On a $\rho_n$-Dilation of Operator in Hilbert Spaces \textsuperscript{†}

A. SALHI, H. ZEROUALI

PB 1014, Departement of Mathematics,
Sciences Faculty, Mohamed V University in Rabat, Rabat, Morocco
radi237@gmail.com, zerouali@fsr.ac.ma

Presented by Mostafa Mbekhta Received March 21, 2016

Abstract: In this paper we define the class of $\rho_n$—dilations for operators on Hilbert spaces. We give various properties of this new class extending several known results $\rho$—contractions. Some applications are also given.

Key words: $\rho_n$—dilation, $\rho$—dilation.


1. Introduction

Sz-Nagy and Foias introduced in [8], the subclass $C_\rho$ of the algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators on a given complex Hilbert space $\mathcal{H}$. More precisely, for each fixed $\rho > 0$, an operator $T \in C_\rho$ if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a unitary transformation $U$ on $\mathcal{K}$ such that:

$$T^n = \rho P_r U^n_{|\mathcal{H}} \quad \text{for all } n \in \mathbb{N}^*.$$ (1)

Where $P_r : \mathcal{K} \to \mathcal{H}$ is the orthogonal projection on $\mathcal{H}$. The unitary operator $U$ is then called a unitary $\rho$—dilation of $T$, and the operator $T$ is a $\rho$—contraction.

Recall that $T$ is power bounded if $\|T^n\| \leq M$ for some fixed $M$ and every nonnegative integer $n$. From Equation (1), it follows that every $\rho$—contraction is power bounded since $\|T^n\| \leq \rho$ for all $n \in \mathbb{N}^*$. Computing the spectral radius of $T$, it comes that the spectrum of the operator $T$ satisfies $\sigma(T) \subset \overline{D}$, where $D = D(0,1)$ is the open unit disc of the set of complex numbers $\mathbb{C}$.

Operators in the class $C_\rho$ enjoy several nice properties, we list below the most known, we refer to [7] for proofs and further information.

\textsuperscript{†}This work is partially supported by Hassan II Academy of Sciences and the CNRST Project URAC 03.
The function $\rho \mapsto C_\rho$ is nondecreasing, that is $C_\rho \subsetneq C_{\rho'}$ if $\rho < \rho'$. We will denote by $C_\infty = \bigcup_{\rho > 0} C_\rho$.

$C_1$ coincides with the class of contractions (see [6]) and $C_2$ is the class of operators $T$ having a numerical radius less or equal to 1 (see [1]). The numerical radius is given by the expression, $w(T) = \sup\{|\langle Th; h\rangle| : \|h\| = 1\}$.

If $T \in C_\rho$ so is $T^n$. It is however not true in general that the product of two operators in $C_\rho$ is in $C_\rho$. Also it is not always true that $\xi T$ belongs to $C_\rho$ when $T \in C_\rho$ for $|\xi| \neq 1$.

For any $M$ a $T$–invariant subspace, the restriction of $T$ to the subspace $M$ is in the class $C_\rho$ whenever $T$ is.

Any operator $T$ such that $\sigma(T) \subset \mathbb{D}$ belongs to $C_\infty$.

Numerous papers have devoted to the study of different aspects of $C_\rho$; we refer to [2, 4, 5] for more information.

The next theorem provides a useful characterization of the class $C_\rho$ in terms of some positivity conditions.

**Theorem 1.1.** Let $T$ be a bounded operator on the Hilbert space $\mathcal{H}$ and $\rho$ be a nonnegative real. The following are equivalent

1. The operator $T$ belongs to the class $C_\rho$;

2. for all $h \in \mathcal{H}; z \in D(0; 1)$
   
   \[
   \left(\frac{2}{\rho} - 1\right)\|zTh\|^2 + \left(2 - \frac{2}{\rho}\right)\text{Re}(zTh, h) \leq \|h\|^2; \quad (2)
   \]

3. for all $h \in \mathcal{H}; z \in D(0; 1)$
   
   \[
   (\rho - 2)\|h\|^2 + 2\text{Re}((I - zT)^{-1}h, h) \geq 0. \quad (3)
   \]

