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Multifractal Formalism and Inequality Involving Packing Dimension

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Abstract: We give a new inequality of the iso-Hölder set's dimension within the framework of the centered multifractal formalism. Besides we develop an example of a class of measure for which this inequality is finer than that established by the classic formalism.

Key words: Multifractal formalism, packing, dimension.

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1. Introduction and preliminaries

One of the targets of the multifractal formalism is to establish the packing dimension of the iso-Hölder set

$$X^\alpha = \left\{ x \in \operatorname{supp}\mu,\ \limsup_{r\to 0} \frac{\log \mu\big(B(x,r)\big)}{\log r} = \alpha \right\},$$

where μ is a Borel probability measure on $\mathbb{R}^d$ and α is a positive number.

By a heuristics approach, physicists [9, 6, 8] obtained for some self-similar measures that $\operatorname{Dim}(X^\alpha)$, where Dim is the packing dimension [16], is equal to the Legendre transform of a free energy function τ, i.e.,

$$\operatorname{Dim}(X^\alpha) = \tau^*(\alpha).$$

This income is generally false. Besides mathematicians [12, 5, 4, 14, 1, 10, 2, 11, 3, 15, 13] proved the inequality below

$$\operatorname{Dim}(X^\alpha) \leqslant \tau^*(\alpha).$$

By considering the nature of X^α, the classic formalism is based on quantities connecting systematically the measure of every ball of a centered packing in its diameter. We present in this paper a new approach that consists in introducing quantities not connecting necessarily the measure of every ball of a centered packing in its diameter.

We were inspired by the work of F. Ben Nasr in [2] to establish a new inequality involving $\mathrm{Dim}(\overline{X}^{\alpha})$, where

$$\overline{X}^{\alpha}=\left\{x\in\operatorname{supp}\mu,\ \limsup_{r\to0}\frac{\log\mu\big(B(x,r)\big)}{\log r}\le\alpha\right\}.$$

Our result would be better if we take place outside of the framework of the classic measure classes as illustrates it the example developed in the third section on a class of of coin-tossing measure. This paper contains three sections. We finish this section by giving some preliminaries. In the second section we establish our main theorems. Finally in the third section we develop an application of our result.

We present afterward the multifractal formalism and the main results established by Olsen in [10]. We shall use these preliminaries in the following sections.

Let μ be a Borel probability measure on $\mathbb{R}^d$. For $E\subset\mathbb{R}^d$, $q,\ t\in\mathbb{R}$ and $\varepsilon>0$, by adopting the convention

$$\begin{cases}0^q=+\infty,\ q<0,\\ 0^0=1,\end{cases}$$

put

$$\overline{P}^{q,t}_{\mu,\varepsilon}(E)=\sup\left\{\sum_i\mu\big(B\left(x_i,r_i\right)\big)^q(2r_i)^t\right\},$$

where the supremum is taken over all the centered $\varepsilon-$packing $\big(B\left(x_i,r_i\right)\big)_{i\in I}$ of E. Also put

$$\overline{P}^{q,t}_{\mu}(E)=\lim_{\varepsilon\to0}\overline{P}^{q,t}_{\mu,\varepsilon}(E).$$

Since $\overline{P}^{q,t}_{\mu}$ is a prepacking-measure, then we consider,

$$P^{q,t}_{\mu}(E)=\inf\left\{\sum_i\overline{P}^{q,t}_{\mu}(E_i),\ E\subset\bigcup_i E_i\right\}.$$

It is clear that

$$P^{q,t}_{\mu}(E)=\inf\left\{\sum_i\overline{P}^{q,t}_{\mu}(E_i),\ E=\bigcup_i E_i\right\}. \tag{1.1}$$

and

$$P_\mu^{q,t}(E) = \inf\left\{\sum_i \overline{P}_\mu^{q,t}(E_i),\ \left(\bigcup_i E_i\right) \text{ is a partition of } E\right\}. \tag{1.2}$$

The prepacking-measure $\overline{P}_\mu^{q,t}$ and the measure $P_\mu^{q,t}$ assign respectively a dimension to each subset E. These dimensions are respectively denoted by $\Delta_\mu^q(E)$ and $\mathrm{Dim}_\mu^q(E)$. They are respectively characterized by

$$\overline{P}_\mu^{q,t}(E) = \begin{cases} \infty \text{ if } t < \Delta_\mu^q(E) \\ 0 \text{ if } t > \Delta_\mu^q(E) \end{cases}$$

and

$$P_\mu^{q,t}(E) = \begin{cases} \infty \text{ if } t < \mathrm{Dim}_\mu^q(E) \\ 0 \text{ if } t > \mathrm{Dim}_\mu^q(E) \end{cases}$$

