\frac{1}{2})}) .$$

**Proof.** Let $(0 \leq c \leq 1)$

$$P_1(x, y) = y^2 ,$$
$$P_2^+(x, y) = x^2 + \frac{1}{4}y^2 + xy ,$$
$$P_2^-(x, y) = x^2 + \frac{1}{4}y^2 - xy ,$$
$$P_3(x, y) = x^2 + \frac{2}{3}y^2 ,$$
$$P_4^+(x, y) = x^2 + \left(\frac{c^2}{4} - 1\right)y^2 + cxy ,$$
$$P_4^-(x, y) = x^2 + \left(\frac{c^2}{4} - 1\right)y^2 - cxy ,$$
$$P_5^+(x, y) = cx^2 + \left(\frac{c^4 + 4\sqrt{4 - c}}{4} - 1\right)y^2 + (c + 2\sqrt{4 - c})xy ,$$
$$P_5^-(x, y) = cx^2 + \left(\frac{c^4 + 4\sqrt{4 - c}}{4} - 1\right)y^2 - (c + 2\sqrt{4 - c})xy .$$

Claim 1: $P_1 = y^2 \in \exp B_{\mathcal{P}}(\mathbb{R}^2_{h(\frac{1}{2})})$.

Let $f \in \mathcal{P}(\mathbb{R}^2_{h(\frac{1}{2})})^*$ be such that

$$\alpha = \frac{1}{5} , \quad \beta = 1 , \quad \gamma = 0 .$$

Indeed,

$$f(P_1) = 1 , \quad |f(P_2)| = \frac{9}{20} , \quad |f(P_3)| = \frac{19}{20} . \quad (*)$$

Note that for all $0 \leq c \leq 1$,

$$|f(P_{4,c}^\pm)| = \frac{4}{5} - \frac{c^2}{4} \leq \frac{4}{5} , \quad (**)$$

$$|f(P_{5,c}^\pm)| = |\sqrt{4 - c} + \frac{9c}{20} - 1| \leq \frac{11}{20} . \quad (***)$$
We will show that \( f \) exposes \( P_1 \). Let \( Q(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(\mathbb{R}_h^2) \) such that \( 1 = \|f\| = f(Q) \). We will show that \( Q = P_1 \). Since \( \mathcal{P}(\mathbb{R}_h^2) \) is a finite dimensional Banach space with dimension 3, by the Krein-Milman Theorem, \( B_{\mathcal{P}(\mathbb{R}_h^2)} \) is the closed convex hull of \( \text{ext}B_{\mathcal{P}(\mathbb{R}_h^2)} \).

Then,

\[
Q(x, y) = uP_1(x, y) + v^+P_2^+(x, y) + v^-P_2^-(x, y) + tP_3(x, y)
\]

for some \( u, v^+, t, \lambda_n^+ \), \( \delta_m^+ \), \( \delta_m^- \), \( \lambda_n^- \), \( \lambda_n^+ \), \( \delta_m^+ \), \( \delta_m^- \), \( \in \mathbb{R} \) with \( 0 \leq c_n, \alpha_m \leq 1 \) and

\[
|u| + |v^+| + |v^-| + |t| + \sum_{n=1}^{\infty} |\lambda_n^+| + \sum_{n=1}^{\infty} |\lambda_n^-| + \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| = 1.
\]

We will show that \( v^+ = t = \lambda_n^+ = \delta_m^+ = 0 \) for every \( n, m \in \mathbb{N} \).

Subclaim: \( v^+ = t = 0 \).

