

## Composition Operators between $\mu$ -Bloch Spaces

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*Abstract:* In this paper we study continuity, boundedness from below and compactness of composition operators between  $\mu$ -Bloch spaces for very general weights  $\mu$ .

*Key words:* Bloch space, composition operators.

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### 1. INTRODUCTION

Let  $\mathbb{D}$  be the unit disk of the complex plane  $\mathbb{C}$  and let  $H(\mathbb{D})$  be the space of all holomorphic functions on  $\mathbb{D}$  with the topology of uniform convergence on compact subsets of  $\mathbb{D}$ . The Bloch space,  $\mathcal{B}$ , consist of all functions  $f \in H(\mathbb{D})$  such that

$$\|f\|_{\mathcal{B}} := \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

It is known that  $\mathcal{B}$  is a Banach space with the norm  $\|f\| := |f(0)| + \|f\|_{\mathcal{B}}$  (see, e.g., [1]). In the last decade, many authors have studied different classes of Bloch type spaces, where the typical weight function,  $v(z) = 1 - |z|^2$ , ( $z \in \mathbb{D}$ ), is replaced by a bounded continuous positive function  $\mu$  defined on  $\mathbb{D}$ . More precisely, a function  $f \in H(\mathbb{D})$  is called a  $\mu$ -Bloch function, denoted as  $f \in \mathcal{B}^{\mu}$ , if

$$\|f\|_{\mu} := \sup_{z \in \mathbb{D}} \mu(z) |f'(z)| < \infty.$$

If  $\mu(z) = v(z)^{\alpha}$  with  $\alpha > 0$ ,  $\mathcal{B}^{\mu}$  is just the  $\alpha$ -Bloch space (see [26]). It is readily seen that  $\mathcal{B}^{\mu}$  is a Banach space with the norm  $\|f\|_{\mathcal{B}^{\mu}} := |f(0)| + \|f\|_{\mu}$ . The  $\mathcal{B}^{\mu}$  spaces appear in the literature in a natural way when one study properties of some operators in certain spaces of holomorphic functions; for instance,

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if  $\mu_1(z) = v(z) \log \frac{2}{v(z)}$  with  $z \in \mathbb{D}$ , Attele in [2] proved that the Hankel operator induced by a function  $f$  in the Bergman space is bounded if and only if  $f \in B^{\mu_1}$ . The space  $B^{\mu_1}$  is also known as the *Log-Bloch space* or the *weighted Bloch space*. Quite recently Stević in [18] introduced, the so called, logarithmic Bloch type space with  $\mu(z) = v(z)^\alpha \ln^\beta \frac{e}{v(z)}$ ,  $\alpha > 0$  and  $\beta \geq 0$ , where some properties of this spaces are studied. Another Bloch type space, using Young's functions, have been recently introduced by Ramos-Fernández in [17].

There is a big interest in the investigation of Bloch type spaces and various concrete linear operators  $L : X \rightarrow Y$ , where at least one of the spaces  $X$  and  $Y$  is Bloch. For some other recent results in the area see, for example, [1]–[26] and a lot of references therein. Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two linear subspaces of  $H(\mathbb{D})$ . If  $\phi$  is a holomorphic self-map of  $\mathbb{D}$ , such that  $f \circ \phi$  belongs to  $\mathcal{H}_2$  for all  $f \in \mathcal{H}_1$ , then  $\phi$  induces a linear operator  $C_\phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  defined as

$$C_\phi(f) := f \circ \phi,$$

called the *composition operator* with *symbol*  $\phi$ . Composition operators has been studied by numerous authors in many subspaces of  $H(\mathbb{D})$  and in particular in Bloch type spaces.