Remark 1. The numbers $\Delta_\mu^q(E)$ and $\mathrm{Dim}_\mu^q(E)$ are respectively the multifractal extensions of the prepacking dimension $\Delta(E)$ and the packing dimension $\mathrm{Dim}(E)$ of E [16], in fact

$$\Delta_\mu^0(E) = \Delta(E) \qquad \text{and} \qquad \mathrm{Dim}_\mu^0(E) = \mathrm{Dim}(E).$$

Note that L. Olsen established in [10] the following results

$$\mathrm{Dim}(E) \leq \Delta(E) \tag{1.3}$$

and

$$\mathrm{Dim}\left(\bigcup_n E_n\right) = \sup_n \mathrm{Dim}(E_n). \tag{1.4}$$

In [10], L. Olsen also proved the proposition and the theorem below.

PROPOSITION 1. *Write* $\Lambda_\mu(q) = \Delta_\mu^q(\operatorname{supp}\mu)$ *and* $B_\mu(q) = \mathrm{Dim}_\mu^q(\operatorname{supp}\mu)$.

(i) $B_\mu \leq \Lambda_\mu$, $B_\mu(1) = \Lambda_\mu(1) = 0$.

(ii) $\Lambda_\mu(0) = \Delta(\operatorname{supp}\mu)$ *and* $B_\mu(0) = \mathrm{Dim}(\operatorname{supp}\mu)$.

(iii) *The functions* $\Lambda_\mu : q \mapsto \Lambda_\mu(q)$ *and* $B_\mu : q \mapsto B_\mu(q)$ *are convex and decreasing.*

THEOREM 1. *For* $\alpha \geq 0$, *if* $\alpha q + B_\mu(q) \geq 0$, *then*

$$\mathrm{Dim}(\overline{X}^\alpha) \leq \inf_{q\geq 0}\big(\alpha q + B_\mu(q)\big).$$

2. Another inequality involving $\mathrm{Dim}(\overline{X}^{\alpha})$

In this section, before giving our new inequality involving $\mathrm{Dim}(\overline{X}^{\alpha})$ for $\alpha > 0$, we start by illustrating our main idea on the set E^{α} defined by

$$E^{\alpha} = \{x \in \operatorname{supp}\mu, \text{ if } r < r_0 \text{ then } (2r)^{\alpha} \leq \mu\big(B(x,r)\big)\}, \tag{2.1}$$

where $r_0 > 0$ is a fixed number. Note that the set E^{α} is very similar to $\overline{X}^{\alpha}$ and that it is established in the classic formalism [10] the inequality

$$\mathrm{Dim}(E^{\alpha}) \leq \inf_{q \geq 0}\big(\alpha q + B_{\mu}(q)\big), \tag{2.2}$$

which proof is based on the inequality

$$\frac{\log \mu\big(B(x_i,r_i)\big)}{\log 2r_i} \leq \alpha,$$

where $(B(x_i,r_i))_{i\in I}$ is a centered ε–packing of E^{α}.

First, we give the definition of a centered ε–k–Besicovich packing of a set $E \subset \mathbb{R}^d$.

Definition 1. Let $\varepsilon > 0$ be a real number and $k \geq 1$ be an integer. A family $\big(B(x_i,r_i)\big)_{i\in I}$ is called a centered ε–k–Besicovich packing of a set E when there exists a finite partition of I such that $I = I_1 \cup \cdots \cup I_s$ with $1 \leq s \leq k$ and $\big(B(x_i,r_i)\big)_{i\in I_j}$ a centered ε–packing of E for all $1 \leq j \leq s$.

This definition is useful to introduce the following quantities not connecting necessarily the measure of every ball of a centered packing in its diameter.

For all $\varepsilon > 0$, let $(u_{\varepsilon})_{\varepsilon>0}$ be a decreasing family of numbers such that $\varepsilon \leq u_{\varepsilon}$ and $\lim_{\varepsilon\to 0} u_{\varepsilon} = 0$. Let $k \geq 1$ be an integer. If $E \subset \operatorname{supp}\mu$, for each centered ε–packing $\big(B(x_i,r_i)\big)_{i\in I}$ of E, we consider all the families $\big(B(y_i,\delta_i)\big)_{i\in I}$ that are centered u_{ε}–k–Besicovich packing, and we define the quantity

$$L^{k}_{\varepsilon,(B(x_i,r_i))_{i\in I}}(E) = \inf\left(\sup_{i\in I}\left(\frac{\log \mu\big(B(y_i,\delta_i)\big)}{\log 2r_i}\right)\right),$$

where the infimum is taken over all the centered u_{ε}–k–Besicovich packing $\big(B(y_i,\delta_i)\big)_{i\in I}$. Now write

$$L^{k}_{\varepsilon}(E) = \sup\left\{L^{k}_{\varepsilon,(B(x_i,r_i))_{i\in I}}(E)\right\},$$

where the supremum is taken over all the centered ε–packing $\big(B\,(x_i, r_i)\,\big)_{i\in I}$ of E. Remark that

$$L^k_{\varepsilon,\left(B(x_i,r_i)\right)_{i\in I}}(E) \leq \sup_{i\in I}\left(\frac{\log\mu\big(B(x_i,r_i)\big)}{\log 2r_i}\right). \tag{2.3}$$