Assume that \( v^+ \neq 0 \). It follows that

\[
1 = f(Q) = uf(P_1) + v^+f(P_2^+) + v^-f(P_2^-) + tf(P_3) + \sum_{n=1}^{\infty} \lambda_n^+ f(P_{4,c_n}^+)
\]

\[
+ \sum_{n=1}^{\infty} \lambda_n^- f(P_{4,c_n}^-) + \sum_{m=1}^{\infty} \delta_m^+ f(P_{5,a_m}^+) + \sum_{m=1}^{\infty} \delta_m^- f(P_{5,a_m}^-)
\]

\[
\leq |u| + |v^+||f(P_2^+)| + |v^-||f(P_2^-)| + |t||f(P_3)| + \sum_{n=1}^{\infty} |\lambda_n^+||f(P_{4,c_n}^+)|
\]

\[
+ \sum_{n=1}^{\infty} |\lambda_n^-||f(P_{4,c_n}^-)| + \sum_{m=1}^{\infty} |\delta_m^+||f(P_{5,a_m}^+)| + \sum_{m=1}^{\infty} |\delta_m^-||f(P_{5,a_m}^-)|
\]

\[
\leq |u| + \frac{9}{20}|v^+| + \frac{9}{20}|v^-| + \frac{19}{20}|t| + \frac{4}{5} \sum_{n=1}^{\infty} |\lambda_n^+|
\]

\[
+ \frac{4}{5} \sum_{n=1}^{\infty} |\lambda_n^-| + \frac{11}{20} \sum_{m=1}^{\infty} |\delta_m^+| + \frac{11}{20} \sum_{m=1}^{\infty} |\delta_m^-| \quad \text{(by (**), (**'), (***))}
\]
which is impossible. Therefore, \( v^+ = 0 \). Using a similar argument as above, we have \( v^- = t = 0 \).

Subclaim: \( \lambda^+_n = \delta^+_m = 0 \) for every \( n, m \in \mathbb{N} \).

Assume that \( \lambda^+_{n_0} \neq 0 \) for some \( n_0 \in \mathbb{N} \). It follows that

\[
1 = f(Q) = u f(P_1) + \lambda^+_{n_0} f(P^+_{4,c_{n_0}}) + \sum_{n \in \mathbb{N}, n \neq n_0} \lambda^+_n f(P^+_{4,c_n})
\]

\[
+ \sum_{n=1}^{\infty} \lambda^-_n f(P^-_{4,c_n}) + \sum_{m=1}^{\infty} \delta^+_m f(P^+_{5,a_m}) + \sum_{m=1}^{\infty} \delta^-_m f(P^-_{5,a_m})
\]

\[
\leq |u| + |\lambda^+_{n_0}| |f(P^+_{4,c_{n_0}})| + \sum_{n \in \mathbb{N}, n \neq n_0} |\lambda^+_n| |f(P^+_{4,c_n})| + \sum_{n=1}^{\infty} |\lambda^-_n| |f(P^-_{4,c_n})| + \sum_{m=1}^{\infty} |\delta^+_m| |f(P^+_{5,a_m})| + \sum_{m=1}^{\infty} |\delta^-_m| |f(P^-_{5,a_m})|
\]

\[
< |u| + \frac{4}{5} \sum_{n \in \mathbb{N}, n \neq n_0} |\lambda^+_n| + \frac{4}{5} \sum_{n=1}^{\infty} |\lambda^-_n| + \frac{11}{20} \sum_{m=1}^{\infty} |\delta^+_m| + \frac{11}{20} \sum_{m=1}^{\infty} |\delta^-_m|
\]

\[
\leq |u| + \sum_{n=1}^{\infty} |\lambda^+_n| + \sum_{n=1}^{\infty} |\lambda^-_n| + \sum_{m=1}^{\infty} |\delta^+_m| + \sum_{m=1}^{\infty} |\delta^-_m| = 1,
\]

which is impossible. Therefore, \( \lambda^+_n = 0 \) for every \( n \in \mathbb{N} \). Using a similar argument as above, we have \( \lambda^-_n = \delta^+_m = 0 \) for every \( n, m \in \mathbb{N} \). Therefore, \( Q(x, y) = uP_1(x, y) \). Hence \( u = 1 \), so \( Q = P_1 \). Therefore, \( f \) exposes \( P_1 \).

Claim 2: \( P_{5,0} = 2xy \in \exp B_{\mathcal{P}(\mathbb{R}^2_{H^{1/2}})} \).