In [14], Madigan and Matheson characterized continuity and compactness for composition operators on the classical Bloch space  $\mathcal{B}$ . In turn, their results have been extended by Xiao [22] to the  $\alpha$ -Bloch spaces and by Yoneda [24] to the *Log-Bloch space*. On the other hand, Gathage, Zheng and Zorboska [10] characterized closed range composition operators on the Bloch space. This result has been extended by Chen and Gauthier [6] to  $\alpha$ -Bloch spaces. Also, in [25], Zhang and Xiao have characterized boundedness and compactness of weighted composition operators that act between  $\mu$ -Bloch spaces on the unit ball of  $\mathbb{C}^n$ . In this case it is required that  $\mu$  be a *normal function*. The results of Zhang and Xiao have been extended by Chen and Gauthier [7] to the  $\mu$ -Bloch spaces being  $\mu$  a positive and non-decreasing continuous function such that  $\mu(t) \rightarrow 0$  as  $t \rightarrow 0$  and  $\mu(t)/t^\delta$  is decreasing for small  $t$  and for some  $\delta > 0$ . Recently, Chen, Stević and Zhou (see [8]) have studied composition operators between Bloch type spaces in the polydisc. While Giménez, Malavé and Ramos-Fernández [11] have extended those results to certain  $\mu$ -Bloch spaces, where the weight  $\mu$  can be extended to non vanishing complex valued holomorphic functions, that satisfy a reasonable geometric condition on the Euclidean disk  $D(1, 1)$ . Ramos-Fernández in [17] have extended all the results mentioned above to the Bloch-Orlicz spaces. In this paper we study properties

such as continuity, boundedness from below and compactness of composition operators acting between  $\mu$ -Bloch spaces, for very general weights  $\mu$ .

The essential norm of a continuous linear operator  $T$  is the distance from  $T$  to the compact operators, that is,  $\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}$ . Notice that  $\|T\|_e = 0$  if and only if  $T$  is compact, so that estimates on  $\|T\|_e$  lead to conditions for  $T$  to be compact. The essential norm of a composition operator on the Bloch space was calculated by Montes-Rodríguez in [15]. Similar results for the essential norms of weighted composition operators between weighted Banach spaces of analytic functions were obtained by Montes-Rodríguez in [16], and by Contreras and Hernández-Díaz in [9], in particular, formulas for the essential norm of weighted composition operators on the  $\alpha$ -Bloch spaces are obtained (see also the paper of McCluer and Zhao in [13]). Recently have appeared many extensions of the above results, for instance, we can mention the paper of Yang and Zhou in [23] and a lot of references therein.

Let us explain the organization of the paper. In Section 2, we summarize preliminaries on spaces  $H_v^\infty$ , associated weight and essential weights. Thus, in Section 3, inspired by the results in [5], we characterize continuity and compactness of the composition operator  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$ , in fact, if we denote by  $\|\delta_z\|$  the norm of the evaluation functional at  $z$  acting on the weighted Banach space of analytic functions  $H_{\mu_1}^\infty$ , then we have the following results:

- The operator  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is continuous if and only if

$$\sup_{z \in \mathbb{D}} \mu_2(z) \|\delta_{\phi(z)}\| |\phi'(z)| < \infty.$$

- The composition operator  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is compact if and only if  $\phi \in \mathcal{B}^{\mu_2}$  and

$$\lim_{|\phi(z)| \rightarrow 1^-} \mu_2(z) \|\delta_{\phi(z)}\| |\phi'(z)| = 0.$$

Finally, in Section 4, we characterize composition operators  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  with closed range in term of certain sampling sets for the space  $\mathcal{B}^{\mu_1}$ .

## 2. THE ASSOCIATED WEIGHT

Let  $v : \mathbb{D} \rightarrow \mathbb{R}_+$  be an arbitrary weight, that is,  $v$  is a bounded, continuous and strictly positive function. A function  $f \in H(\mathbb{D})$  belongs to the space  $H_v^\infty$  if

$$\|f\|_{H_v^\infty} := \sup_{z \in \mathbb{D}} v(z) |f(z)| < \infty. \quad (2.1)$$

It is known (see [3]) that  $H_v^\infty$  is a Banach space with the norm defined in (2.1). The space  $H_v^\infty$  is connected with the study of growth conditions of analytic functions and were studied in detail in [3, 4]. When  $v(z) = (1 - |z|^2)^\alpha$  with  $\alpha > 0$ , we get the Korenblum spaces  $\mathcal{A}^{-\alpha}$ . The relation between the  $\mu$ -Bloch space and the space  $H_v^\infty$  is evident, in fact,  $f \in \mathcal{B}^\mu$  if and only if  $f' \in H_\mu^\infty$  and