2. **Unitary $\rho_n$-dilation**

We extend the notion of $\rho$-contractions to a more general setting. More precisely, let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence of nonnegative numbers. We will say that the operator $T$ on a complex Hilbert space $\mathcal{H}$ belongs to the class $C_{\rho_n}$ if, there
exists a Hilbert space $K$ containing $H$ as a subspace and a unitary operator $U$ such that
\[
T^n = \rho_n P r U^n_{|H} \text{ for all } n \in \mathbb{N}^*.
\] (4)

We say in this case that the unitary operator $U$ is a $\rho_n$-dilation for the operator $T$ and the operator $T$ will be called a $\rho_n$-contraction.

**Remark 2.1.**

1. For any bounded operator $T$, the operator $\frac{T}{\|T\|}$ is a contraction and hence admits a unitary dilation. We deduce that,
\[
T \in C_{\rho_n} \text{ for } \rho_n = \|T\|^n \text{ for all } n \in \mathbb{N}.
\]
We notice at this level that, without additional restrictive assumptions on the sequence $(\rho_n)_{n \in \mathbb{N}}$, there is no hope to construct a reasonable $\rho_n$-dilation theory. Our goal will be to extend the most useful properties of $\rho$-contraction to this more general setting.

2. From Equation 4, for $T \in C_{\rho_n}$ with $U$ a $\rho_n$-dilation, we obtain
\[
\|T^n\| \leq \|\rho_n P r U^n_{|H}\| \leq \rho_n.
\]
Therefore the condition $\lim_{n \to \infty} (\rho_n)^\frac{1}{n} \leq 1$ will ensure that $\sigma(T) \subseteq \overline{D(0;1)}$.

3. In contrast with the class $C_{\rho}$, the class $C_{(\rho_n)}$ is not stable by powers. However, if $T \in C_{\rho_n}$ and $k \geq 1$ is a given integer, we obtain $T^k \in C_{\rho_{kn}}$. This latter fact can be seen as a trivial extension of the case $\rho_n = \rho_0$ for every $n$.

In the remaining part of this paper, we will assume that $(\rho_n)_{n \in \mathbb{N}}$ is a sequence of nonnegative numbers satisfying
\[
\lim_{n \to \infty} (\rho_n)^\frac{1}{n} \leq 1.
\] (5)

We associate with the sequence $(\rho_n)_{n \in \mathbb{N}}$, the following function,
\[
\rho(z) = \sum_{n \geq 0} \frac{z^n}{\rho_n}.
\]
It is easy to see that condition $\lim_{n \to \infty} (\rho_n)^\frac{1}{n} \leq 1$ implies that $\rho \in \mathcal{H}(D)$. Here $\mathcal{H}(D)$ is the set of holomorphic functions on the open unit disc $D$. Also, the valued-operators function
\[
\rho(zT) = \sum_{n \geq 0} \frac{z^n T^n}{\rho_n}
\]
is well defined and converges in norm for every $|z| < 1$.

We give next a necessary and sufficient condition to the membership to the class $C_{\rho_n}$:

**THEOREM 2.2.** Let $T$ be an operator on a Hilbert space $H$ and $(\rho_n)_{n \in \mathbb{N}}$ is a sequence of nonnegative numbers. The operator $T$ has a $\rho_n$-dilation if and only if

$$
(1 - \frac{2}{\rho_0})||h||^2 + 2 \text{Re} \langle \rho(zT)(h); h \rangle \geq 0 \text{ for all } h \in H; z \in D(0; 1).
$$

(6)

We recall first the next well known lemma from [7, Theorem 7.1] that will be needed in the proof of the previous theorem.

**LEMMA 2.3.** Let $H$ be a Hilbert space, $G$ be a multiplicative group and $\Psi$ be an operator valued function $s \in G \mapsto \Psi(s) \in \mathcal{L}(H)$ such that

$$
\begin{cases}
\Psi(e) = I, \ e \text{ is the identity element of } G \\
\Psi(s^{-1}) = \Psi(s)^* \\
\sum_{s \in G} \sum_{t \in G} (\Psi(t^{-1}s)h(s); h(t)) \geq 0
\end{cases}
$$

for finitely non-zero function $h(s)$ from $G$.

