Three-operator Problems in Banach Spaces

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Received May 31, 2018

Abstract: We study the analogue of 3-space problems for classes of operators acting on Banach spaces. We show examples of classes of operators having or failing the 3-operator property, and give several methods to obtain classes with this property.

Key words: Three-space property, extending operators, lifting operators, semigroup, operator ideal.

AMS Subject Class. (2000): 46B03, 46B08, 46B10.

1. Introduction

The 3-space problem for a class or property $\mathcal{P}$ of Banach spaces is the question of whether a Banach space $X$ has $\mathcal{P}$ provided that a certain subspace $Y$ and the corresponding quotient space $X/Y$ have $\mathcal{P}$. If so, it is then said that $\mathcal{P}$ is a 3-space property. For instance, reflexivity, separability or having density character $\aleph_0$ are 3-space properties. Three-space problems have been a popular topic of research because a positive answer for $\mathcal{P}$ yields a technique to get spaces with $\mathcal{P}$; while a negative answer necessarily provides a new insight into Banach space constructions. Moreover, to get either positive results or counterexamples more often than not requires a blend of different techniques. The monograph [12] contains a thorough, not-too-outdated, treatment of the 3-space problem in Banach spaces.

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*The research of the first author was supported in part by Project IB16056 de la Junta de Extremadura; that of the first and third authors was supported in part by MINECO (Spain), Project MTM2016-76958. This paper benefited from a stay in 2016 of Castillo and González at Kanagawa University invited by Prof. Cho.
In this paper we consider the analogue of 3-space problem for operators by means of what we will call the 3-operator property (see Definition 1), which was introduced in [29]. We also consider some weak versions of the 3-operator property that were introduced in [13]. Some of them can be enjoyed by an operator ideal $A$ and maintain close connections with the fact that the space ideal of $A$ enjoys the 3-space property. We first observe that no (nontrivial) operator ideal can enjoy the 3-operator property, and so we turn our attention to other classes, most remarkably semigroups. Our main result (Theorem 1) gives a characterization of the semigroups that satisfy the 3-operator property. As a consequence we show in Corollary 1 that several semigroups considered in [13] have the 3-operator property. We also describe some categorical methods to obtain new classes with the 3-operator property from a class that have that property. Finally we consider some operator ideals related with the separably injective spaces studied in [5] that provide examples satisfying or failing some 3-operator-like properties considered in the paper.

2. Three-operator properties

A class $A$ of operators is called an operator ideal if it contains the class $F$ of finite-rank operators, is closed under addition, and the composition of an element of $A$ with any operator is in $A$. Typical examples of operator ideals are the classes $L$ of all operators, $K$ of compact operators and $W$ of weakly compact operators. A class $S$ of operators is a semigroup if it contains the bijective operators, it is stable under composition, and given $S \in L(U, X)$ and $T \in L(V, Y)$,

$$S, T \in S \iff S \circ T \in S,$$

where $S \circ T : U \oplus V \to X \oplus Y$ is defined by $S \circ T(u, v) = (Su, Tv)$.

The notion of operator ideal was thoroughly studied by Pietsch [24] (see [25] for details) while the notion of semigroup was considered in [1] and [18, Chapter 6].

An operator ideal $A$ has associated a class of Banach spaces $Sp(A) = \{X : id_X \in A\}$, where $id_X$ is the identity in $X$, which is called the space ideal of $A$ [24]. Space ideals are, according to [24], classes of Banach spaces containing the finite dimensional spaces and stable under products and complemented subspaces. Observe that $X \in Sp(A)$ if and only if $L(X, X) = A(X, X)$. Similarly, a semigroup $S$ has associated a class of spaces $ker(S) = \{X : 0_X \in S\}$, and $X \in ker(S)$ if and only if $L(X) = S(X)$. We will see in Proposition 1 that, for an operator ideal $A$, $Sp(A)$ coincides with the kernel of the semi-
groups $A_+$ and $A_-$ associated to $A$. Given a class of operators $A$, we denote $A^d = \{ T \in L : T^* \in A \}$, the dual class of $A$. Note that if $A$ is an operator ideal or a semigroup then so does $A^d$ [24, Theorem 4.4.2], [18, Proposition 6.1.4].