$$\|f\|_\mu = \|f'\|_{H_\mu^\infty}.$$

From the relation (2.1), it is clear that, for  $z \in \mathbb{D}$  fixed, there exists a constant  $K_z$ , depending on  $z$  and  $v$ , such that

$$|f(z)| \leq K_z \|f\|_{H_v^\infty},$$

for all  $f \in H_v^\infty$ . This means that the evaluation functional at  $z$ , denoted as  $\delta_z$ , is continuous on  $H_v^\infty$  and we can define the *associated weight* with  $v$ , denoted as  $\tilde{v}$ , by

$$\tilde{v}(z) = \frac{1}{\|\delta_z\|} = \frac{1}{\sup\{|f(z)| : \|f\|_v \leq 1\}},$$

where  $z \in \mathbb{D}$  and  $\|\delta_z\|$  denotes the norm of the evaluation functional at  $z$ . In [3], it is shown that  $\tilde{v}$  satisfies the following useful properties:

1.  $\tilde{v}$  is a weight and  $0 < v(z) < \tilde{v}(z)$  for all  $z \in \mathbb{D}$ ,
2. for every  $z \in \mathbb{D}$ , there exists  $f_z \in H_v^\infty$  such that  $\|f\|_{H_v^\infty} \leq 1$  and

$$\tilde{v}(z)|f_z(z)| = 1,$$

3.  $H_v^\infty$  is isometrically equal to  $H_{\tilde{v}}^\infty$  and  $\|f\|_v = \|f\|_{\tilde{v}}$  for all  $f \in H_v^\infty$ .

A weight  $v$  is called *essential* if there exists a constant  $C > 0$  such that  $\tilde{v}(z) \leq Cv(z)$  for all  $z \in \mathbb{D}$ . For instance, if  $v(z) = 1/M(f, |z|)$  for some analytic function  $f \in H(\mathbb{D})$ , then  $v = \tilde{v}$ , where  $M(f, r) = \sup_{|z|=r} |f(z)|$ . The following are examples of essential weights (see [3] for a reasonable amount of examples of essential weights):

- $v_\alpha(z) = (1 - |z|)^\alpha$  with  $\alpha > 0$ ,
- $v(z) = \exp(-1/(1 - |z|)^\alpha)$ , with  $\alpha > 0$ ,
- $v(z) = 1/\max\{1, -\log(1 - |z|)\}$ .

In general, is not easy to calculate the associated weight  $\tilde{v}$ ; however, in [3, Section 3] Bierstedt, Bonet and Taskinen give some estimations of  $\tilde{v}$ .

Finally, we like to comment that some of the properties of composition operators acting on  $H_v^\infty$  spaces has been studied by Bonet, Domański, Lindström and Taskinen in [5], Contreras and Hernández-Díaz in [9] and Wolf in [20].

3. CONTINUITY AND COMPACTNESS OF COMPOSITION OPERATORS  
BETWEEN  $\mu$ -BLOCH SPACES

In this section, we study continuity and compactness of composition operators between  $\mu$ -Bloch spaces. Throughout this section,  $\phi$  is a holomorphic self-map of  $\mathbb{D}$  and  $C_\phi$  denote its associated composition operator.  $\mu_1$  and  $\mu_2$  are weight functions defined on  $\mathbb{D}$  and  $\tilde{\mu}_1$  is the associated weight of  $\mu_1$ ;  $\mathcal{B}^{\mu_1}$  and  $\mathcal{B}^{\mu_2}$  are their respective  $\mu$ -Bloch spaces. Also,  $\|\delta_z\|$  denotes the norm of the evaluation functional at  $z$  on the weighted Banach space of analytic functions  $H_{\mu_1}^\infty$ . With this notations we have the following results.

3.1. CONTINUITY. The following, generalize many results about continuity of composition operators acting on Bloch type spaces. A similar result was obtained recently by Wolf in [21], while this article was under review.