Then, there exists a Hilbert space $K$ containing $H$ as a subspace and a unitary representation $U$ of $G$, such that

$$
\Psi(s) = \text{Pr}(U(s)) \quad (s \in G)
$$

and

$$
K = \bigvee_{s \in G} U(s)H
$$

**Proof of Theorem 2.2.** Let $T$ be a bounded operator in the class $C_{\rho_n}$ and $U$ be the unitary $\rho_n$-dilation of $T$, given by the expression 4. We have clearly,

$$
I + 2 \sum_{n \geq 1} z^n U^n \text{ converges to } (I + zU)(I - zU)^{-1}
$$

for all complex numbers $z$ such that $|z| < 1$.

And

$$
\text{Pr}(I + 2 \sum_{n \geq 1} z^n U^n) = I + 2 \sum_{n \geq 1} \frac{z^n}{\rho_n} T^n.
$$
By writing,

\[ I + 2 \sum_{n \geq 1} \frac{z^n}{\rho_n} T^n = (1 - \frac{2}{\rho_0})I + 2 \sum_{n \geq 0} \frac{z^n}{\rho_n} T^n = (1 - \frac{2}{\rho_0})I + 2\rho(zT), \]

we get

\[ Pr((I + zU)(I - zU)^{-1}) = (1 - \frac{2}{\rho_0})I + 2\rho(zT). \]

On the other hand,

\[ \langle (I + zU)k; (I - zU)k \rangle = \|k\|^2 + \langle zUk; k \rangle - \langle k; zUk \rangle - \|zUk\|^2 \]

It follows that for every \( k \in \mathcal{K} \), we have

\[ \text{Re} \langle (I + zU)k; (I - zU)k \rangle = \|k\|^2 - \|zUk\|^2 = \|k\|^2(1 - |z|^2) \geq 0 \text{ since } |z| < 1. \]

Now if we take \( h = (I - zU)k \) we will find,

\[ \text{Re} \langle (I + zU)(I - zU)^{-1}h; h \rangle = \text{Re} \langle Pr(I + zU)(I - zU)^{-1}h; h \rangle = \text{Re} \langle (1 - \frac{2}{\rho_0})h + 2\rho(zT)(h); h \rangle, \]

and hence for every \( h \in \mathcal{H} \), we obtain

\[ \text{Re} \langle (1 - \frac{2}{\rho_0})h + 2\rho(zT)(h); h \rangle \geq 0 \]

or equivalently,

\[ (1 - \frac{2}{\rho_0})\|h\|^2 + 2\text{Re} \langle \rho(zT)(h); h \rangle \geq 0 \]

for every \( h \in \mathcal{H} \) and all complex number \( z \) such that \( |z| < 1 \).

Conversely, let us show that condition (6) implies that the operator \( T \) belongs to the class \( C_{\rho_0} \). To this aim, assume that (6) is satisfied and take \( 0 \leq r < 1 \) and \( 0 \leq \phi < 2\pi \). We introduce the next operator valued function

\[ Q(r; \phi) = I + \sum_{n \geq 1} \frac{r^n}{\rho_n}(e^{in\phi}T^n + e^{-in\phi}T^*) \]
Then $Q(r; \phi)$ converges in the norm operator for every $r$ and $\phi$. Moreover, from the inequality 6, we have

$$\langle Q(r; \phi)l; l \rangle \geq 0$$

for every $l \in \mathcal{H}$. Therefore

$$J = \frac{1}{2\pi} \int_{0}^{2\pi} \langle Q(r; \phi)h(\phi); h(\phi) \rangle d\phi \geq 0$$

for every $h(\phi) = \sum_{n} h_{n} e^{-i n \phi}$ where $(h_{n})_{n \in \mathbb{Z}}$ is a sequence with only finite number of nonzero elements in $\mathcal{H}$. We have

$$J =: \sum_{-\infty}^{+\infty} \|h_{n}\|^2 + \sum_{m \in \mathbb{Z}} \sum_{n \geq m} \frac{r^{n-m}}{\rho_{n-m}} \langle T^{n-m} h_{n}; h_{m} \rangle + \sum_{m \in \mathbb{Z}} \sum_{n < m} \frac{r^{m-n}}{\rho_{m-n}} \langle T^{* (m-n)} h_{n}; h_{m} \rangle$$

