

## The Commutant of an Operator with Bounded Conjugation Orbits and $C_0$ -Contractions

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*Abstract:* Let  $A$  be an invertible bounded linear operator on a complex Banach space,  $\{A\}'$  the commutant of  $A$  and  $B_A$  the set of all operators  $T$  such that  $\sup_{n \geq 0} \|A^n T A^{-n}\| < +\infty$ . Equality  $\{A\}' = B_A$  was studied by many authors for different classes of operators. In this paper we investigate a local version of this equality and the case where  $A$  is a  $C_0$ -contraction.

*Key words:* Operators, commutant, bounded conjugation orbit,  $C_0$ -contraction.

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### 1. INTRODUCTION

Denote by  $\mathcal{L}(X)$  the algebra of all bounded linear operators on the complex Banach space  $X$ . Let  $A \in \mathcal{L}(X)$ . We denote by  $\sigma(A)$  the spectrum of  $A$  and by  $\{A\}'$  the commutant of  $A$ . Throughout this paper we assume that  $A$  is invertible and we set

$$B_A = \left\{ T \in \mathcal{L}(X) : \sup_{n \geq 0} \|A^n T A^{-n}\| < +\infty \right\}.$$

Clearly we have  $\{A\}' \subset B_A$ . When  $X$  is of finite dimension, Deddens showed in [1] that equality  $\{A\}' = B_A$  holds if and only if  $\sigma(A)$  is a singleton. He conjectured that the above result remains true when  $X$  is any Hilbert space. P. G. Roth gave a negative answer to this conjecture. Let  $V$  be the integral Volterra operator on  $H = L^2([0, 1])$  and let  $A$  be the operator acting on  $H \oplus H$  defined by

$$A = \begin{pmatrix} 1 + V & 0 \\ 0 & 1 \end{pmatrix}.$$

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It is shown in [8, Example 2.7] that  $\{A\}' \neq B_A$ . Moreover we have

$$\|(A-1)^n\| = \|V^n\| = \frac{1}{(n-1)!},$$

which implies that

$$\|(A-1)^n\|^{1/n} = O(1/n), \quad n \rightarrow +\infty. \quad (1)$$

On the other hand side one can easily obtain from Theorem 2.3 and Theorem 2.5 of [8] the following result (see also Corollary 2.4).

**THEOREM 1.1.** *Assume that for some complex number  $\lambda$ ,*

$$\|(A-\lambda)^n\|^{1/n} = o(1/n), \quad n \rightarrow +\infty. \quad (2)$$

*Then  $B_A = \{A\}'$ .*

The example given above shows that if we replace "o" by "O" in the condition (2) then the conclusion of the theorem is false in general. Notice that Williams showed in [11] that if  $\sigma(A)$  is a singleton then  $B_A \cap B_{A^{-1}} = \{A\}'$ . This was improved by Drissi and Mbekhta in [2], replacing  $B_{A^{-1}}$  by the set  $\{T \in \mathcal{L}(X) : \forall \epsilon > 0, \|A^{-n}TA^n\| = o(e^{\epsilon\sqrt{n}}), n \rightarrow +\infty\}$ . Notice that equality  $B_A \cap B_{A^{-1}} = \{A\}'$  was extensively studied by Drissi and Mbekhta for different classes of operators and elements in Banach algebras (see [2], [3] and [4]).

In this note, we are interested in a local version of the above results. To describe the results obtained here let us introduce some notations. For a complex number  $\lambda$ , we set

$$V_{(A,\lambda)} = \left\{ x \in X : \|(A-\lambda)^n x\|^{1/n} = o(1/n), n \rightarrow +\infty \right\}.$$

Notice that  $V_{(A,\lambda)}$  is a linear subspace of  $X$ , invariant by  $A$  and is the zero set if  $\lambda$  is not in the spectrum of  $A$ . We denote by  $C_A$  the set of all operators  $T \in \mathcal{L}(X)$  such that, for all  $\lambda \in \sigma(A)$  and  $x \in V_{(A,\lambda)}$ ,

$$\|(A-\lambda)^i T (A^{-1} - \lambda^{-1})^j x\|^{1/(i+j)} = o\left(\frac{1}{i+j}\right), \quad i+j \rightarrow +\infty.$$

Finally if  $(V_i)_{i \in I}$  is a family of subsets of  $X$  we denote by  $\text{Span}(V_i, i \in I)$  the linear subspace of  $X$  generated by  $\cup_{i \in I} V_i$ .