The homological notation will be useful in this paper: recall that an exact sequence

$$0 \rightarrow Y \rightarrow i \rightarrow X \rightarrow \pi \rightarrow Z \rightarrow 0 \quad (2.1)$$

of Banach spaces and (linear continuous) operators is a diagram in which the kernel of each arrow coincides with the image of the preceding one. The open mapping theorem yields that $Y$ is isomorphic to a subspace of $X$ such that $X/i(Y)$ is isomorphic to $Z$. Consequently, $P$ is a 3-space property if whenever one has a short exact sequence (2.1) in which the spaces $Y,Z$ have $P$ then also $X$ has $P$. Still according to [24], given a space ideal $A$, the class of all operators that factorize through a space in $A$ form an operator ideal $Op(A)$. Thus, if the spaces with $P$ form a space ideal (or even if not with some ad hoc amendments), $P$ is a 3-space property if given a commutative diagram

$$0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0$$

$$\begin{array}{ccc}
0 & \rightarrow & Y \\
\downarrow id_Y & & \downarrow id_X \\
0 & \rightarrow & X \\
\downarrow id_Y & & \downarrow id_Z \\
0 & \rightarrow & Z \\
\end{array} \quad (2.2)$$

with exact rows, if $id_Y, id_Z \in Op(P)$ then also $id_X \in Op(P)$. Consequently, the following concept makes sense.

**Definition 1.** A class $A$ of operators is said to have the 3-operator property if given a commutative diagram with exact rows

$$0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0$$

$$\begin{array}{ccc}
0 & \rightarrow & Y' \\
\downarrow \alpha & & \downarrow \beta \\
0 & \rightarrow & X' \\
\downarrow \gamma & & \downarrow \gamma \\
0 & \rightarrow & Z' \\
\end{array} \quad (2.3)$$

if $\alpha, \gamma \in A$ then also $\beta \in A$.

This notion was introduced by Zeng and Zhong in [29], where they prove that the classes of upper and lower semi-Fredholm operators satisfy it (a new proof will be given in Corollary 3), and Zeng proved in [28] that some classes of operators defined in terms of spectral properties satisfy the 3-operator property.
Unfortunately, $\mathcal{L}$ is the only operator ideal enjoying the 3-operator property: the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & X
\end{array}
\xrightarrow{id}
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array}
\]

shows that an operator ideal satisfying the 3-operator property contains the identity of every Banach space. Nevertheless, classes of operators with the 3-operator property do exist:

**Example 1.** The classical 3-lemma from homological algebra [12, p. 3] shows that the following classes of operators have the 3-space property:

(a) The class $\text{inj}$ of injective operators.
(b) The class $\text{dens}$ of dense range operators.
(c) The class $\text{emb}$ of (into) embedding operators.
(d) The class $\text{surj}$ of surjective operators.
(e) The class $\text{iso}$ of bijective operators.

Two “3-operator-like” properties were introduced in [13, Definitions 2 and 5 and Propositions 10 and 16].

**Definition 2.** Let $\mathcal{A}$ be a class of operators.

(a) We say that $\mathcal{A}$ satisfies the 3$S_-$ property if given a push-out diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & Y \\
\downarrow & \alpha & \downarrow \\
0 & \longrightarrow & X \\
\downarrow & \beta & \downarrow \\
0 & \longrightarrow & Z
\end{array}
\xrightarrow{j}
\begin{array}{ccc}
X & \longrightarrow & Z
\end{array}
\xrightarrow{q}
\begin{array}{ccc}
0 & \longrightarrow & 0
\end{array}
\]

with exact rows then $\alpha, q \in \mathcal{A} \Rightarrow \beta \in \mathcal{A}$.

(b) We say that $\mathcal{A}$ satisfies the 3$S_+$ property if given a pull-back diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & Y \\
\downarrow & \beta & \downarrow \\
0 & \longrightarrow & PB \\
\downarrow & \gamma & \downarrow \\
0 & \longrightarrow & Z
\end{array}
\xrightarrow{j'}
\begin{array}{ccc}
Y' & \longrightarrow & Z'
\end{array}
\xrightarrow{q'}
\begin{array}{ccc}
0 & \longrightarrow & 0
\end{array}
\]

with exact rows then $\gamma, j' \in \mathcal{A} \Rightarrow \beta \in \mathcal{A}$. 
While $\mathcal{L}$ is the only operator ideal that satisfies the 3-operator property, some operator ideals satisfy the $3S_-$ and $3S_+$ properties:

(a) The operator ideals of strictly singular and strictly cosingular operators enjoy respectively properties $3S_-$ and $3S_+$.

This fact is essentially proved in [14, Proposition 3.2], and explicitly in [10, Lemma 8] and [13, Proposition 11].

(b) The operator ideal of $p$-converging operators $C_p$ studied in [11] satisfies the $3S_-$ and $3S_+$ properties for $1 \leq p \leq \infty$ [13, Propositions 13 and 17].