for every $0 \leq r < 1$. Now taking $r \rightarrow 1^{-}$ will imply

$$\sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \langle \Psi_{(\rho_{n})} (n - m) h_{n}; h_{m} \rangle \geq 0,$$

where $\Psi_{(\rho_{n})} : \mathbb{Z} \rightarrow \mathcal{L}(\mathcal{H})$ is defined by $\Psi_{(\rho_{n})}(0) = I$, $\Psi_{(\rho_{n})}(n) = \frac{1}{\rho_{n}} T^{n}$ and $\Psi_{(\rho_{n})}(-n) = \frac{1}{\rho_{n}} T^{* n}$ for every $n > 0$.

It is immediate that $\Psi_{(\rho_{n})}(n)$ is nonnegative on the additive group $\mathbb{Z}$ of integers. Using Lemma 2.3, there exists a unitary operator $U$ on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and such that $\Psi_{(\rho_{n})}(n) = P r U(n)$ for all $n \in \mathbb{Z}$.

Therefore for all $n \in \mathbb{N}^{*}$

$$T^{n} = \rho_{n} P r U_{n}^{n}_{|\mathcal{H}}.$$  

The proof is completed.

Remark 2.4. In the case where $(\rho_{n})_{n \in \mathbb{N}}$ is a constant sequence, that is $\rho_{n} = \rho$ for all $n \in \mathbb{N}$ with $\rho > 0$, we obtain

$$\rho(z) = \frac{1}{1 - z}$$

and hence, the inequality 6 becomes

$$(1 - \frac{2}{\rho}) \|h\|^2 + 2 \text{Re} \langle (I - z T)^{-1} h; h \rangle \geq 0$$

for all $h \in \mathcal{H}$ and $z \in \mathbb{D}$. We substitute $h$ by $l = (I - z T)^{-1} h$ to retrieve relation 3 and by Theorem (2.1) we obtain $T$ is a $\rho$-contraction.
The next two corollaries are immediate consequences of Equation (6) and are related to analogous results of ρ–contraction.

**Corollary 2.5.** Let $T \in C_{\rho_n}$ and $M$ a $T$–invariant subspace. Then $T|_M \in C_{\rho_n}$.

*Proof.* It suffices to see that Equation 6 is close to restrictions. □

**Corollary 2.6.** Let $T$ be in the class $C_{\rho_n}$ and $r \geq 1$ be a real number, then $T$ is in the class $C_{r\rho_n}$.

*Proof.* The inequality 6 is equivalent to

$$(\rho_0 - 2)\|h\|^2 + 2\rho_0 \text{Re} \langle \rho(zT)(h); h \rangle \geq 0.$$

Plugging $r\rho_n$ instead of $\rho_n$, we get

$$(r\rho_0 - 2)\|h\|^2 + 2r\rho_0 \text{Re} \langle \frac{1}{r} \rho(zT)(h); h \rangle \geq 0,$$

and thus

$$(1 - \frac{2}{r\rho_0})\|h\|^2 + 2 \text{Re} \langle \frac{1}{r} \rho(zT)(h); h \rangle \geq 0.$$

Therefore $T \in C_{(r\rho_n)}$. □

We also have,

**Proposition 2.7.** Let $T$ be a bounded operator on a Hilbert space $\mathcal{H}$. Then for every $\alpha > 2$, there exists $\Gamma(\alpha) > 0$ such that the operator $T$ belongs to $C_{(\rho_n)}$, where $\rho_n$ is a sequence given by $\rho_n = \Gamma(\alpha).\|T^n\|(1 + n^\alpha)$.

*Proof.* Let $\Gamma > 0$ and $\rho_\alpha(z) = \sum_{n \geq 1} \frac{z^n}{\Gamma.\|T^n\|(1 + n^\alpha)}$ for all $|z| \leq 1$. Then

$$\rho_\alpha(zT) = \sum_{n \geq 1} \frac{z^n T^n}{\Gamma.\|T^n\|(1 + n^\alpha)}$$

for all $|z| \leq 1$.