**THEOREM 3.1.** *The operator  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is continuous if and only if*

$$\sup_{z \in \mathbb{D}} \mu_2(z) \|\delta_{\phi(z)}\| |\phi'(z)| < \infty. \quad (3.1)$$

*Proof.* Suppose first that

$$L = \sup_{z \in \mathbb{D}} \mu_2(z) \|\delta_{\phi(z)}\| |\phi'(z)| < \infty.$$

Then, for each  $f \in \mathcal{B}^{\mu_1}$ , since  $\tilde{\mu}_1(s) \|\delta_s\| = 1$  for all  $s \in \mathbb{D}$ , we have the following estimate

$$\begin{aligned} \|f \circ \phi\|_{\mu_2} &= \sup_{z \in \mathbb{D}} \mu_2(z) \|\delta_{\phi(z)}\| |\phi'(z)| \tilde{\mu}_1(\phi(z)) |f'(\phi(z))| \\ &\leq L \|f\|_{\tilde{\mu}_1} = L \|f\|_{\mu_1}. \end{aligned}$$

Also, since  $\mu_1$  is continuous and positive on the compact set  $[0, \phi(0)]$ , there exists a constant  $K_{\mu_1, \phi} > 0$ , depending on  $\mu_1$  and  $\phi(0)$ , such that

$$\int_0^{\phi(0)} \frac{|ds|}{\mu_1(s)} \leq K_{\mu_1, \phi}. \quad (3.2)$$

Hence, we have

$$|f(\phi(0))| \leq |f(0)| + \int_0^{\phi(0)} |f'(s)| |ds| \leq |f(0)| + K_{\mu_1, \phi} \|f\|_{\mu_1}.$$

We conclude that

$$|f(\phi(0))| + \|f \circ \phi\|_{\mu_2} \leq |f(0)| + (L + K_{\mu_1, \phi})\|f\|_{\mu_1}$$

and the composition operator  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is continuous.

Now, suppose that there exists a constant  $L > 0$  such that  $\|f \circ \phi\|_{\mu_2} \leq L\|f\|_{\mu_1}$  for all function  $f \in \mathcal{B}^{\mu_1}$  with  $f(0) = 0$  and let us fix  $z \in \mathbb{D}$ . By definition of associated weight, for  $a = \phi(z) \in \mathbb{D}$ , there exists a function  $f_a \in H(\mathbb{D})$  such that  $\sup_{w \in \mathbb{D}} \mu_1(w)|f_a(w)| \leq 1$  and  $\tilde{\mu}_1(a)|f_a(a)| = 1$ , hence the function  $g_a$  given by

$$g_a(w) = \int_0^w f_a(s) ds,$$

with  $w \in \mathbb{D}$  belongs to  $\mathcal{B}_1^\mu$  and satisfies  $g_a(0) = 0$ . Thus, applying the hypothesis with  $f = g_a$ , we have  $\|g_a \circ \phi\|_{\mu_2} \leq L$ ; that is,

$$\sup_{w \in \mathbb{D}} \mu_2(w)|g'_a(\phi(w))|\|\phi'(w)\| \leq L.$$

This last, implies that

$$\mu_2(z)\|\delta_{\phi(z)}\|\|\phi'(z)\| \leq L.$$

The proof of the theorem is complete. ■

As an immediate consequence of Theorem 3.1, we have the following result, which has been obtained by many authors for various type of weight  $\mu$ .

**COROLLARY 3.2.** *If  $\mu_1$  is a essential weight, then the composition operator  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is continuous if and only if*

$$\sup_{z \in \mathbb{D}} \frac{\mu_2(z)}{\mu_1(\phi(z))} |\phi'(z)| < \infty.$$

*Proof.* It follows from the fact that  $\tilde{\mu}_1(s)\|\delta_s\| = 1$  for all  $s \in \mathbb{D}$  and  $\tilde{\mu}_1 \sim \mu_1$ . ■

**EXAMPLE 3.3.** If  $\mu_1(z) = (1 - |z|)^\alpha$  and  $\mu_2(z) = (1 - |z|)^\beta$  with  $\alpha, \beta > 0$ , we have the result of Xiao in [22]. If  $\mu_1(z) = \mu_2(z) = (1 - |z|) \log\left(\frac{2}{1-|z|}\right)$ , we have the result of Yoneda in [24].

3.2. COMPACTNESS. Now, we are going to characterize compactness of composition operators that act between  $\mu$ -Bloch spaces. Our goal is to obtain genuine extensions of the results in [17, 11, 14]. In [19], Tjani showed the following result.