Note that $C_\infty = C$, the completely continuous operators, and $C_1 = U$, the unconditionally converging operators.

(c) Let $\mathcal{K} \mathcal{W}, \mathcal{R}, \mathcal{U}, \mathcal{C}$ and $\mathcal{WC}$ denote the operator ideals of compact, weakly compact, Rosenthal, unconditionally convergent, completely continuous, and weakly completely continuous operators (see [24] for their definitions), and let $\mathcal{A}$ be one of these operator ideals. Then $\mathcal{A}$ satisfies the $3S_+$ property and its dual $\mathcal{A}^d$ satisfies the $3S_-$ property [13, Propositions 15 and 9].

The following result will be useful later:

**Lemma 1.** If $\mathcal{A}$ has the property $3S_-$ then $\mathcal{A}^d$ has the property $3S_+$; and similarly, if $\mathcal{A}$ has the property $3S_+$ then $\mathcal{A}^d$ has the property $3S_-$. 

**Proof.** For the proof of the first result, it is enough to observe that

\[
\begin{array}{cccccc}
0 & \longrightarrow & Z^* & \xrightarrow{q^*} & X^* & \xrightarrow{j^*} & Y^* & \longrightarrow & 0 \\
\downarrow{\gamma^*} & & \downarrow{\beta^*} & & \| & & \| \\
0 & \longrightarrow & Z^{**} & \longrightarrow & \text{PB}^* & \longrightarrow & Y^* & \longrightarrow & 0
\end{array}
\]

which is the conjugate of diagram (2.5), is also a push-out diagram like (2.4). Indeed, PB is the pull-back of $\gamma$ and $q$, and $\text{PB}^*$ can be identified with the push-out of $\gamma^*$ and $q^*$. We refer to [13, Section 2] for the details.

The proof of the second result is similar. □

Those properties are interesting for the study of 3-space problems. Indeed, as it was shown in [13, Proposition 18], if an operator ideal $\mathcal{A}$ satisfies one of the properties $3S_-$ or $3S_+$ then the space ideal $Sp(\mathcal{A})$ satisfies the 3-space property.
For applications of the $3S_+$ and $3S_-$ properties to the analysis of commutative diagrams of operators we refer to [13, Propositions 14 and 19] and [14, Proposition 3.3].

3. Semigroups of operators

The classes of operators appearing in Example 1 satisfy the definition of semigroup presented at the beginning of Section 2. This notion of semigroup if closely related with that of operator ideal. It was proved in [1] and [18, Chapter 6] that every operator ideal $\mathcal{A}$ has associated two semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$ defined as follows.

**Definition 3.** Let $\mathcal{A}$ be an operator ideal and let $T \in L(X,Y)$.

(a) $T \in \mathcal{A}_+$ if for every $S \in L(Z,X)$, $TS \in \mathcal{A}$ implies $S \in \mathcal{A}$.

(b) $T \in \mathcal{A}_-$ if for every $S \in L(Y,Z)$, $ST \in \mathcal{A}$ implies $S \in \mathcal{A}$.

As a direct consequence of the definitions we obtain the following equalities.

**Proposition 1.** For every operator ideal $\mathcal{A}$, $Sp(\mathcal{A}) = \ker(\mathcal{A}_+) = \ker(\mathcal{A}_-)$.  

The following two 3-operator-like properties for semigroups were introduced in [13, Definition 1] in a slightly different way.

**Definition 4.** Let $\mathcal{A}$ be a class of operators. We say that:

(a) $\mathcal{A}$ satisfies the 3 PO property if given a push-out diagram like (2.4), if $\alpha \in \mathcal{A}$ then also $\beta \in \mathcal{A}$.

(b) $\mathcal{A}$ satisfies the 3 PB property if given a pull-back diagram like (2.5), if $\gamma \in \mathcal{A}$ then also $\beta \in \mathcal{A}$.

As in the case of the 3-operator property, an operator ideal satisfying the 3 PO property or the the 3 PB property contains the identity of every Banach space, hence it is $L$. For the 3 PO property, note that we can construct diagrams like (2.4) with $Z$ arbitrary; and we can similarly argue for the 3 PB property.
Proposition 2. Let $\mathcal{A}$ be an operator ideal.

(a) If $\mathcal{A}$ is injective then $\mathcal{A}^+$ satisfies the 3 PB property.

(b) If $\mathcal{A}$ is surjective then $\mathcal{A}^-$ satisfies the 3 PO property.