Let \( f \in \mathcal{P}(\mathbb{R}^2_{H^{1/2}})^* \) be such that

\[
\alpha = \beta = 0, \quad \gamma = \frac{1}{2}.
\]
We will show that $f$ exposes $P_{5,0}$. Indeed, $f(P_{5,0}) = 1$, $f(P_1) = 0$, $f(P_2^+) = \pm \frac{1}{2}$, $f(P_3) = 0$,

$$-\frac{1}{2} \leq f(P_{4,c}^+) = \pm \frac{c}{2} \leq \frac{1}{2} \quad (0 \leq c \leq 1).$$

Note that, for $0 < c \leq 1$,

$$-1 < f(P_{5,c}^\pm) = \pm \frac{c + 2\sqrt{1-c}}{2} < 1.$$  \(\dagger\)

Hence, by Theorem 2.3, $1 = \|f\|$. Let

$$Q(x, y) = uP_1(x, y) + v^+P_2^+(x, y) + v^-P_2^-(x, y) + tP_3(x, y)$$

$$+ \sum_{n=1}^{\infty} \lambda_n^+ P_{4,c_n}^+(x, y) + \sum_{n=1}^{\infty} \lambda_n^- P_{4,c_n}^-(x, y)$$

$$+ \sum_{m=1}^{\infty} \delta_m^+ P_{5,a_m}^+(x, y) + \sum_{m=1}^{\infty} \delta_m^- P_{5,a_m}^-(x, y),$$

for some $u, v^+, t, \lambda_n^+, \delta_m^+ \in \mathbb{R}$ ($n, m \in \mathbb{N}$) with $0 \leq c_n^+, a_m^+ \leq 1$ and

$$|u| + |v^+| + |v^-| + |t| + \sum_{n=1}^{\infty} |\lambda_n^+| + \sum_{n=1}^{\infty} |\lambda_n^-| + \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| = 1.$$  

We will show that $v^\pm = t = \lambda_n^\pm = \delta_m^\pm = 0$ for every $n, m \in \mathbb{N}$.

Subclaim: $v^+ = 0$.

Assume that $v^+ \neq 0$. It follows that

$$1 = f(Q) = v^+ f(P_2^+) + v^- f(P_2^-) + \sum_{n=1}^{\infty} \lambda_n^+ f(P_{4,c_n}^+)$$

$$+ \sum_{n=1}^{\infty} \lambda_n^- f(P_{4,c_n}^-) + \sum_{m=1}^{\infty} \delta_m^+ f(P_{5,a_m}^+) + \sum_{m=1}^{\infty} \delta_m^- f(P_{5,a_m}^-)$$

$$< |v^+| + \frac{1}{2} |v^-| + \sum_{n=1}^{\infty} |\lambda_n^+||f(P_{4,c_n}^+)| + \sum_{n=1}^{\infty} |\lambda_n^-||f(P_{4,c_n}^-)|$$

$$+ \sum_{m=1}^{\infty} |\delta_m^+||f(P_{5,a_m}^+)| + \sum_{m=1}^{\infty} |\delta_m^-||f(P_{5,a_m}^-)|$$

$$\leq |v^+| + |v^-| + \sum_{n=1}^{\infty} |\lambda_n^+| + \sum_{n=1}^{\infty} |\lambda_n^-| + \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| \leq 1,$$
which is impossible. Therefore, \( v^+ = 0 \). Using a similar argument as Claim 1, we have \( v^- = \lambda_n^+ = 0 \) for every \( n \in \mathbb{N} \). Hence,

\[
Q(x, y) = uP_1(x, y) + tP_3(x, y) + \sum_{m=1}^{\infty} \delta_m^+ P_{5,a_m^+}(x, y) + \sum_{m=1}^{\infty} \delta_m^- P_{5,a_m^-}(x, y).
\]