LEMMA 3.4. *Let  $X, Y$  be two Banach spaces of analytic functions on  $\mathbb{D}$ . Suppose that*

1. *The point evaluation functionals on  $X$  are continuous.*
2. *The closed unit ball of  $X$  is a compact subset of  $X$  in the topology of uniform convergence on compact sets.*
3.  *$T : X \rightarrow Y$  is continuous when  $X$  and  $Y$  are given the topology of uniform convergence on compact sets.*

*Then,  $T$  is a compact operator if and only if given a bounded sequence  $\{f_n\}$  in  $X$  such that  $f_n \rightarrow 0$  uniformly on compact sets, then the sequence  $\{Tf_n\}$  converges to zero in the norm of  $Y$ .*

Observe that for  $z \in \mathbb{D}$  fixed, since  $\mu_1$  is positive and continuous on the compact set  $[0, z]$ , there exists a constant  $K_{\mu_1, z} > 0$  such that

$$|f(z)| \leq |f(0)| + \int_0^z \frac{|ds|}{\mu_1(s)} \leq K_{\mu_1, z} \|f\|_{\mathcal{B}^{\mu_1}}$$

and the point evaluation functionals on  $\mathcal{B}^{\mu_1}$  are continuous. Thus, as a consequence of Lemma 3.4, we have the following result.

LEMMA 3.5. *The composition operator  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is compact if and only if given a bounded sequence  $\{f_n\}$  in  $\mathcal{B}^{\mu_1}$  such that  $f_n \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$ , then  $\|C_\phi(f_n)\|_{\mu_2} \rightarrow 0$  as  $n \rightarrow \infty$ .*

Next we establish our criterion for the compactness of  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$ . It generalizes a result of Madigan and Matheson in [14].

THEOREM 3.6. *The composition operator  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is compact if and only if  $\phi \in \mathcal{B}^{\mu_2}$  and*

$$\lim_{|\phi(z)| \rightarrow 1^-} \mu_2(z) \|\delta_{\phi(z)}\| |\phi'(z)| = 0. \quad (3.3)$$

*Proof.* Let us suppose first that  $\phi \in \mathcal{B}^{\mu_2}$  and (3.3) holds. Let  $\{f_n\}$  be a bounded sequence in  $\mathcal{B}^{\mu_1}$  converging to 0 uniformly on compact subsets of  $\mathbb{D}$ . Then, by Lemma 3.5, it suffices to show that  $\|C_\phi(f_n)\|_{\mu_2} \rightarrow 0$  as  $n \rightarrow \infty$ . To this end, we set  $K = \sup_n \|f_n\|_{\mu_1} = \sup_n \|f_n\|_{\tilde{\mu}_1}$ . Then, for  $\epsilon > 0$  we can find an  $r \in (0, 1)$  such that

$$\mu_2(z) \|\delta_{\phi(z)}\| |\phi'(z)| < \frac{\epsilon}{K},$$

for any  $z \in \mathbb{D}$  satisfying  $r < |\phi(z)| < 1$ . Hence, we have

$$\begin{aligned} \mu_2(z) |(f_n \circ \phi)'(z)| &= \mu_2(z) \|\delta_{\phi(z)}\| |\phi'(z)| \tilde{\mu}_1(\phi(z)) |f'_n(\phi(z))| \\ &\leq \frac{\epsilon}{K} K = \epsilon \end{aligned}$$

whenever  $r < |\phi(z)| < 1$ . Here, we have used the fact that  $\|\delta_s\| \tilde{\mu}_1(s) = 1$  for all  $s \in \mathbb{D}$ .

On the other hand, since  $\phi \in \mathcal{B}^{\mu_2}$  and  $\tilde{\mu}_1$  is continuous and positive on the compact set  $\{w \in \mathbb{D} : |w| \leq r\}$ , we can find a constant  $C > 0$ , depending only on  $r$  and  $\mu_1$ , such that

$$\sup_{|\phi(z)| \leq r} \frac{\mu_2(z)}{\tilde{\mu}_1(\phi(z))} |\phi'(z)| \leq C \|\phi\|_{\mu_2}.$$

