A Note on a Paper of S.G. Kim

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Abstract: We answer a question posed by S.G. Kim in [3] and show that some of the results of his paper are immediate consequences of known results.

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The recent paper [3] deals with extreme multilinear forms and polynomials and the constants of the Bohnenblust-Hille inequalities. In this note we answer a question posed in [3] and show that two theorems stated in [3] are immediate corollaries of well known results of this field.

Let $K$ be $\mathbb{R}$ or $\mathbb{C}$. The multilinear Bohnenblust-Hille inequality asserts that, given a positive integer $m$, there is an optimal constant $C (m : K) \geq 1$ such that

$$\left( \sum_{i_1, \ldots, i_m = 1}^{\infty} |U(e_{i_1}, \ldots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C (m : K) \|U\|,$$

for all bounded $m$-linear forms $U : c_0 \times \cdots \times c_0 \to K$. The case of complex scalars was first investigated in [1] and the case of real scalars seems to have been just explored more recently. It is well known that the exponent $2m/(m+1)$ is sharp, so one of the main goals of the research in this field is to investigate the constants involved. The following result was proved in [4]:

**Theorem 1.** ([4, Corollary 5.4, 2018]) Let $m \geq 2$ be a positive integer. If the optimal constant $C (m : \mathbb{R})$ is attained in a certain $T : c_0 \times \cdots \times c_0 \to \mathbb{R}$, then the quantity of non zero monomials of $T$ is bigger than $4^{m-1} - 1$.

As an immediate corollary we conclude that if $N_1, \ldots, N_m \geq 1$ are positive integers such that

$$\prod_{j=1}^{m} N_j \leq 4^{m-1} - 1,$$

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then
\[
\sup \left( \sum_{i_1, \ldots, i_m = 1}^{N_1, \ldots, N_m} \left| T(e_{i_1}, \ldots, e_{i_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} < C(m : \mathbb{R}),
\]
where the sup runs over all norm one $m$-linear forms $T : \ell_\infty^{N_1} \times \ldots \times \ell_\infty^{N_m} \to \mathbb{R}$. In particular,
\[
\sup \left( \sum_{i,j,k=1}^{2} \left| T(e_i, e_j, e_k) \right|^6 \right)^{\frac{1}{6}} < C(3 : \mathbb{R}),
\]
where the sup runs over all norm one $m$-linear forms $T : \ell_2^{N_1} \times \ell_2^{N_2} \times \ell_2^{N_3} \to \mathbb{R}$, and this is the content of [3, Theorem 4.9].

The polynomial Bohnenblust-Hille inequality for real scalars asserts that, given a positive integer $m$, there is an optimal constant $C_p(m : \mathbb{R}) \geq 1$ such that
\[
\left( \sum_{|\alpha| = m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_p(m : \mathbb{R}) \|Q\|,
\]
for all $N \geq 1$ and for all $m$-homogeneous polynomials $Q : \ell_\infty^N(\mathbb{R}) \to \mathbb{R}$ given by
\[
Q(z) = \sum_{|\alpha| = m} a_\alpha z^\alpha.
\]

To the best of our knowledge, the case of real scalars became unexplored until the publication of the paper [2] in 2015, where it is proved that the constants $C_p(m : \mathbb{R})$ cannot be chosen with a sub-exponential growth. More precisely,

**Theorem 2. ([2, Theorem 2.2], 2015)**

\[
C_p(m : \mathbb{R}) > \left( \frac{\sqrt{2/3}}{\sqrt{5}} \right)^m > (1.177)^m,
\]
for all positive integers $m \geq 2$.

The above result is, obviously, by far, rather precise than [3, Theorem 4.5], which states that
\[
C_p(m : \mathbb{R}) \geq 2^{\frac{m+1}{2m}},
\]
for all positive integers $m \geq 2$. The only case that deserves a little bit more of attention is the case $m = 2$, since

$$\left( \frac{2\sqrt{3}}{\sqrt{5}} \right)^2 < 2^\frac{3}{4},$$

but in the case $m = 2$ a quick look at the proof of [2, Proof of Theorem 2.2] shows that

$$C_p(2 : \mathbb{R}) \geq \frac{3^\frac{3}{4}}{4} \approx 1.8236 > 2^\frac{3}{4},$$

and this answers in the negative the Question (2) posed by the author in [3, Question (2)].

References