We prove (Theorem 2.3) that if  $\text{Span}(V_{(A,\lambda)}, \lambda \in \sigma(A))$  is dense in  $X$  then  $B_A \cap C_A = \{A\}'$ . To prove this we introduce a local version of the sets  $\{A\}'$

and  $B_A$  and we study the relationship between these sets, using a theorem of [5] about local properties of powers of an operator.

In Section 3 we study the above equalities when  $A$  is a  $C_0$ -contraction. Let  $H$  be a separable Hilbert space and let  $H^\infty$  be the classical Hardy space of all bounded and holomorphic functions on  $\mathbb{D}$ . A contraction  $A$  on  $H$  is called a  $C_0$ -contraction (or in class  $C_0$ ) if it is completely nonunitary and there exists a nonzero function  $\theta \in H^\infty$  such that  $\theta(A) = 0$ . Notice that for a  $C_0$ -contraction  $A$  on  $H$ , there exists a minimal inner function  $\Theta_A$  that annihilates  $A$ , i.e., is such that  $\Theta_A(A) = 0$  (see [7] and [6]). Let  $A$  be an invertible  $C_0$ -contraction. We prove (Proposition 3.2) that if  $\Theta_A$  is a Blaschke product then we have  $B_A \cap C_A = \{A\}'$ . We obtain also (Proposition 3.3) that if  $\Theta_A$  is a Blaschke product then the equality  $B_A = \{A\}'$  holds if and only if  $\Theta_A$  is a Blaschke product with a unique root. If the singular part of  $\Theta_A$  is not constant, we do not know when the above equalities holds.

2. BOUNDED CONJUGATION ORBIT

Let  $A \in \mathcal{L}(X)$  be invertible,  $x \in X$  and  $\lambda \in \mathbb{C} \setminus \{0\}$ . We denote by  $C_{(A,\lambda,x)}$  the set of all operators  $T \in \mathcal{L}(X)$  such that

$$\|(A - \lambda)^i T (A^{-1} - \lambda^{-1})^j x\|^{\frac{1}{i+j}} = o\left(\frac{1}{i+j}\right), \quad i + j \rightarrow +\infty.$$

and  $C_{(A,\lambda)} = \bigcap_{x \in V_{(A,\lambda)}} C_{(A,\lambda,x)}$ . Then we have  $C_A = \bigcap_{\lambda \in \sigma(A)} C_{(A,\lambda)}$ .

We set

$$B_{(A,x)} = \left\{ T \in \mathcal{L}(X) : \sup_{n \geq 0} \|A^n T A^{-n} x\| < +\infty \right\}$$

and

$$\{(A, x)\}' = \{T \in \mathcal{L}(X) : ATx = TAx\}.$$

Clearly we have  $\bigcap_{n \geq 0} \{(A^{-n}, x)\}' \subset B_{(A,x)}$ . We shall need the following elementary result.

LEMMA 2.1. *Let  $n, i$  and  $j$  be integers such that  $0 \leq i \leq n$  and  $0 \leq j \leq n$ .*

We set

$$F_n(i, j) = \sum_{\max\{i,j\} \leq k \leq n} (-1)^{n-k} \binom{n}{k} \binom{k}{i} \binom{k}{j}.$$

Then for every integer  $n \geq 0$ , we have

$$F_n(i, j) = \begin{cases} 0 & \text{if } i + j < n, \\ \binom{n}{2n-i-j} \binom{2n-i-j}{n-j} & \text{if } i + j \geq n, \end{cases}$$

and

$$\sum_{0 \leq i, j \leq n} F_n(i, j) = 3^n.$$

*Proof.* We set  $Q(x, y) = (x + y + xy)^n$ . Using the binomial formula twice, we have

$$\begin{aligned} Q(x, y) &= \sum_{k=0}^n \binom{n}{k} (x + y)^k x^{n-k} y^{n-k} \\ &= \sum_{k=0}^n \sum_{l=0}^k \binom{n}{k} \binom{k}{l} x^{n+l-k} y^{n-l} \\ &= \sum_{\substack{0 \leq i, j \leq n \\ n \leq i+j}} \binom{n}{2n-i-j} \binom{2n-i-j}{n-j} x^i y^j. \end{aligned}$$