Thus, since  $\{f_n\}$  converges to 0 uniformly on compact subsets of  $\mathbb{D}$  and  $\tilde{\mu}_1$  is bounded on the compact set  $|s| \leq r$ , we have  $\sup_{|s| \leq r} \tilde{\mu}_1(s) |f'_n(s)| \rightarrow 0$ , as  $n \rightarrow \infty$ . Hence, for the given  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that

$$\begin{aligned} \sup_{|\phi(z)| \leq r} \mu_2(z) |(f_n \circ \phi)'(z)| &= \sup_{|\phi(z)| \leq r} \frac{\mu_2(z)}{\tilde{\mu}_1(\phi(z))} |\phi'(z)| \tilde{\mu}_1(\phi(z)) |f'_n(\phi(z))| \\ &\leq C \|\phi\|_{\mu_2} \epsilon \end{aligned}$$

whenever  $n \geq N$ . Finally, since  $f_n \circ \phi(0) \rightarrow 0$  as  $n \rightarrow \infty$ , we conclude that

$$\|f_n \circ \phi\|_{\mu_2} = |f_n \circ \phi(0)| + \sup_{z \in \mathbb{D}} \mu_2(z) |(f_n \circ \phi)'(z)| < (1 + C \|\phi\|_w) \epsilon$$

whenever  $n \geq N$ , which means that  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is a compact operator.

To prove the converse, suppose that there exists an  $\epsilon_0 > 0$  such that

$$\sup_{|\phi(z)| \geq r} \frac{\mu_2(z)}{\tilde{\mu}_1(\phi(z))} |\phi'(z)| \geq \epsilon_0$$



for any  $r \in (0, 1)$ . Then, given a sequence of real numbers  $\{r_n\} \subset (0, 1)$  such that  $r_n \rightarrow 1$  as  $n \rightarrow \infty$ , we can find a sequence  $\{z_n\} \subset \mathbb{D}$  such that  $|\phi(z_n)| > r_n$  and

$$\frac{\mu_2(z_n)}{\tilde{\mu}_1(w_n)} |\phi'(z_n)| \geq \frac{1}{2} \epsilon_0,$$

where  $w_n = \phi(z_n)$ . By taking a subsequence, if necessary, we may suppose that  $w_n \rightarrow w_0 \in \partial\mathbb{D}$ . Also, since  $|w_n| \rightarrow 1$  as  $n \rightarrow \infty$ , we can find an increasing sequence  $\{\alpha(n)\}$  of positive integers such that  $\alpha(n) \rightarrow \infty$  as  $n \rightarrow \infty$  and  $|w_n|^{\alpha(n)} \geq \frac{1}{2}$  for all  $n \in \mathbb{N}$ .

Now, for  $n \in \mathbb{N}$ , we choose a function  $f_n \in H(\mathbb{D})$  such that

$$\sup_{w \in \mathbb{D}} \mu_1(w) |f_n(w)| \leq 1$$

and  $\tilde{\mu}_1(w_n) |f_n(w_n)| = 1$  and we set

$$g_n(z) = \int_0^z s^{\alpha(n)} f_n(s) ds,$$

with  $z \in \mathbb{D}$ . We can see that  $\{g_n\}$  is a bounded sequence in  $\mathcal{B}^{\mu_1}$ , in fact,

$$\|g_n\|_{\mu_1} = \sup_{z \in \mathbb{D}} \mu_1(z) |g_n'(z)| = \sup_{z \in \mathbb{D}} \mu_1(z) |z|^{\alpha(n)} |f_n(z)| \leq 1$$

for all  $n \in \mathbb{N}$ . Furthermore, because of the factor  $z^{\alpha(n)}$ , the sequences  $\{g_n'\}$  converges to 0 uniformly on compact subsets of  $\mathbb{D}$ , therefore, since

$$g_n(z) = \int_0^z g_n'(s) ds$$

for all  $z \in \mathbb{D}$ , we can see that  $\{g_n\}$  is a sequence converging to 0 uniformly on compact subsets of  $\mathbb{D}$  and satisfying

$$\begin{aligned} \|C_\phi(g_n)\|_{\mu_2} &\geq \mu_2(z_n) |g_n'(w_n)| |\phi'(z_n)| \\ &= \mu_2(z_n) |w_n|^{\alpha(n)} |f_n(w_n)| |\phi'(z_n)| \\ &= \frac{1}{2} \frac{\mu_2(z_n)}{\tilde{\mu}_1(w_n)} |\phi'(z_n)| > \frac{1}{4} \epsilon_0 > 0, \end{aligned}$$

where, we have used the fact that  $|f_n(w_n)| = 1/\tilde{\mu}_1(w_n)$ . Therefore,  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is not a compact operator. This completes the proof of the theorem.  $\blacksquare$

As an immediate consequence, we have.