(c) If $\mathcal{A}^+$ satisfies the 3 PO property or $\mathcal{A}^-$ satisfies the 3 PB property then $\text{Sp}(\mathcal{A})$ has the 3-space property.

Proof. (a) Suppose $\mathcal{A}$ is injective and $\gamma$ in diagram (2.5) belongs to $\mathcal{A}^+$. We have to show that $\beta \in \mathcal{A}^+$. Let $S : W \to \text{PB}$ such that $\beta S \in \mathcal{A}$. From $\gamma \in \mathcal{A}^+$ and $q\beta S = \gamma q'S \in \mathcal{A}$, we get $q'S \in \mathcal{A}$. The map $J : \text{PB} \to X \oplus Z'$ given by $Jv = (\beta v, q'v)$ is an (into) embedding, and $\beta S, q'S \in \mathcal{A}$ implies $JS \in \mathcal{A}$; hence $S \in \mathcal{A}$ because $\mathcal{A}$ is injective.

(b) Suppose $\mathcal{A}$ is surjective and $\alpha$ in diagram (2.4) belongs to $\mathcal{A}^-$. We have to show that $\beta \in \mathcal{A}^-$. Let $S : \text{PO} \to W$ such that $S\beta \in \mathcal{A}$. From $\alpha \in \mathcal{A}^-$ and $S\beta \mathcal{A} = Sj\alpha \in \mathcal{A}$, we get $Sj \in \mathcal{A}$. The map $Q : Y' \oplus X \to \text{PO}$ given by $Q(y', x) = j'y' + \beta x$ is surjective, and $S\beta, Sj \in \mathcal{A}$ implies $SQ \in \mathcal{A}$; hence $S \in \mathcal{A}$ because $\mathcal{A}$ is surjective.

(c) See [13, Proposition 7].

Clearly the 3 PO property implies the 3$S_+$, the 3 PB property implies the 3$S_-$ property, and Lemma 1 has its counterpart admitting a similar proof:

Lemma 2. If $\mathcal{A}$ has the 3 PB property then $\mathcal{A}^d$ has the 3 PO property; and similarly, if $\mathcal{A}$ has the 3 PO property then $\mathcal{A}^d$ has the 3 PB property.

The following result is useful to understand the 3-operator property.

Theorem 1. Let $\mathcal{A}$ be a class of operators stable by composition and containing the bijective operators. Then $\mathcal{A}$ has the 3-operator property if and only if it has properties 3 PO and 3 PB.

Proof. The direct implication is trivial, because each identity is in $\mathcal{A}$. For the converse implication, given the commutative diagram in Definition 1

$$
\begin{array}{c}
0 \longrightarrow Y \overset{j}{\longrightarrow} X \overset{q}{\longrightarrow} Z \longrightarrow 0 \\
\alpha \downarrow \quad \beta \downarrow \quad \gamma \\
0 \longrightarrow Y' \overset{j'}{\longrightarrow} X' \overset{q'}{\longrightarrow} Z' \longrightarrow 0
\end{array}
$$

(3.1)
we consider the push-out diagram of $\alpha$ and $j$:

\[
\begin{array}{c}
0 \rightarrow Y \xrightarrow{j} X \xrightarrow{q} Z \rightarrow 0 \\
\downarrow \quad \downarrow \pi \quad \downarrow \ \\
0 \rightarrow Y' \xrightarrow{j'} PO \xrightarrow{q'} Z \rightarrow 0
\end{array}
\]

where

\[
PO = (Y' \times X)/\Delta \quad \text{with} \quad \Delta = \{(\alpha y, -jy) : y \in Y\},
\]

\[
\tilde{j}y' = (y', 0) + \Delta, \quad \tilde{\alpha}x = (0, x) + \Delta \quad \text{and} \quad \tilde{\eta}((y', 0) + \Delta) = qx.
\]

Moreover we consider the pull-back diagram of $\gamma$ and $q'$:

\[
\begin{array}{c}
0 \rightarrow Y' \xrightarrow{j'} PB \xrightarrow{q'} Z \rightarrow 0 \\
\downarrow \quad \downarrow \gamma \quad \downarrow \ \\
0 \rightarrow Y' \xrightarrow{j'} X' \xrightarrow{q'} Z' \rightarrow 0
\end{array}
\]

where

\[
PB = \{(x', z) \in X' \times Z : q'x' = \gamma z\},
\]

\[
\tilde{j}'y' = (j'y', 0), \quad \gamma(x', z) = x' \quad \text{and} \quad q'(x', z) = z.
\]

Let us show that the map $\Psi : PO \rightarrow PB$ defined by

\[
\Psi((y', x) + \Delta) = (j'y' + \beta x, qx)
\]

is a bijective operator such that $\beta = \gamma \Psi \pi$. We do it in several steps:

(1) $\Psi$ takes values in PB: $q'(j'y' + \beta x) = q'\beta x = \gamma qx$.