On the other hand side, writing  $Q(x, y) = ((1+x)(1+y) - 1)^n$  and using again the binomial formula, we get

$$\begin{aligned} Q(x, y) &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (1+x)^k (1+y)^k \\ &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \sum_{\substack{0 \leq i \leq k \\ 0 \leq j \leq k}} \binom{k}{i} \binom{k}{j} x^i y^j \\ &= \sum_{0 \leq i, j \leq n} F_n(i, j) x^i y^j. \end{aligned} \tag{3}$$

Now, by identifying the coefficients of  $x^i y^j$  in the two polynomial expressions of  $Q(x, y)$  obtained above, we get the first equality of the lemma. To prove the second one it suffices to take  $x = y = 1$  in (3). ■

**THEOREM 2.2.** *Let  $A \in \mathcal{L}(X)$  be invertible,  $x \in X$  and  $\lambda \in \mathbb{C} \setminus \{0\}$ . Then we have*

$$B_{(A,x)} \cap C_{(A,\lambda,x)} = \bigcap_{n \geq 0} \{(A^{-n}, x)\}' \text{ if and only if } x \in V_{(A,\lambda)}.$$

*Proof.* Assume that the equality  $B_{(A,x)} \cap C_{(A,\lambda,x)} = \bigcap_{n \geq 0} \{(A^{-n}, x)\}'$  holds. Then the identity operator belongs to  $C_{(A,\lambda,x)}$  which implies that  $x \in V_{(A,\lambda)}$ .

To prove the converse, we denote by  $W$  the bounded linear operator defined on  $\mathcal{L}(X)$  by  $W(T) = ATA^{-1}$ . We have

$$(W - I)^n Tx = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} A^k T A^{-k} x,$$

where  $I$  is the identity operator defined on  $\mathcal{L}(X)$ . We write  $A^k = (A - \lambda + \lambda)^k$ ,  $A^{-k} = (A^{-1} - \lambda^{-1} + \lambda^{-1})^k$  and we use the binomial formula. Then we get

$$(W - I)^n Tx = \sum_{0 \leq i, j \leq n} \lambda^{j-i} F_n(i, j) (A - \lambda)^i T (A^{-1} - \lambda^{-1})^j x, \quad (4)$$

where  $F_n(i, j)$  is defined in Lemma 2.1.

Suppose that  $T \in C_{(A,\lambda,x)}$ . For every  $\epsilon > 0$ , there exists a positive constant  $C_\epsilon$  such that for all non-negative integers  $i$  and  $j$ , we have

$$\|(A - \lambda)^i T (A^{-1} - \lambda^{-1})^j x\| \leq C_\epsilon \frac{\epsilon^{i+j}}{(i+j)^{i+j}}.$$

Since by Lemma 2.1 we have  $F_n(i, j) = 0$  for  $i + j < n$ , it follows from the last inequality and (4) that

$$\|(W - I)^n Tx\| \leq C_\epsilon \max(|\lambda|^n, |\lambda|^{-n}) \sum_{\substack{0 \leq i, j \leq n \\ n \leq i+j}} F_n(i, j) \frac{\epsilon^{i+j}}{(i+j)^{i+j}}. \quad (5)$$

The function  $t \rightarrow \frac{\epsilon^t}{t^t}$  decreases on the set  $[\frac{\epsilon}{e}, +\infty[$ . So for  $i + j \geq n \geq \epsilon/e$ , we have  $\frac{\epsilon^{i+j}}{(i+j)^{i+j}} \leq \frac{\epsilon^n}{n^n}$ . Combining this observation with the second equality in Lemma 2.1 and (5) we get

$$\|(W - I)^n Tx\| \leq C_\epsilon \frac{(3 \max(|\lambda|, |\lambda|^{-1}) \epsilon)^n}{n^n}.$$