It follows that

\[
1 = f(Q) = \sum_{m=1}^{\infty} \delta_m^+ f(P_{5,a_m^+}) + \sum_{m=1}^{\infty} \delta_m^- f(P_{5,a_m^-})
\]

\[
\leq \sum_{m=1}^{\infty} |\delta_m^+||f(P_{5,a_m^+})| + \sum_{m=1}^{\infty} |\delta_m^-||f(P_{5,a_m^-})|
\]

\[
\leq \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| \leq 1,
\]

which shows that

\[
f(P_{5,a_m^+}) = f(P_{5,a_m^-}) = \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| = 1, \quad u = t = 0 \quad \text{for all } m \in \mathbb{N}.
\]

By (1), \( P_{5,a_m^\pm} = P_{5,0} \) for every \( m \in \mathbb{N} \) and \( \sum_{m=1}^{\infty} \delta_m^+ + \sum_{m=1}^{\infty} \delta_m^- = 1 \). Therefore, \( Q = P_{5,0} \). Hence, \( f \) exposes \( P_{5,0} \).

Claim 3: \( P_2^+ = x^2 + \frac{1}{4} y^2 + xy \in \exp B_{P(\mathbb{R}^2_{h(\frac{1}{2})})} \).

Let \( f \in \mathcal{P}(\mathbb{R}^2_{h(\frac{1}{2})})^* \) be such that

\[
\alpha = \frac{1}{2} = \beta, \quad \gamma = \frac{3}{8}.
\]

We will show that \( f \) exposes \( P_2 \). Indeed, \( f(P_2^+) = 1, f(P_2^-) = \frac{1}{4}, f(P_3^+) = \frac{1}{2} \), \( f(P_3^-) = \frac{7}{4} \). By some calculation, we have

\[
|f(P_{4,c}^\pm)| \leq \frac{1}{2}, \quad |f(P_{5,c}^\pm)| \leq \frac{57}{64} \quad \text{for } 0 \leq c \leq 1.
\]

Hence, by Theorem 2.3, \( 1 = \|f\| \). By similar arguments as Claims 1 and 2, \( f \) exposes \( P_2^+ \). Obviously, \( P_2^- \in \exp B_{P(\mathbb{R}^2_{h(\frac{1}{2})})} \).

Claim 4: \( P_{4,0}^+ = x^2 - y^2 \in \exp B_{P(\mathbb{R}^2_{h(\frac{1}{2})})} \).
Let $f \in \mathcal{P}(\mathbb{R}^2_{h(\frac{1}{2})})^*$ be such that
\[
\alpha = \frac{1}{2} = -\beta, \quad \gamma = 0.
\]
We will show that $f$ exposes $P_{4,0}$. Indeed,
\[
f(P_{4,0}^+) = 1, \quad |f(P_1)| = \frac{1}{2}, \quad |f(P_2^\pm)| = \frac{3}{8}, \quad |f(P_3)| = \frac{1}{8}.
\]
Note that
\[
|f(P_{4,c}^\pm)| = 1 - \frac{c^2}{8} < 1 \quad \text{for } 0 < c \leq 1.
\]
Note that, for $0 \leq c \leq 1$,
\[
|f(P_{5,c}^\pm)| = \frac{3c + 4 - 4\sqrt{\frac{1-c}{8}}}{8} \leq \frac{7}{8}.
\]
Hence, by Theorem 2.3, $1 = \|f\|$. By similar arguments as Claims 1 and 2, $f$ exposes $P_{4,0}^+$.

Claim 5: $P_3 = x^2 + \frac{3}{4}y^2 \in \exp B_{P_3}^{\mathbb{R}^2_{h(\frac{1}{2})}}$.

Let $f \in \mathcal{P}(\mathbb{R}^2_{h(\frac{1}{2})})^*$ be such that
\[
\alpha = \frac{5}{8}, \quad \beta = \frac{1}{2}, \quad \gamma = 0.
\]
We will show that $f$ exposes $P_3$. Indeed,
\[
f(P_3) = 1, \quad |f(P_1)| = \frac{1}{2}, \quad |f(P_2^\pm)| = \frac{3}{4}.
\]
Note that
\[
|f(P_{4,c}^\pm)| \leq \frac{1}{4}, \quad |f(P_{5,c}^\pm)| \leq \frac{1}{3} \quad \text{for } 0 \leq c \leq 1.
\]
Hence, by Theorem 2.3, $1 = \|f\|$. By similar arguments as Claims 1 and 2, $f$ exposes $P_3$.