For every vector $h$ in $\mathcal{H}$, we set

$$A(z) = \langle \rho_\alpha(zT)h; h \rangle$$
$|A(z)| = \left| \sum_{n \geq 0} \left( \frac{t^n}{\Gamma \|T^n\|(1 + n^\alpha)} T^n h; h \right) \right|$

\begin{align*}
&\leq \sum_{n \geq 0} \left| \frac{(T^n h; h)}{\Gamma \|T^n\|(1 + n^\alpha)} z^n \right|.
\end{align*}

Setting $a_n = \frac{(T^n h; h)}{\Gamma \|T^n\|(1 + n^\alpha)}$, we have

$$|a_n| \leq \frac{\|T^n\|^2 \|h\|^2}{\Gamma \|T^n\|(1 + n^\alpha)} = \frac{\|h\|^2}{\Gamma(1 + n^\alpha)} < \infty.$$ 

We conclude that $A(z)$ is holomorphic in the unit disc and continuous on the boundary. Since the maximum is attained of the circle $|z| = 1$, we obtain

$$|A(z)| = \left| \sum_{n \geq 0} a_n z^n \right|$$

\begin{align*}
&\leq \sum_{n \geq 0} |a_n| |z|^n = \sum_{n \geq 1} |a_n| \\
&\leq \sum_{n \geq 0} \frac{\|h\|^2}{\Gamma(1 + n^\alpha)}
\end{align*}

Now, since $\sum_{n \geq 0} \frac{1}{1 + n^\alpha}$ is a convergent sequence ($\alpha > 2$), then choosing $\Gamma = 2 \sum_{n \geq 0} \frac{1}{1 + n^\alpha}$ will leads to

$$|A(z)| \leq \frac{1}{2} \|h\|^2,$$

and then

$$\|h\|^2 + 2 \text{Re} \langle \rho(zT)h; h \rangle \geq 0 \text{ for all } h \in \mathcal{H} \text{ and } z \in D.$$ 

Finally, Inequality (6) is satisfied and the operator $T$ belongs to the class $C_{(\rho_n)}$. \(\blacksquare\)

3. The Bergmann shift

We devote this section to the membership of the Bergmann shift to the class $C_{(\rho_n)}$ for some suitable sequence $\rho_n$. Let $\mathcal{H}$ be a Hilbert space and $(e_i)_{i \in \mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Recall that for a given sequence $(\omega_n)_{n \in \mathbb{N}}$ of non negative numbers; the weighted shift $S_\omega$ associated with $\omega_n$ is defined on the
basis by \( S_\omega(e_n) = \omega_n e_{n+1} \). A detailed study on weighted shifts can be found in the survey [9]. On the other hand; the membership of weighted shifts to the class \( C_\rho \) is investigated in [3].

The Bergman shift is the weighted shift defined on the basis by the expression \( T e_n = w_n e_{n+1} \), where

\[
  w_n = \frac{n+1}{n} \quad \text{for all integer } n \in \mathbb{N}^*.
\]

It is easy to see that,

- \( \|T\| = \sup(w_n)_{n \in \mathbb{N}^*} = 2. \)
- The weight \( (w_n)_{n \in \mathbb{N}^*} \) is decreasing and then

\[
  \|T^n\| = \prod_{i=1}^{n} w_i = n + 1.
\]

In particular \( T \) is not power bounded and hence does not belong to the class \( C_\rho \) for any \( \rho > 0. \)

We have

**Proposition 3.1.** Let \( T \) be the Bergman shift and \( \rho_n \) be the sequence given by \( \rho_n = n^\alpha \) for some \( \alpha > 0. \) Then for every \( \alpha > 2, \) there exists \( \Gamma(\alpha) \) such that \( T \in C_{\Gamma(\alpha)\rho_n}. \)

**Proof.** Let \( \Gamma > 0 \) and \( \rho_\alpha(z) = \sum_{n \geq 1} \frac{z^n}{n^{\alpha}} \) for all \( |z| \leq 1 \) in that

\[
  \rho(zT) = \sum_{n \geq 1} \frac{z^n T^n}{\Gamma n^\alpha} \text{ for all } |z| \leq 1.
\]