This shows that

$$\|(W - I)^n Tx\|^{1/n} = o(1/n), \quad n \rightarrow \infty. \quad (6)$$

Let  $\ell$  be an element in the dual  $X^*$  of  $X$  and let  $L$  be the continuous linear functional on  $\mathcal{L}(X)$  defined by  $L(R) = \ell(Rx)$ ,  $R \in \mathcal{L}(X)$ . By (6) we have

$$|L((W - I)^n T)|^{1/n} = o(1/n), \quad n \rightarrow \infty.$$

Suppose in addition that  $T \in B_{(A,x)}$ . Then  $|L(W^n T)| = O(1)$ ,  $n \rightarrow \infty$ . It follows from Theorem 1.1 of [5] that  $L((W - I)^n T) = 0$ ,  $n \geq 1$ . Since  $\ell$  is any element of  $X^*$  we obtain by the Hahn-Banach theorem that  $(W - I)^n T x = 0$ ,  $n \geq 1$ . Now by induction on  $n$ , one checks easily that  $T \in \{(A^{-n}, x)\}'$  for every  $n \geq 1$ . So the inclusion  $B_{(A,x)} \cap C_{(A,\lambda,x)} \subset \bigcap_{n \geq 0} \{(A^{-n}, x)\}'$  holds for every  $x \in X$ .

Assume now that  $x \in V_{(A,\lambda)}$ . Let  $T \in \bigcap_{n \geq 0} \{(A^{-n}, x)\}'$ . Obviously  $T \in B_{(A,x)}$ . On the other hand side, for all non-negative integers  $n, m$ , we have

$$A^{-n} T A^{-m} x = A^{-(n+m)} T x = T A^{-(n+m)} x.$$

Therefore we obtain

$$\begin{aligned} (A - \lambda)^i T (A^{-1} - \lambda^{-1})^j x &= (-\lambda)^i A^i (A^{-1} - \lambda^{-1})^i T (A^{-1} - \lambda^{-1})^j x \\ &= (-\lambda)^i A^i T (A^{-1} - \lambda^{-1})^{i+j} x \\ &= (-\lambda)^{-j} A^i T A^{-i-j} (A - \lambda)^{i+j} x. \end{aligned}$$

Thus

$$\begin{aligned} \|(A - \lambda)^i T (A^{-1} - \lambda^{-1})^j x\|^{\frac{1}{i+j}} &\leq \|(-\lambda)^{-j} A^i T A^{-i-j}\|^{\frac{1}{i+j}} \|(A - \lambda)^{i+j} x\|^{\frac{1}{i+j}} \\ &= o\left(\frac{1}{i+j}\right), \quad i+j \rightarrow \infty \end{aligned}$$

and so  $T \in C_{(A,\lambda,x)}$ . Therefore we have  $\bigcap_{n \geq 0} \{(A^{-n}, x)\}' \subset B_{(A,x)} \cap C_{(A,\lambda,x)}$ , which concludes the proof. ■

**THEOREM 2.3.** *Let  $A \in \mathcal{L}(X)$  be invertible. If  $\text{Span}(V_{(A,\lambda)}, \lambda \in \sigma(A))$  is dense in  $X$  then*

$$B_A \cap C_A = \{A\}'.$$

*Proof.* Let  $T \in \{A\}'$ . For every  $x \in X$ , we have  $T \in \bigcap_{n \geq 0} \{(A^{-n}, x)\}'$ . It follows from Theorem 2.2 that if  $\lambda \in \sigma(A)$  and if  $x \in V_{(A,\lambda)}$  then  $T \in C_{(A,\lambda,x)}$ . Thus  $T \in \bigcap_{\lambda \in \sigma(A)} \left( \bigcap_{x \in V_{(A,\lambda)}} C_{(A,\lambda,x)} \right) = C_A$ . So  $\{A\}' \subset B_A \cap C_A$ . Suppose now that  $T \in B_A \cap C_A$ . For every  $\lambda \in \sigma(A)$  and  $x \in V_{(A,\lambda)}$ , we have  $T \in B_{(A,x)} \cap C_{(A,\lambda,x)}$ . It follows again from Theorem 2.2 that for every  $x \in V_{(A,\lambda)}$ ,  $A^{-1} T x = T A^{-1} x$ . Clearly this equality holds for every  $x \in X$ , since  $\text{Span}(V_{(A,\lambda)}, \lambda \in \sigma(A))$  is dense in  $X$ . So  $A^{-1}$  and  $A$  commute with  $T$ . ■