Claim 6: $P_{5,1}^+ = x^2 - \frac{3}{4}y^2 + xy \in \exp B_{P_5}^{\mathbb{R}^2_{h(\frac{1}{2})}}$.

Let $f \in \mathcal{P}(\mathbb{R}^2_{h(\frac{1}{2})})^*$ be such that
\[
\alpha = \frac{11}{16}, \quad \beta = -\frac{1}{4}, \quad \gamma = \frac{1}{8}.
\]
We will show that \( f \) exposes \( P_{5,1}^+ \). Indeed,

\[
f(P_{5,1}^+) = 1, \quad |f(P_1)| = \frac{1}{4}, \quad |f(P_2^+)| \leq \frac{3}{4}, \quad |f(P_3)| = \frac{1}{2}.
\]

Note that

\[
\frac{3}{4} \leq f(P_{4,c}^+) < 1, \quad -\frac{1}{4} \leq f(P_{5,c}^+) < 1 \quad \text{for } 0 \leq c < 1.
\]

Hence, by Theorem 2.3, \( 1 = \|f\| \). By similar arguments as Claims 1 and 2, \( f \) exposes \( P_{5,1}^+ \). Obviously, \( P_{5,1}^- \in \exp B_p(2\mathbb{R}^2_{h(\frac{1}{2})}) \).

Claim 7: \( P_{4,c}^+ = x^2 + (\frac{c^2}{4} - 1)y^2 + cxy \in \exp B_p(2\mathbb{R}^2_{h(\frac{1}{2})}) \) for \( 0 < c < 1 \).

Let \( f \in \mathcal{P}(2\mathbb{R}^2_{h(\frac{1}{2})})^* \) be such that

\[
\alpha = \frac{3}{4} - \frac{c^2}{16}, \quad \beta = -\frac{1}{4}, \quad \gamma = \frac{c}{8}.
\]

Indeed,

\[
f(P_{4,c}^+) = 1, \quad \frac{3}{4} \leq f(P_{4,c}^-) = 1 - \frac{c^2}{4} < 1, \quad |f(P_1)| = \frac{1}{4}, \quad \frac{1}{2} \leq f(P_2^+) \leq \frac{3}{4}, \quad \frac{1}{2} \leq f(P_3) < \frac{9}{16}.
\]

Note that for every \( t \in [0,1] \) with \( t \neq c \),

\[
f(P_{4,t}^+) = -\frac{1}{16}t^2 + \frac{c}{8}t + \left(1 - \frac{c^2}{16}\right)
\]

and

\[
f(P_{4,t}^-) = -\frac{1}{16}t^2 - \frac{c}{8}t + \left(1 - \frac{c^2}{16}\right).
\]

Hence, we have, for every \( t \in [0,1] \) with \( t \neq c \),

\[
1 < \min \left\{1 - \frac{c^2}{16}, 1 - \frac{(1-c)^2}{16}\right\} \leq f(P_{4,t}^+) < 1 \quad (**)
\]

and

\[-1 < 1 - \frac{(1+c)^2}{16} \leq f(P_{4,t}^-) \leq 1 - \frac{c^2}{16} < 1.
\]
Note that, for every $t \in [0, 1]$, 

$$f(P_{5,t}^+) = \left(-\frac{c^2 + 2c + 11}{16}\right)t + \left(\frac{c-1}{4}\right)\sqrt{1-t} + \frac{1}{4}$$

and

$$f(P_{5,t}^-) = \left(-\frac{c^2 - 2c + 11}{16}\right)t + \left(\frac{c+1}{4}\right)\sqrt{1-t} + \frac{1}{4}.$$ 