We set, \( S = \rho(zT) \) and let \( x \) be a vector in \( \mathcal{H}. \) Therefore

\[
  S(x) = \rho(zT)(x) = \sum_{i \geq 1} \frac{z^n T^n x_i}{\Gamma n^\alpha}.
\]

Writing \( x = \sum_{i \geq 1} x_i e_i, \) we get

\[
  S(x) = \sum_{i \geq 1} \langle S(x); e_i \rangle e_i = \sum_{i \geq 1} \left( \sum_{j \geq 1} x_j \langle Se_j; e_i \rangle \right) e_i,
\]
and

\[
\langle S e_j; e_i \rangle = \sum_{n \geq 1} \frac{z^n}{\Gamma_n} T^n (e_j; e_i) = \sum_{n \geq 1} \frac{z^n}{\Gamma_n} \langle T^n (e_j); e_i \rangle
\]

\[
= \sum_{n \geq 1} \frac{z^n}{\Gamma_n} \langle (\prod_{p=j}^{n-1} w_p) e_{j+n}; e_i \rangle.
\]

It follows that

\[
\langle S e_j; e_i \rangle = \frac{z^{i-j}}{\Gamma} (i-j)^\alpha \prod_{p=j}^{i-1} w_p,
\]

and then

\[
S(x) = \sum_{i \geq 2} \left( \sum_{j=1}^{i-1} \left( \prod_{p=j}^{i-1} w_p \right) x_j \frac{z^{i-j}}{\Gamma(i-j)^\alpha} \right) e_i.
\]

For the Bergman shift, we have \( \prod_{p=j}^{i-1} w_p = \frac{i}{j} \) and thus

\[
\rho(zT)(x) = \sum_{i \geq 2} \left( \sum_{j=1}^{i} \frac{i}{\Gamma j(i-j)^\alpha} x_j z^{i-j} \right) e_i.
\]

Finally, we conclude that the inequality (6) is equivalent to

\[
\sum_{i \geq 1} |x_i|^2 + 2\text{Re} \left( \sum_{i \geq 2} \frac{i}{\Gamma j(i-j)^\alpha} x_i x_j z^{i-j} \right) \geq 0.
\]

If we consider the function

\[
A(z) = 2 \sum_{i \geq 2} \sum_{j=1}^{i-1} \frac{i}{\Gamma j(i-j)^\alpha} x_i x_j z^{i-j}
\]

and we write \( n = i - j \), we will obtain,

\[
A(z) = 2 \sum_{i \geq 2} \sum_{n=1}^{i-1} \frac{i}{\Gamma(i-n)\Gamma^\alpha} x_i x_{i-n} z^n = 2 \sum_{n \geq 1} \sum_{i \geq n+1} \frac{i}{\Gamma(i-n)\Gamma^\alpha} x_i x_{i-n} z^n.
\]

We denote by \((A(n)) = (a_n)_{n \in \mathbb{N}}^\ast\) the sequence of coefficients of \( A \),