(2) $\Psi$ is well-defined since $(y', x) \in \Delta$ implies $y' = \alpha y$ and $x = -jy$ for some $y \in Y$. Then $j'y' + \beta x = j'y + \beta jy = 0$ and $qx = -qjy = 0$.

(3) $\Psi$ is bijective because the diagram

\[
\begin{array}{c}
0 \rightarrow Y' \xrightarrow{j'} PO \xrightarrow{\eta} Z \rightarrow 0 \\
\downarrow \Psi \quad \downarrow \ \\
0 \rightarrow Y' \xrightarrow{j'} PB \xrightarrow{q'} Z \rightarrow 0
\end{array}
\]
is commutative. Indeed,
\[
q' \Psi((y', x + \Delta)) = q'(j'y' + \beta x, qx) = qx = \eta((y', x + \Delta))
\]
and
\[
\Psi j'y' = \Psi((y', 0) + \Delta) = (j'y', 0) = j'y'.
\]

(4) For every \( x \in X \),
\[
\hat{\gamma} \Psi \pi x = \hat{\gamma} \Psi((0, x) + \Delta) = \hat{\gamma}(\beta x, qx) = \beta x.
\]

To conclude the proof, since \( A \) has properties 3 PO and 3 PB, \( \alpha \in A \) implies \( \pi \in A \) and \( \gamma \in A \) implies \( \gamma \in A \), hence \( \beta = \hat{\gamma} \Psi \pi \in A \).

Several examples of semigroups satisfying the 3 PO or the 3 PB property were given in [13].

**Proposition 3.** ([13, Theorem 1]). Let \( A \) denote one of the operator ideals \( K, W, R, U, C \) or \( WC \). Then the semigroup \( A_+ \) satisfies the 3 PO property and the semigroup \( (A_+\d) \) satisfies the 3 PB property.

Note that \( K = K_\d \) by Schauder’s theorem and \( W = W_\d \) by Gantmacher’s theorem. As a consequence,

**Corollary 1.** Let \( A \) denote one of the operator ideals \( K, W, R, U, C \) or \( WC \). Then the semigroups \( A_+ \) and \( (A_+)\d \) satisfy the 3-operator property.

**Proof.** The operator ideal \( A \) is injective, hence the semigroup \( A_+ \) satisfies the 3 PB property by Proposition 2. Since \( A_+ \) also satisfies the 3 PO property (Proposition 3), it has the 3-operator property by Theorem 1.

Similarly, since \( A_\d \) is surjective and \( (A_\d)_+ \) satisfies the 3 PB property, \( (A_\d)_- \) has the 3-operator property. 

Since \( C_\d \) is injective (hence \( C_\d_+ \) is surjective), \( (C_\d)_- \) satisfies the 3 PB property and \( (C_\d)_- \) the 3 PO property. However the following questions remain open:

(a) Do the semigroups \( (C_\d)_+ \) and \( (C_\d)_- \) satisfy the the 3-operator property?

(b) Does \( (C_\d)_+ \) satisfy the 3 PO property, or \( (C_\d)_- \) satisfy the 3 PB property?

For the obtention of lifting result for sequences when the quotient map belongs to one of the semigroups \( W_+, R_+, C_+ \) or \( WC_+ \) we refer to [19].
4. Categorical methods to obtain 3-operator classes

A functor $F$ acting on the category of Banach spaces is said to be exact if it transforms exact sequences into exact sequences. Examples of exact functors can be found in [12, Section 2.2] and include the following ones:

- The duality functor, $F(X) = X^*$ and $F(T) = T^*$ (as well as the Biduality functor, $\alpha$-transfinite dual, etc).
- The residual functor, $F(X) = X^{**}/X$ and $F(T) = T^{**}/T$, where if $T : X \to Y$ then $T^{**}/T : X^{**}/X \to Y^{**}/Y$ is the operator induced by $T^{**}$.
- The ultrapower functor, $F(X) = X_{\mathcal{U}}$ and $F(T) = T_{\mathcal{U}}$, where $\mathcal{U}$ is a non-trivial ultrafilter.
- The ultra-residual functor, $F(X) = X_{\mathcal{U}}/X$ and $F(T) = T_{\mathcal{U}}/T$.
- The $C(K, \cdot)$ functor, $K$ a compact space: $C(K, X)$ is the Banach space of all continuous functions $f : K \to X$, and for each $T : X \to Y$, the operator $C(K, T) : C(K, X) \to C(K, Y)$ is defined by $C(K, T)(f)(k) = T(f(k))$.