Hence, we have that, for every $t \in [0, 1]$,

$$-1 < \frac{c}{4} \leq f(P_{5,t}^+) \leq \frac{-c^2 + 2c + 15}{16} < 1 \quad (***).$$

and

$$-1 < \frac{c+2}{4} \leq f(P_{5,t}^-) \leq \frac{-c^2 - 2c + 15}{16} < 1.$$ 

Hence, by Theorem 2.3, $1 = \|f\|$. We will show that $f$ exposes $P_{4,c}^+$. Let $Q(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(2\mathbb{R}_+^2)$ such that $1 = \|Q\| = f(Q)$. We will show that $Q = P_{4,c}^+$. By the Krein-Milman Theorem,

$$Q(x, y) = uP_1(x, y) + v^+P_2^+(x, y) + v^-P_2^-(x, y) + tP_3(x, y)$$

$$+ \sum_{n=1}^{\infty} \lambda^+_n P_{4,c_n^+}^+(x, y) + \sum_{n=1}^{\infty} \lambda^-_n P_{4,c_n^-}^-(x, y)$$

$$+ \sum_{m=1}^{\infty} \delta^+_m P_{5,a_m^+}^+(x, y) + \sum_{m=1}^{\infty} \delta^-_m P_{5,a_m^-}^-(x, y),$$

for some $u, v^+, v^-, t, \lambda^+_n, \lambda^-_n, \delta^+_m, \delta^-_m \in \mathbb{R}$ ($n, m \in \mathbb{N}$) with $0 \leq c_n^+, a_n^+ \leq 1$ and

$$|u| + |v^+| + |v^-| + |t| + \sum_{n=1}^{\infty} |\lambda^+_n| + \sum_{n=1}^{\infty} |\lambda^-_n| + \sum_{m=1}^{\infty} |\delta^+_m| + \sum_{m=1}^{\infty} |\delta^-_m| = 1.$$ 

We will show that $u = v^+ = t = \lambda^+_n = \delta^+_m = 0$ for every $n, m \in \mathbb{N}$. Assume
that $\delta_{m_0}^+ \neq 0$ for some $m_0 \in \mathbb{N}$. It follows that

$$1 = f(Q) = uf(P_1) + v^+ f(P_2^+) + v^- f(P_2^-) + tf(P_3) + \sum_{n=1}^{\infty} \lambda_n^+ f(P_{4,c_n^+}^+)$$

$$+ \sum_{n=1}^{\infty} \lambda_n^- f(P_{4,c_n^-}^-) + \sum_{m=1}^{\infty} \delta_{m_0}^+ f(P_{5,a_m^+}^+) + \sum_{m=1}^{\infty} \delta_{m}^- f(P_{5,a_m^-}^-)$$

$$< \frac{1}{4} |u| + \frac{3}{4} |v^+| + \frac{3}{4} |v^-| + \frac{9}{40} |t| + \sum_{n=1}^{\infty} |\lambda_n^+|$$

$$+ \sum_{n=1}^{\infty} |\lambda_n^-| + |\delta_{m_0}^+| + \sum_{m \neq m_0} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| \quad \text{(by (**), (**), (***)}) \leq 1,$$

which is impossible. Therefore, $\delta_{m}^+ = 0$ for every $m \in \mathbb{N}$. Using a similar argument as above, we have $u = v^+ = t = \lambda_n^- = 0$. Therefore,

$$Q(x, y) = \sum_{n=1}^{\infty} \lambda_n^+ P_{4,c_n^+}^+ (x, y).$$

We will show that if $c_{n_0}^+ \neq c$ for some $n_0 \in \mathbb{N}$, then $\lambda_{n_0}^+ = 0$. Assume that $\lambda_{n_0}^+ \neq 0$. It follows that

$$1 = f(Q) = \lambda_{n_0}^+ f(P_{4,c_{n_0}^+}^+) + \sum_{n \neq n_0} \lambda_n^+ f(P_{4,c_n^+}^+)$$

$$< |\lambda_{n_0}^+| + \sum_{n \neq n_0} |\lambda_n^+| = 1,$$

which is impossible. Therefore, $\lambda_{n_0}^+ = 0$ for every $n \in \mathbb{N}$. Therefore,

$$Q(x, y) = \left( \sum_{c_n^+ = c} \lambda_n^+ \right) P_{4,c}^+ (x, y) = P_{4,c}^+ (x, y).$$

Therefore, $f$ exposes $P_{4,c}^+$. Obviously, $P_{4,c}^+ \in \exp B_{P}(2R_{h(\frac{1}{4})}^2)$ for $0 < c \leq 1$.