COROLLARY 2.4. *Let  $A$  be an invertible operator on  $X$  such that*

$$\|(A - \lambda)^n\|^{1/n} = o(1/n), \quad n \rightarrow +\infty.$$

*for some complex number  $\lambda$ . Then  $B_A = \{A\}'$ .*

*Proof.* We have  $V_{(A,\lambda)} = X$ . By Theorem 2.3 we only need to prove that  $C_A = \mathcal{L}(X)$ . Let  $T \in \mathcal{L}(X)$  and let  $\epsilon > 0$ . There exists  $C_\epsilon$  such that for every nonnegative integer  $i$

$$\|(A - \lambda)^i\| \leq C_\epsilon \frac{\epsilon^i}{i^i}$$

Notice that the hypothesis implies that  $\sigma(A) = \{\lambda\}$  and so  $\lambda \neq 0$ . We have

$$\begin{aligned} \|(A^{-1} - \lambda^{-1})^i\| &\leq \|\lambda^{-i} A^{-i} (\lambda - A)^i\| \\ &\leq C_\epsilon \left( \frac{\|A^{-1}\|}{|\lambda|} \right)^i \frac{\epsilon^i}{i^i}. \end{aligned}$$

Thus for every  $x \in X$  and every nonnegative integers  $i$  and  $j$ , we get

$$\|(A - \lambda)^i T (A^{-1} - \lambda^{-1})^j x\| \leq \|T\| \|x\| C_\epsilon^2 \left( \frac{\|A^{-1}\|}{|\lambda|} \right)^i \frac{\epsilon^{i+j}}{i^i j^j}.$$

Since  $(i + j)^{i+j} \leq e^{i+j} i^i j^j$ , we obtain

$$\|(A - \lambda)^i T (A^{-1} - \lambda^{-1})^j x\| \leq C'_\epsilon \frac{(e c \epsilon)^{i+j}}{(i + j)^{i+j}},$$

where  $c = \max \left\{ \frac{\|A^{-1}\|}{|\lambda|}, 1 \right\}$  and  $C'_\epsilon$  is a constant independent of  $i$  and  $j$ . Hence  $C_A = \mathcal{L}(X)$ . ■

*Remark.* For a family of subspaces  $(V_i)_{i \in I}$  of  $X$ , we set

$$\text{Alg } (V_i, i \in I) = \{T \in \mathcal{L}(X) : TV_i \subset V_i, (i \in I)\}.$$

If  $\lambda \in \sigma(A)$  and if  $V_{(A,\lambda)}$  is closed, it follows from the Banach–Steinhaus theorem that

$$\left\| \left( A|_{V_{(A,\lambda)}} - \lambda \right)^n \right\|^{1/n} = o(1/n), \quad n \rightarrow +\infty.$$

Then it follows that  $C_{(A,\lambda)} = \{T \in \mathcal{L}(X) : TV_{(A,\lambda)} \subset V_{(A,\lambda)}\}$ . So, if for every  $\lambda \in \sigma(A)$ ,  $V_{(A,\lambda)}$  is closed, then we have  $C_A = \text{Alg } (V_{(A,\lambda)}, \lambda \in \sigma(A))$ .

Let  $A \in \mathcal{L}(X)$  be invertible and with finite spectrum,  $\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_p\}$ . It is well known that  $X = F_1 + \dots + F_p$ , where  $F_1, \dots, F_p$  are closed subspaces which are invariant by  $A$  and such that  $\sigma(A|_{F_i}) = \{\lambda_i\}$ ,  $1 \leq i \leq p$ .

Assume that

$$\|Q(A)^n\|^{1/n} = o(1/n), \quad n \rightarrow +\infty,$$

where  $Q(x) = \prod_{1 \leq i \leq p} (x - \lambda_i)$ . It is easily seen that  $F_i = V_{(A, \lambda_i)}$ ,  $1 \leq i \leq p$ . We deduce from Corollary 2.4 that

$$B_A \cap \text{Alg}(V_{(A, \lambda_i)}, 1 \leq i \leq p) = \{A\}'.$$

### 3. THE CASE OF $C_0$ -CONTRACTIONS

In this section we investigate the equalities  $B_A = \{A\}$  and  $B_A \cap C_A = \{A\}'$  when  $A$  is a  $C_0$ -contraction. Notice that when the spectrum of  $A$  is reduced to a single point, say  $\sigma(A) = \{\lambda\}$ , then  $A - \lambda$  is nilpotent if  $|\lambda| < 1$  and  $\|(A - \lambda)^n\|^{1/n} = O(1/n)$  ( $n \rightarrow +\infty$ ), if  $|\lambda| = 1$  (see the proof at the end of this section).