One obviously has:

**Proposition 4.** Let $\mathcal{A}$ be a class of operators with the 3-operator property and let $F$ be an exact functor. Then the class

$$F^{-1}(\mathcal{A}) = \{T : F(T) \in \mathcal{A}\}$$

has the 3-operator property.

Let us show that nontrivial results can be obtained with this method. It was proved in [20] that $W_+$ coincides with the class of tauberian operators introduced in [22] and $W_-$ coincides with the class of cotauberian operators introduced in [27]. Since (see [18]),

$W_+ = \{T : T^{**}/T \in \mathbf{inj}\}$ and $W_- = \{T : T^{**}/T \in \mathbf{dens}\}$

one has

**Corollary 2.** The semigroups $W_+$ of tauberian and $W_-$ of cotauberian operators have the 3-operator property.
It was proved in [3] that \((W_+)^{dd} = \{T : T^{**} \in W_+\}\) is properly contained in \(W_+\). It follows from Proposition 4 and Corollary 2 that \((W_+)^{dd}\) has the 3-operator property.

In [21] (see also [4]) the operators \(S\) that can be represented as \(T^{**}/T\) for some operator \(T\) are studied. Recall that many Banach spaces (such as separable of weakly compactly generated [8]) are linearly isometric to some \(X^{**}/X\). The operators \(T\) such that \(T^{**}/T \in \text{emb}\) are called strongly tauberian in [26]. In a similar way as in the case of tauberian operators, it can be shown that the class of strongly tauberian operators has the 3-operator property.

The semigroup \(\mathcal{K}_+\) coincides with the class of upper semi-Fredholm operators (operators with closed range and finite dimensional kernel) while \(\mathcal{K}_-\) coincides with the class of lower semi-Fredholm operators (operators with closed and finite codimensional range) (see [17]). Since

\[
\mathcal{K}_+ = \{T : T_{\#}/T \in \text{inj}\} = \{T : T_{\#}/T \in \text{emb}\}
\]

\[
\mathcal{K}_- = \{T : T_{\#}/T \in \text{dens}\} = \{T : T_{\#}/T \in \text{surj}\}
\]

one has

\textbf{Corollary 3.} The semigroups \(\mathcal{K}_+\) and \(\mathcal{K}_-\) have the 3-operator property.

As in Corollaries 2 and 3, it is possible to show other classes with the 3-operator property by applying exact functors to known classes that satisfy that property. It would be interesting to identify some of them with known classes of operators.

\textbf{5. Separable injectivity revisited}

The \textit{density character} of a Banach space \(X\), \(\text{dens}(X)\), is the smallest cardinal \(\aleph\) for which \(X\) has a subset of cardinality \(\aleph\) spanning a dense subspace.

\textbf{Definition 5.} Let \(\aleph\) be a cardinal.

A Banach space \(X\) is said to be \(\aleph\)-\textit{injective} if every operator \(t : Y \to X\) admits an extension \(T : E \to X\) to a superspace \(E \supset Y\) whenever \(\text{dens}(E) < \aleph\).

The space \(X\) is said to be \(\textit{universally }\aleph\text{-injective} if every operator \(t : Y \to X\) admits an extension \(T : E \to X\) to a superspace \(E \supset Y\) whenever \(\text{dens}(Y) < \aleph\).
The separably injective and universally separably injective Banach spaces (corresponding to the choice $\aleph = \aleph_1$) have been recently studied in the monograph [5]. We present now an operator approach to those ideas.

**Definition 6.** Let $\aleph$ be a cardinal and let $T : X \to Y$ be an operator.

(i) $T \in \mathcal{E}_0(\aleph)$ if for every super-space $E \supseteq X$ with $\text{dens}(E/X) < \aleph$ there exists an extension $\widetilde{T} : E \to Y$.

(ii) $T \in \mathcal{E}(\aleph)$ if for every subspace $M \subseteq X$ with $\text{dens}(M) < \aleph$ and every superspace $E \supseteq M$, the restriction $T|_M : M \to Y$ admits an extension $\widetilde{T}|_M : E \to Y$.