COROLLARY 3.7. *If  $\mu_1$  is an essential weight, then the composition operator  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is compact if and only if  $\phi \in \mathcal{B}^{\mu_2}$  and*

$$\lim_{|\phi(z)| \rightarrow 1^-} \frac{\mu_2(z)}{\mu_1(\phi(z))} |\phi'(z)| = 0.$$

EXAMPLE 3.8. If  $\mu_1(z) = \mu_2(z) = 1 - |z|$ , we get the result of Madigan and Matheson in [14]. When  $\mu_1(z) = (1 - |z|^2)^\alpha$  and  $\mu_2(z) = (1 - |z|^2)^\beta$  with  $\alpha, \beta > 0$ , we obtain a criterion for the compactness of composition operators between  $\alpha$ -Bloch type spaces, similar result was found by Montes-Rodríguez in [16], later, independently, by Xiao in [22] and by McCluer and Zhao in [13]. If  $\mu_1(z) = \mu_2(z) = (1 - |z|) \log(2/(1 - |z|))$  we obtain the result of Yoneda in [24].

#### 4. COMPOSITION OPERATORS WITH CLOSED RANGE BETWEEN $\mu$ -BLOCH SPACES

In this section, we characterize the composition operators  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  with closed range in terms of certain sampling sets for the  $\mu$ -Bloch space. The purpose here is to generalize the results in [6, 10, 11, 17] for the  $\mu$ -Bloch spaces. Recall (see [12]) that a subset  $G$  of the unit disk  $\mathbb{D}$  is said to be a *sampling set* for the Korenblum space  $\mathcal{A}^{-\alpha}$  if there exists a positive constant  $L > 0$  such that

$$\sup_{z \in G} (1 - |z|^2)^\alpha |f(z)| \geq L \|f\|_{\mathcal{A}^{-\alpha}}$$

for all  $f \in \mathcal{A}^{-\alpha}$ . Observe that  $v(z) = (1 - |z|^2)^\alpha = \tilde{v}(z)$  and for this reason, we introduce the following definition.

DEFINITION 4.1. Let  $v$  be a weight defined on  $\mathbb{D}$ . A subset  $G$  of the unit disk  $\mathbb{D}$  is said to be a *sampling set* for  $\mathcal{B}^v$  if there exists a positive constant  $L > 0$  such that

$$\sup_{z \in G} \tilde{v}(z) |f'(z)| \geq L \|f\|_v \tag{4.1}$$

for all  $f \in \mathcal{B}^v$ .

*Remark 4.2.* When  $v$  is an essential weight, we can replace, in the preceding definition, the weight  $\tilde{v}$  by  $v$ .

Let  $\mu_1$  and  $\mu_2$  two weights. For  $\varepsilon > 0$ , let us denote

$$\Omega_\varepsilon := \left\{ z \in \mathbb{D} : \frac{\mu_2(z)}{\mu_1(\phi(z))} |\phi'(z)| \geq \varepsilon \right\}.$$

With this notation, we have the following result.

**THEOREM 4.3.** *Let  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  be a continuous composition operator.  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is bounded below if and only if there exists  $\varepsilon > 0$  such that  $G_\varepsilon = \phi(\Omega_\varepsilon)$  is a sampling set for  $\mathcal{B}^{\mu_1}$ .*