Henceforth we denote by  $A$  a  $C_0$ -contraction on a separable Hilbert space  $H$  and by  $\Theta_A$  its minimal inner function (see [6]). We have

$$\sigma(A) = \text{Clos } \Theta_A^{-1}(0) \cup \text{supp } \mu,$$

where  $\mu$  is the singular measure associated to the singular part of  $\Theta_A$  and  $\text{supp } \mu$  is the closed support of  $\mu$  (see [6, p. 63]). The set of eigenvalues of  $A$  is given by the formula

$$\sigma_p(A) = \sigma(A) \cap \mathbb{D} = \Theta_A^{-1}(0).$$

We see that  $A$  is invertible if and only if  $\Theta_A(0) \neq 0$ . For  $\lambda \in \mathbb{D}$  we set  $b_\lambda(z) = \frac{|\lambda|}{\lambda} \frac{\lambda - z}{1 - \bar{\lambda}z}$ ,  $z \in \mathbb{D}$ . If  $\Theta_A(\lambda) = 0$  we set  $k_\lambda$  to be the multiplicity of  $\lambda$ .

A typical example of a  $C_0$ -contraction is the so-called model operator  $S_\Theta$ , where  $\Theta$  is an inner function. Let  $K_\Theta = H^2 \ominus \Theta H^2$ , where  $H^2$  is the classical Hardy space.  $S_\Theta$  is the compression of the shift, defined on  $K_\Theta$  by:  $f \rightarrow P_\Theta(zf)$ , where  $P_\Theta : H^2 \rightarrow K_\Theta$  is the orthogonal projection. Notice that the minimal inner function of  $S_\Theta$  is  $\Theta$ .

LEMMA 3.1. *We have*

$$V_{(A, \lambda)} = \begin{cases} \{0\} & \text{if } \lambda \in \sigma(A) \cap \mathbb{T}, \\ \ker(A - \lambda)^{k_\lambda} & \text{if } \lambda \in \sigma(A) \cap \mathbb{D}. \end{cases}$$



*Proof.* Let  $\lambda \in \sigma(A) \cap \mathbb{T}$  and  $x \in V_{(A,\lambda)}$ . Since  $A$  is a contraction, it follows from [5, Theorem 1.1] that  $Ax = \lambda x$ . Since  $A$  has no eigenvalue on  $\mathbb{T}$ , we have  $x = 0$ . So  $V_{(A,\lambda)} = \{0\}$ .

Now let  $\lambda \in \sigma(A) \cap \mathbb{D} = \sigma_p(A)$ . Notice that  $\lambda$  is an isolated point in  $\sigma(A)$ . Denote by  $F_\lambda$  the (maximal) spectral subspace of  $A$  such that  $\sigma(A|_{F_\lambda}) = \{\lambda\}$  (see [6, p. 84–86]). The space  $H$  decomposes into the direct sum of two subspaces  $H_\lambda$  and  $H'_\lambda$ , hyperinvariant for  $A$  and so that  $H_\lambda = \ker(A - \lambda)^{k_\lambda}$  and  $\lambda \notin \sigma(A|_{H'_\lambda})$  (see [7, Theorem 7.1, p. 135]). So  $F_\lambda = \ker(A - \lambda)^{k_\lambda}$ .

On the other hand  $F_\lambda$  is the set of all  $x \in H$  such that the map  $z \rightarrow (z - A)^{-1}x$  admits an analytic extension to  $\mathbb{C} \setminus \{\lambda\}$ . We have  $(z - A)^{-1}x = \sum_{n \geq 0} (z - \lambda)^{-n-1} (A - \lambda)^n x$  for  $|z|$  sufficiently large. So  $F_\lambda = \{x \in H : \lim_{n \rightarrow +\infty} \|(A - \lambda)^n x\|^{1/n} = 0\}$ . Thus we get  $\ker(A - \lambda)^{k_\lambda} = V_{(A,\lambda)} = F_\lambda$ . ■

**PROPOSITION 3.2.** *Suppose that  $\Theta_A$  is a Blaschke product with  $\Theta_A(0) \neq 0$ . Then*

$$B_A \cap C_A = \{A\}'.$$

*Proof.* By Lemma 3.1 we have

$$\text{Span} \{V_{(A,\lambda)}, \lambda \in \sigma(A)\} = \text{Span} \{ \ker(A - \lambda)^{k_\lambda}, \lambda \in \Lambda_\Theta \}.$$

Now the result follows from the completeness theorem ([7, Proposition 7.2, p. 135]) and Theorem 2.3. ■

For  $u$  and  $v$  in  $H$  we denote by  $v \otimes u$  the operator defined on  $H$  by  $x \rightarrow \langle x, v \rangle u$ .