**Proposition 5.** The classes $\mathcal{E}_0(\aleph)$ and $\mathcal{E}(\aleph)$ are operator ideals.

**Proof.** It is obvious that $\mathcal{E}_0(\aleph) + \mathcal{E}_0(\aleph) \subseteq \mathcal{E}_0(\aleph)$, $\mathcal{E}(\aleph) + \mathcal{E}(\aleph) \subseteq \mathcal{E}(\aleph)$, and both classes contain the finite rank operators. We prove that given $T \in \mathcal{E}_0(\aleph)(X, X')$, $R \in \mathcal{L}(X', Y')$ and $S \in \mathcal{L}(Y, X)$ one gets $RTS \in \mathcal{E}_0(\aleph)(Y, Y')$.

Let $E$ be a superspace of $Y$ such that $\text{dens}(E/Y) < \aleph$ and consider the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & E/Y & \longrightarrow & 0 \\
\downarrow S & & \downarrow s & & \parallel & & \parallel & & \\
0 & \longrightarrow & X & \longrightarrow & PO & \longrightarrow & E/Y & \longrightarrow & 0 \\
\downarrow T & & \downarrow \quad & & \quad & & \quad & & \quad \\
& & X' & & & & & & \\
\downarrow R & & \downarrow & & \downarrow & & \downarrow & & \\
& & X' & & & & & & 
\end{array}
\]

Since $T \in \mathcal{E}_0$, it admits an extension $t : PO \to X'$, thus the operator $RTS$ admits the extension $Rts$. The proof for $\mathcal{E}(\aleph)$ is analogous.

Our interest in these uncommon operator ideals appears explained in the next result.

**Proposition 6.** Let $\aleph$ be a cardinal.

(a) $X \in \text{Sp}(\mathcal{E}_0(\aleph))$ if and only if $X$ is $\aleph$-injective.

(b) $X \in \text{Sp}(\mathcal{E}(\aleph))$ if and only if $X$ is universally $\aleph$-injective.
Proof. It is clear that $Id_X \in \mathcal{E}_0(\aleph)$ if and only if $X$ is complemented in every superspace $E$ so that $\text{dens}(E/X) < \aleph$; namely, $X$ is $\aleph$-injective. It is also clear that if $X$ is universally $\aleph$-injective then $Id_X \in \mathcal{E}(\aleph)$. To prove the converse, let $Y$ be a Banach space with $\text{dens}(Y) < \aleph$ which is a subspace of a space $E$, and let $t : Y \to X$ be an operator. Consider the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & E/Y & \longrightarrow & 0 \\
\downarrow t & & \downarrow t' & & \| & & \| & & \\
0 & \longrightarrow & \overline{t(Y)} & \longrightarrow & \text{PO} & \longrightarrow & E/Y & \longrightarrow & 0 \\
\downarrow i & & \| & & \| & & \| & & \\
 & & X & & & & & & \\
\end{array}
\]

Since $\text{dens}(\overline{t(Y)}) \leq \text{dens}(Y)$, the canonical inclusion $i$ can be extended to PO, therefore $t$ can be extended to $E$, which shows that $X$ is universally $\aleph$-injective. \[\square\]

It is therefore clear that $\mathcal{E}_0(\aleph)$ (resp. $\mathcal{E}(\aleph)$) are non-injective operator ideals containing (although probably different from) the class of all operators that factorize through an $\aleph$-injective (resp. universally $\aleph$-injective) space. They are likely not to be surjective either. More ad hoc versions of these operator ideals have been introduced and studied by Domański [16]: he fixes a Banach space $Z$ and considers the ideals $E_Z$ of those operators $T : X \to Y$ such that for every super-space $E \supset X$ with $E/X \simeq Z$ there exists an extension $\widetilde{T} : E \to Y$. Therefore, it will turn out (see [5] for details) that $\mathcal{E}_0(\aleph) = \bigcup_Z E_Z$ when the intersection runs over all spaces $Z$ with density character $< \aleph$. One has

**Proposition 7.** The class $\mathcal{E}_0(\aleph)$ satisfies the $3S_\rightarrow$ property.

**Proof.** We consider a push-out diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\
\downarrow \alpha & & \downarrow \beta & & \| & & \| & & \\
0 & \longrightarrow & Y' & \longrightarrow & \text{PO} & \longrightarrow & Z & \longrightarrow & 0 \\
\end{array}
\]

with $q, \alpha \in \mathcal{E}_0(\aleph)$, and we have to show that $\beta \in \mathcal{E}_0(\aleph)$. Let $X'$ be a superspace of $X$ so that $\text{dens}(X'/X) < \aleph$. Our goal is to extend $\beta$ to an operator $B : X' \to \text{PO}$. 