On the other hand, when $\varepsilon < \varepsilon'$, $L^k_{\varepsilon'}(E) > L^k_{\varepsilon}(E)$, then we define

$$L^k(E) = \lim_{\varepsilon\to 0} L^k_{\varepsilon}(E).$$

As the sequence $\big(L^k(E)\big)_k$ is decreasing, write

$$L(E) = \lim_{k\to+\infty} L^k(E).$$

Proposition 2. *For $\alpha > 0$,*

$$L(E^\alpha) \leq \alpha.$$

Proof. For $\varepsilon < 2r_0$ and $\big(B(x_i,r_i)\big)_{i\in I}$ a centered ε–packing of E^α, thanks to the characteristic property of E^α (2.1), we have for all $i \in I$,

$$\frac{\log\mu\big(B(x_i,r_i)\big)}{\log 2r_i} \leq \alpha,$$

hence

$$\sup_{i\in I}\frac{\log\mu\big(B(x_i,r_i)\big)}{\log 2r_i} \leq \alpha,$$

from the inequality (2.3), we deduce that

$$L^k_{\varepsilon,(B(x_i,r_i))_{i\in I}}(E^\alpha) \leq \alpha,$$

while considering the supremum over all the centered ε–packing, it results that $L^k_\varepsilon(E^\alpha) \leq \alpha$. Letting $\varepsilon \to 0$, we obtain that $L^k(E^\alpha) \leq \alpha$, then letting $k \to +\infty$, it follows that $L(E^\alpha) \leq \alpha$. ∎

Theorem 2. *Assume $s := \inf_q B_\mu(q) < 0$. Then for $\alpha > 0$,*

$$\mathrm{Dim}(E^\alpha) \leq \frac{L(E^\alpha)}{\alpha}\inf_{q\geq 1}\big(\alpha q + B_\mu(q)\big).$$

Proof. The proof of the Theorem 2 is organized in two steps.

1^{st} *Step*: For $z > s$, put $\psi(z) = \inf B_\mu^{-1}(]-\infty, z[)$. Then

$$\inf\left\{\psi(\alpha t) + t,\ \frac{s}{\alpha} < t < 0\right\} = \frac{1}{\alpha}\inf_{q\geq 1}\big(\alpha q + B_\mu(q)\big).$$

In fact, as B_μ is convex, decreasing on $[0, +\infty[$ and taking strictly negative values for $\frac{s}{\alpha} < t < 0$, there exists an unique $q > 1$ such that $\alpha t = B_\mu(q)$. We deduce that for all $n \in \mathbb{N} \setminus \{0\}$, $B_\mu(q + \frac{1}{n}) < B_\mu(q) = \alpha t$. Thus $q + \frac{1}{n} \in B_\mu^{-1}(]-\infty, \alpha t[)$, therefore $\psi(\alpha t) \leq q$.

On the other hand, from the equalities

$$\psi(\alpha t) = \inf B_\mu^{-1}(]-\infty, \alpha t[) = \inf\{\theta,\ B_\mu(\theta) < \alpha t\} = \inf\{\theta,\ \theta > q\},$$

it follows that $\psi(\alpha t) \geq q$. So $\psi(\alpha t) = q$.

Two possible cases appear :

If $s = -\infty$, then it is clear that

$$\inf\left\{\psi(\alpha t) + t,\ \frac{s}{\alpha} < t < 0\right\} = \inf\left\{q + \frac{B_\mu(q)}{\alpha},\ q > 1\right\}.$$

If $s > -\infty$, put $q_s = \inf_{q>1}\{q,\ B_\mu(q) = s\}$. As B_μ is convex, it follows that for all $q \geq q_s$, $B_\mu(q) = s$. Then $q + \frac{B_\mu(q)}{\alpha} \geq q_s + \frac{s}{\alpha}$. It results that

$$\inf\left\{q + \frac{B_\mu(q)}{\alpha},\ q > 1\right\} = \inf\left\{q + \frac{B_\mu(q)}{\alpha},\ 1 < q \leq q_s\right\}.$$