\[
a_n = \frac{1}{2} \sum_{i \geq n+1} \frac{i}{\Gamma^\alpha(i-n)} x_i x_{i-n}
\]
Since \( \frac{i}{n(n-i)} = \frac{1}{n} + \frac{1}{i-n} \leq 2 \) for every \( i \geq n+1 \), we obtain
\[
|a_n| = \left| \frac{1}{2} \sum_{i \geq n+1} \frac{i}{\Gamma n^{\alpha}(i-n)} x_i x_{i-n} \right|
\leq \frac{1}{\Gamma n^{\alpha-1}} \sum_{i \geq n+1} |x_i x_{i-n}| \leq \frac{1}{\Gamma n^{\alpha-1}} \sum_{i \geq n+1} |x_i x_{i-n}|;
\]
and by the Cauchy-Schwartz inequality, it follows,
\[
|a_n| \leq \frac{1}{\Gamma n^{\alpha-1}} \|x\|^2 \leq \infty.
\]
We deduce that \( A(z) \) is holomorphic in the open unit disc and continuous on the closed unit disc. As the maximum is attained on the circle \( |z| = 1 \), we have
\[
|A(z)| = \left| \frac{1}{2} \sum_{n \geq 1} \left( \sum_{i \geq n+1} \frac{i}{i \Gamma(i-n)} x_i x_{i-n} \right) z^n \right|
\leq \sum_{n \geq 1} |a_n| |z|^n = \sum_{n \geq 1} |a_n|.
\]
Now, since \( \sum_{n \geq 1} \frac{1}{(n)^{\alpha-1}} \) is a convergent sequence \( (\alpha \geq 2) \), choosing \( \Gamma = \sum_{n \geq 1} \frac{1}{(n)^{\alpha-1}} \) would lead us to
\[
|A(z)| \leq \|x\|^2 = \sum_{i \geq 1} |x_i|^2.
\]
We derive that,
\[
|\text{Re} (A(z))| \leq |A(z)| \leq \|x\|^2 = \sum_{i \geq 1} |x_i|^2,
\]
and hence
\[
\frac{1}{2} \text{Re} \left( \sum_{i \geq 2} \sum_{j=1}^{i-1} \frac{i}{i \Gamma(i-j)} x_i x_j z^{i-j} \right) \leq \sum_{i \geq 1} |x_i|^2.
\]
Therefore for all \( x \in \mathcal{H} \) and a complex \( z \) such that \( |z| \leq 1 \) we have
\[
\sum_{i \geq 1} |x_i|^2 + 2\text{Re} \left( \sum_{i \geq 2} \sum_{j=1}^{i-1} \frac{i}{i \Gamma(i-j)} x_i x_j z^{i-j} \right) \geq 0.
\]
We conclude that the weighted shift \( \{w_n\} \) is a \( \rho_n \)-contraction with \( \rho_n = \Gamma n^\alpha \).
Remark 3.2. We claim that for every $\alpha \geq 1$ the Bergmann shift belongs to a class $C_{\infty,n^\alpha}$. A proof is not available for this claim; however it is motivated by the incomplete computations below.

Let us set, for example, $\rho_n = 4n$ for all integer $n \geq 1$, Let $H$ be a Hilbert space and $(e_i)_{i \in \mathbb{N}^*}$ be an orthonormal basis for the Hilbert space $H$. Consider the Bergmann shift defined on the basis by $T e_n = n + 1 \cdot (n + 1) e_n$ for all $n \in \mathbb{N}^*$.

Then as in the proof of the previous proposition, we show that inequality (6) is equivalent to the next

$$\sum_{i \geq 1} |x_i|^2 + \Re \left( \sum_{i \geq 2} \sum_{j=1}^{i-1} \frac{i}{2j(i-j)} x_i x_{i-j} z^{i-j} \right) \geq 0. \quad (7)$$

We write

$$\sum_{i \geq 1} |x_i|^2 + \Re \left( \sum_{i \geq 2} \sum_{j=1}^{i-1} \frac{i}{2j(i-j)} x_i x_{i-j} z^{i-j} \right) \geq \sum_{i \geq 1} |x_i|^2 - \sum_{i \geq 2} \sum_{j=1}^{i-1} \frac{i}{2j(i-j)} |x_i||x_j|,$$

and

$$\sum_{i \geq 1} |x_i|^2 + \sum_{i \geq 2} \sum_{j=1}^{i-1} \frac{i}{2j(i-j)} |x_i||x_j| = \sum_{i,j \geq 1} a_{i,j} |x_i||x_j|,$$

with

$$\begin{cases} a_{i,i} = 1 & \text{for all } i \geq 1 \\ a_{i,j} = \frac{i}{4j(t-j)} & \text{for all } j \neq i \end{cases}$$

Then to show inequality (7), it suffices to prove that the infinite symmetric matrix with the real entries $M = [a_{i,j}]$ is nonnegative. To this aim, we compute the determinant of the first $n \times n$-corner, to check if it is nonnegative. An attempt on classical softwares allow to show this fact for $n \leq 150$. It is hence reasonable to conjecture that the Bergman shift belongs to $C_{\infty,n^\alpha}$.

References