**PROPOSITION 3.3.** *Suppose that  $\Theta_A$  is a Blaschke product with  $\Theta_A(0) \neq 0$ . Then  $B_A = \{A\}'$  if and only if  $\Theta_A$  is a Blaschke product with a unique root.*

*Proof.* If  $\Theta_A$  has the form  $b_\lambda^{k_\lambda}$ ,  $|\lambda| < 1$ , then  $H = \ker(A - \lambda)^{k_\lambda}$  and the equality  $B_A = \{A\}'$  follows from Corollary 2.4.

For the converse assume that  $\Theta_A$  vanishes at two distinct points  $\lambda_1$  and  $\lambda_2$  with  $|\lambda_1| \leq |\lambda_2| < 1$ . Notice that  $A^*$  is a  $C_0$ -contraction and  $\Theta_{A^*}(z) = \overline{\Theta_A(\bar{z})}$ ,  $z \in \mathbb{D}$ . So  $\lambda_1$  is an eigenvalue of  $A$  and  $\bar{\lambda}_2$  is an eigenvalue of  $A^*$ . Let  $u$  and  $v$  be nonzero vectors in  $H$  such that  $Au = \lambda_1 u$  and  $A^*v = \bar{\lambda}_2 v$ . We have  $A^n(v \otimes u)A^{-n} = \left(\frac{\lambda_1}{\lambda_2}\right)^n v \otimes u$  and so  $v \otimes u \in B_A \setminus \{A\}'$ . ■

When  $\Theta$  is not a Blaschke product, we do not know when the equalities  $B_A \cap C_A = \{A\}'$  and  $B_A = \{A\}'$  hold. It is interesting to study the particular case when  $\Theta_A(z) = e^{\frac{z+1}{z-1}}$ ,  $z \in \mathbb{D}$ . Notice that in this case  $\sigma(A) = \{1\}$  and  $A$  satisfies condition (1). Indeed, let, for  $\zeta \in \mathbb{C} \setminus \{1\}$ ,

$$\Delta_\zeta(z) = \frac{\Theta_A(z) - \Theta_A(\zeta)}{z - \zeta}, \quad |z| < 1.$$

Clearly  $\Delta_\zeta \in H^\infty$  and since  $\Theta_A(A) = 0$ , we have  $\Delta_\zeta(A) = -\Theta_A(\zeta)(A - \zeta)^{-1}$ . Hence

$$\|(A - \zeta)^{-1}\| \leq \frac{\|\Delta_\zeta\|_\infty}{\Theta_A(\zeta)},$$

where  $\|\cdot\|_\infty$  is the supremum norm on  $\mathbb{D}$ .

If  $|z - \zeta| \geq \frac{1}{2}|\zeta - 1|$  then  $|\Delta_\zeta(z)| \leq \frac{4}{|\zeta - 1|}$ . On the other hand if  $|z - \zeta| \leq \frac{1}{2}|\zeta - 1|$ , then  $|\Delta_\zeta(z)| \leq \sup_{w \in [z, \zeta]} |\Theta'_A(w)| \leq \frac{8}{|\zeta - 1|^2}$ . So

$$\|\Delta_\zeta\|_\infty \leq \frac{8}{|\zeta - 1|^2} \quad \text{for } |\zeta - 1| \leq 1.$$

Thus

$$\|(A - \zeta)^{-1}\| \leq \frac{8}{|\zeta - 1|^2} e^{\frac{2}{|\zeta - 1|}} \quad \text{for } |\zeta - 1| \leq 1. \quad (7)$$

Since  $\sigma(A) = \{1\}$ , we obtain from the Dunford–Riesz functional calculus that for every  $r > 0$ ,

$$(A - I)^n = \frac{1}{2i\pi} \int_{|\zeta - 1| = r} (\zeta - 1)^n (\zeta - A)^{-1} d\zeta.$$

It follows from this and inequality (7) that

$$\|(A - I)^n\| \leq 8r^{n-1} e^{\frac{2}{r}}, \quad 0 < r \leq 1.$$

Taking  $r = \frac{1}{n}$ , we see that  $A$  satisfies condition (1).

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