Claim 8: $P_{5,c}^+ = cx^2 + \left( \frac{c+4\sqrt{1-c}}{4} - 1 \right) y^2 + (c+2\sqrt{1-c})xy \in \exp B_{P}(2R_{h(\frac{1}{4})}^2)$ for $0 < c < 1$. 

Let \( f \in \mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})^* \) be such that
\[
\alpha = \frac{1}{2} \left( 1 - \frac{c + 4\sqrt{1 - c}}{4} \right), \quad \beta = -\frac{c}{2}, \quad \gamma = \frac{c + 2\sqrt{1 - c}}{4}.
\]
Note that
\[
0 \leq \alpha < \frac{3}{8}, \quad -\frac{1}{2} < \beta \leq 0, \quad \frac{1}{4} < \gamma \leq \frac{1}{2}.
\]
We will show that \( f \) exposes \( P_{5,c}^+ \). Indeed,
\[
f(P_{5,c}^+) = 1, \quad |f(P_1)| < \frac{1}{2}, \quad 0 < f(P_2^+) < \frac{1}{2}, \quad -1 < f(P_2^-) < -\frac{1}{8}, \quad -\frac{1}{8} \leq f(P_3) < 0.
\]
Note that for every \( t \in [0, 1] \),
\[
f(P_{4,t}^+) = -\frac{c}{8} t^2 + \left( \frac{c + 2\sqrt{1 - c}}{4} \right) t + \frac{1}{2} + \frac{3c}{8} - \sqrt{1 - c}
\]
and
\[
f(P_{4,t}^-) = -\frac{c}{8} t^2 - \left( \frac{c + 2\sqrt{1 - c}}{4} \right) t + \frac{1}{2} + \frac{3c}{8} - \sqrt{1 - c}.
\]
Hence, we have for every \( t \in [0, 1] \),
\[
-1 < \frac{1}{2} + \frac{3c}{8} - \sqrt{1 - c} \leq f(P_{4,t}^+) \leq \frac{c + 1}{2} < 1, \quad (**)
\]
\[
-1 < \frac{1}{2} - \sqrt{1 - c} \leq f(P_{4,t}^-) \leq \frac{1}{2} + \frac{3c}{8} - \sqrt{1 - c} < 1.
\]
Note that for every \( t \in [0, 1] \) with \( t \neq c \),
\[
f(P_{5,t}^+) = \frac{1}{2} t + \sqrt{1 - c} \sqrt{1 - t} + \frac{c}{2}
\]
and
\[
f(P_{5,t}^-) = \left( \frac{1 - c - \sqrt{1 - c}}{2} \right) t - (c + \sqrt{1 - c}) \sqrt{1 - t} + \frac{c}{2}.
\]
Hence, we have for every \( t \in [0, 1] \) with \( t \neq c \),
\[
-1 < \min \left\{ \frac{c}{2} + \sqrt{1 - c}, \frac{c + 1}{2} \right\} \leq f(P_{5,t}^+) < 1, \quad (***)
\]
\[
-1 < -\left( \frac{c}{2} + \sqrt{1 - c} \right) \leq f(P_{5,t}^-) \leq \frac{1}{2} - \sqrt{1 - c} < 1.
\]
Hence, by Theorem 2.3, \( 1 = \|f\| \). Let \( Q(x, y) = ax^2 + by^2 + cxy \) in \( \mathcal{P}(\mathbb{R}^2_{h(2)}) \) such that \( 1 = \|Q\| = f(Q) \). By the Krein-Milman Theorem,

\[
Q(x, y) = uP_1(x, y) + v^+ P^+_2(x, y) + v^- P^-_2(x, y) + tP_3(x, y)
\]