*Proof.* Let us suppose first that there exists  $\varepsilon > 0$  such that  $G_\varepsilon = \phi(\Omega_\varepsilon)$  is a sampling set for  $\mathcal{B}^{\mu_1}$ . In this case, we can find a constant  $L > 0$  such that

$$\|f\|_{\mu_1} \leq L \sup_{z \in G_\varepsilon} \tilde{\mu}_1(z) |f'(z)|$$

for all functions  $f \in \mathcal{B}^{\mu_1}$ . Hence, we have that

$$\begin{aligned} \|f\|_{\mu_1} &\leq L \sup_{z \in \Omega_\varepsilon} \tilde{\mu}_1(\phi(z)) |f'(\phi(z))| \\ &= L \sup_{z \in \Omega_\varepsilon} \frac{\tilde{\mu}_1(\phi(z))}{\mu_2(z) |\phi'(z)|} \mu_2(z) |(f \circ \phi)'(z)| \\ &\leq \frac{L}{\varepsilon} \|f \circ \phi\|_{\mu_2}, \end{aligned}$$

and since

$$|f(0)| \leq |f(\phi(0))| + K_{\mu_1, \phi} \|f\|_{\mu_1},$$

where  $K_{\mu_1, \phi}$  is the constant in (3.2), we conclude that

$$|f(0)| + \|f\|_{\mu_1} \leq |f(\phi(0))| + (1 + K_{\mu_1, \phi}) \frac{L}{\varepsilon} \|f \circ \phi\|_{\mu_2}$$

and the operator  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is bounded below.

To prove the converse, suppose that  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is bounded below. For any non constant function  $g \in \mathcal{B}^{\mu_1}$ , we set

$$f(z) = \frac{1}{\|g\|_{\mu_1}} (g(z) - g(\phi(0)))$$

and we have that  $f(\phi(0)) = 0$  and  $\|f\|_{\mu_1} = 1$ . Hence, by hypothesis, there exists a constant  $K > 0$  (not depending on  $g$ ), such that  $\|C_\phi(f)\|_{\mathcal{B}^{\mu_2}} \geq K \|f\|_{\mathcal{B}^{\mu_1}}$ ; this last, implies that

$$\|C_\phi(f)\|_{\mu_2} = \sup_{z \in \mathbb{D}} \mu_2(z) |f'(\phi(z))| |\phi'(z)| \geq K.$$

Thus, by definition of supremum, we can find  $z_f \in \mathbb{D}$ , such that

$$\mu_2(z_f)|f'(\phi(z_f))||\phi'(z_f)| \geq \frac{K}{2},$$

which, in turn, implies that

$$\frac{\mu_2(z_f)}{\tilde{\mu}_1(\phi(z_f))} |\phi'(z_f)| \tilde{\mu}_1(\phi(z_f)) |f'(\phi(z_f))| \geq \frac{K}{2}. \quad (4.2)$$

Thus, since  $\tilde{\mu}_1(\phi(z_f))|f'(\phi(z_f))| \leq 1$ , it must be

$$\frac{\mu_2(z_f)}{\tilde{\mu}_1(\phi(z_f))} |\phi'(z_f)| \geq \frac{K}{2}.$$

Therefore, putting  $\varepsilon := \frac{K}{2}$ , we have  $z_f \in \Omega_\varepsilon$ .

Now, since  $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is continuous, Theorem 3.1 implies that there is a constant  $M > 0$ , such that

$$\frac{\mu_2(z_f)}{\tilde{\mu}_1(\phi(z_f))} |\phi'(z_f)| \leq M.$$

From (4.2) we conclude that

$$\tilde{\mu}_1(\phi(z_f))|f'(\phi(z_f))| \geq \frac{K}{2M}.$$

Finally, since  $\phi(z_f) \in G_\varepsilon$ , it must be

$$\sup_{z \in G_\varepsilon} \tilde{\mu}_1(z)|f'(z)| \geq \frac{K}{2M}.$$

That is,

$$\sup_{z \in G_\varepsilon} \tilde{\mu}_1(z) \frac{|g'(z)|}{\|g\|_{\mu_1}} \geq \frac{K}{2M}$$

and therefore  $G_\varepsilon$  is a sampling set for  $\mathcal{B}^{\mu_1}$ . The proof of the theorem is complete. ■

EXAMPLE 4.4. If  $\mu_1(z) = \mu_2(z) = 1 - |z|^2$ , we have the result of Ghatage, Zheng and Zorboska in [10]. If  $\mu_1(z) = (1 - |z|)^\alpha$  and  $\mu_2(z) = (1 - |z|)^\beta$  with  $\alpha, \beta > 0$ , we obtain the result of Chen and Gauthier in [6].

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