Let $Q : X' \to Z$ be an extension of $q$. The commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \ker Q & \longrightarrow & X' & \longrightarrow & Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ker Q/Y & \longrightarrow & X'/X & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & & 0 & & 0 & & 0
\end{array}
\]

shows that $\ker Q$ is a superspace of $Y$ with $\text{dens}(\ker Q/Y) < \aleph_0$. Let $A : \ker Q \to Y'$ be an extension of $\alpha$. Given $x' \in X'$ pick $x \in X$ so that $Qx' = qx$ and define $B : X' \to \text{PO}$ by means of

\[
B(x') = (A(x' - x), x) + \Delta.
\]

To check it is well defined, observe that if $Q(x') = q(x) = q(w)$ then

\[
(A(x' - x), x) - (A(x' - w), w) = (A(w - x), x - w) \in \Delta
\]

since $x - w \in Y$. The operator $B$ is continuous since

\[
\|Bx'\| = \inf_{y \in Y} \|(A(x' - x), x) - (Ay, -y)\| = \inf_{y \in Y} \|(A(x' - x - y), x + y)\| \\
\leq \inf_{y \in Y} \|A(x' - x - y)\| + \|x + y\| \\
\leq \|Ax'\| + 2\|qx\| \\
\leq \|A\|\|x'\| + 2\|Qx'\| \\
\leq (\|A\| + 2\|Q\|)\|x'\|.
\]

Finally, $B$ is an extension of $\beta$ since when $x \in X$ one has

\[
B(x) = (A(x - x), x) + \Delta = (0, x) + \Delta = \beta(x).
\]
We do not know however if the operator ideal $E_0$ satisfies the $3S_+$. Thus, by the result [13, Proposition 18] above mentioned, we get that the class $Sp(E_0(\aleph))$ of $\aleph$-injective Banach spaces has the 3-space property. This fact is well-known (see [6, 5]). The homological argument can be found in [9]: every property having the form $\text{Ext}(\cdot, X) = 0$ or $\text{Ext}(X, \cdot) = 0$ is a 3-space property. A forerunner can be found in [15]. By Lemma 1, the class $E_0(\aleph)^d$ has property $3S_+$, from where it follows also that $Sp(E_0(\aleph)^d)$ has the 3-space property, something we already knew since it is a standard fact that if $Sp(A)$ has the 3-space property then also $Sp(A^d)$ has the 3-space property. Moreover, to identify the class $Sp(E_0(\aleph)^d)$ is easy and, surprisingly, the class turns out to be independent of $\aleph$:

**Proposition 8.** For every $\aleph$, $Sp(E_0(\aleph)^d)$ is the class of $L_1$-spaces.

*Proof.* Recall [5] that $\aleph$-injective spaces are $L_\infty$-spaces and that dual $L_\infty$-spaces are injective. Thus, $X^*$ is $\aleph$-injective if and only if it is an $L_\infty$-space, which occurs if and only if $X$ is an $L_1$-space.

Curiously enough, if one defines the apparently dual classes $L(\aleph)$ of operators that can be lifted to any superquotient having kernel with density character strictly lesser than $\aleph$ the identification of $Sp(L(\aleph))$ is not so simple. Indeed, observe that a Banach space in $Sp(L(\aleph))$ must be $\aleph$-projective, with the obvious meaning that $\text{Ext}(X, Y) = 0$ for every Banach space $Y$ with $\text{dens}(Y) < \aleph$. It is clear that a separable separably projective must be $\ell_1$. It is also clear that an $\aleph$-projective space is such that any $\aleph$-projective space must be an $L_1$-space with the Schur property (see [5] for details). There are however uncountably many non-mutually isomorphic $L_1$-subspaces of $\ell_1$. It is likely that that the answer to the following question be positive.

**Question 1.** Is a separably projective space projective? Equivalently [24, Theorem C.3.8], is a separably projective space isomorphic to $\ell_1(I)$ for some set $I$?

Turning to the main topic of this paper, observe that the case of the ideal $E(\aleph)$ is quite different form that of $E_0(\aleph)$ since, surprisingly, one has

**Lemma 3.** $E(\aleph)$ does not enjoy either $3S_-$ or $3S_+$.

*Proof.* Otherwise, the associated space class $Sp(E_0(\aleph))$ of universally $\aleph$-injective spaces would enjoy the 3-space property, something that, under CH, has been shown to be false in [7].
References