\[
+ \sum_{n=1}^{\infty} \lambda^+_n P^+_{4,c_n}(x, y) + \sum_{n=1}^{\infty} \lambda^-_n P^-_{4,c_n}(x, y)
\]

\[
+ \sum_{m=1}^{\infty} \delta^+_m P^+_{5,a_m}(x, y) + \sum_{m=1}^{\infty} \delta^-_m P^-_{5,a_m}(x, y),
\]

for some \( u, v^+, t, \lambda^+_n, \delta^+_m, \in \mathbb{R} \) \( (n, m \in \mathbb{N}) \) with \( 0 \leq c^+_n, a^+_m \leq 1 \) and

\[
|u| + |v^+| + |v^-| + |t| + \sum_{n=1}^{\infty} |\lambda^+_n| + \sum_{n=1}^{\infty} |\lambda^-_n| + \sum_{m=1}^{\infty} |\delta^+_m| + \sum_{m=1}^{\infty} |\delta^-_m| = 1.
\]

We will show that \( u = v^+ = t = \lambda^+_n = \delta^+_m = 0 \) for every \( n, m \in \mathbb{N} \). Assume that \( \lambda_{n_0} \neq 0 \) for some \( n_0 \in \mathbb{N} \). It follows that

\[
1 = f(Q) = uf(P_1) + v^+ f(P^+_2) + v^- f(P^-_2) + tf(P_3) + \sum_{n=1}^{\infty} \lambda^+_n f(P^+_{4,c_n})
\]

\[
+ \sum_{n=1}^{\infty} \lambda^-_n f(P^-_{4,c_n}) + \sum_{m=1}^{\infty} \delta^+_m f(P^+_{5,a_m}) + \sum_{m=1}^{\infty} \delta^-_m f(P^-_{5,a_m})
\]

\[
\leq \frac{1}{2} |u| + \frac{1}{2} |v^+| + \frac{1}{2} |v^-| + \frac{1}{2} |t| + |\lambda^+_{n_0}| + \sum_{n \neq n_0} |\lambda^-_n| + \sum_{m=1}^{\infty} |\delta^+_m| + \sum_{m=1}^{\infty} |\delta^-_m| 
\]

\[
\leq 1 \quad (\text{by } (*), (**), (***)�)
\]

which is impossible. Therefore, \( \lambda^+_n = 0 \) for every \( n \in \mathbb{N} \). Using a similar argument as above, we have \( u = v^+ = t = \lambda^-_n = \delta^-_m = 0 \) for every \( n, m \in \mathbb{N} \). Therefore,

\[
Q(x, y) = \sum_{m=1}^{\infty} \delta^+_m P^+_{5,a_m}(x, y)
\]

We will show that if \( a^+_{m_0} \neq c \) for some \( m_0 \in \mathbb{N} \), then \( \delta^+_{m_0} = 0 \). Assume that
\[ \delta_{m_0}^+ \neq 0. \] It follows that
\[
1 = f(Q) = \delta_{m_0}^+ f(P_{5,a_{m_0}}^+) + \sum_{m \neq m_0} \delta_m^+ f(P_{5,a_m}^+).
\]
\[
< |\delta_{m_0}^+| + \sum_{m \neq m_0} |\delta_m^+| = 1
\]
which is impossible. Therefore, \( \delta_{m_0}^+ = 0. \) Therefore,
\[
Q(x, y) = \left( \sum_{a_m = 0} \delta_m^+ \right) P_{5,c}^+(x, y) = P_{5,c}^+(x, y).
\]
Therefore, \( f \) exposes \( P_{5,c}^+ \). Obviously, \( P_{5,c}^- \in \exp B_p(\mathbb{R}^2 h_{\frac{1}{2}}) \) for \( 0 < c < 1. \)

Therefore, we complete the proof. \( \blacksquare \)

**References**


[7] Y.S. Choi, S.G. Kim, Smooth points of the unit ball of the space \( \mathcal{P}(2l_1^2) \), *Results Math.* **36** (1999), 26–33.