Consider a sequence (q_n) such that $q_n \to q_s$ and $1 < q_n < q_s$. As B_μ is continuous, surely we obtain that $q_n + \frac{B_\mu(q_n)}{\alpha} \to q_s + \frac{B_\mu(q_s)}{\alpha}$. It follows that

$$\inf\left\{q + \frac{B_\mu(q)}{\alpha},\ q > 1\right\} = \inf\left\{q + \frac{B_\mu(q)}{\alpha},\ 1 < q < q_s\right\},$$

i.e.,

$$\inf\left\{q + \frac{B_\mu(q)}{\alpha},\ q > 1\right\} = \inf\left\{\psi(\alpha t) + t,\ \frac{s}{\alpha} < t < 0\right\}.$$

Otherwise, $(1 + \frac{1}{n}) + \frac{B_\mu(1+\frac{1}{n})}{\alpha} \to 1 + \frac{B_\mu(1)}{\alpha}$, so

$$\inf\left\{q + \frac{B_\mu(q)}{\alpha},\ q > 1\right\} = \inf\left\{q + \frac{B_\mu(q)}{\alpha},\ q \geq 1\right\}.$$

Finally,

$$\inf\Big\{\psi(\alpha t)+t,\ \frac{s}{\alpha}<t<0\Big\}=\frac{1}{\alpha}\inf_{q\geq 1}\big(\alpha q+B_\mu(q)\big).$$

2^{nd} *Step*: From the first step, it follows that

$$\text{if } \frac{s}{\alpha}<t<0, \text{ then } \psi(\alpha t)+t\geq 0.$$

For $\gamma>0$ and $\frac{s}{\alpha}<t<0$, if $\gamma>\psi(\alpha t)+t$, then $B_\mu(\gamma-t)<\alpha t$. It results that $P_\mu^{\gamma-t,\alpha t}(\operatorname{supp}\mu)=0$. Then $P_\mu^{\gamma-t,\alpha t}(E^\alpha)=0$. According to the equality (1.1), we can write

$$E^\alpha=\bigcup_{m\in M}E_m \tag{2.4}$$

such that for all $m\in M$,

$$\overline{P}_\mu^{\gamma-t,\alpha t}(E_m)<\infty. \tag{2.5}$$

Let $\lambda>L\left(E^\alpha\right)$. First of all let us prove that for all $m\in M$, $\triangle(E_m)\leq\gamma\lambda$. As $E_m\subset E^\alpha$ and $\lambda>L(E_m)$, then there exists an integer $k\geq 1$ and $\varepsilon_0<2r_0$ such that for all $\varepsilon<\varepsilon_0$,

$$L_\varepsilon^k(E_m)<\lambda.$$

It follows that for all $\big(B(x_i,r_i)\big)$ a centered ε–packing of E_m, there exists a centered u_ε–k–Besicovich packing $\big(B(y_i,\delta_i)\big)_{i\in I}$ of E_m such that for $i\in I$,

$$\frac{\log\mu\big(B(y_i,\delta_i)\big)}{\log 2r_i}<\lambda.$$

So that

$$\mu\big(B(y_i,\delta_i)\big)>(2r_i)^\lambda. \tag{2.6}$$

Let us recall that thanks to the characteristic property of E^α (2.1), we can write

$$(2\delta_i)^\alpha<\mu\big(B(y_i,\delta_i)\big). \tag{2.7}$$

Thus, from the inequalities (2.6) and (2.7) we obtain that for $\gamma>0$ and $\frac{s}{\alpha}<t<0$,

$$(2r_i)^{\gamma\lambda}\leq\mu\big(B(y_i,\delta_i)\big)^{\gamma-t}(2\delta_i)^{\alpha t}.$$

Then, using the equality $I=I_1\cup\cdots\cup I_s$ with $1\leq s\leq k$ and $(B(x_i,r_i))_{i\in I_j}$ a centered $\varepsilon-$packing of E_m for all $1\leq j\leq s$, it follows that

$$\sum_{i\in I}(2r_i)^{\gamma\lambda}\leq\sum_{i\in I}\mu\big(B(y_i,\delta_i)\big)^{\gamma-t}(2\delta_i)^{\alpha t}=\sum_{j=1}^{s}\sum_{i\in I_j}\mu\big(B(y_i,\delta_i)\big)^{\gamma-t}(2\delta_i)^{\alpha t},$$

Therefore, it results that

$$\sum_{i\in I}(2r_i)^{\gamma\lambda} \le k\overline{P}^{\gamma-t,\alpha t}_{\mu,u_\varepsilon}(E_m). \tag{2.8}$$

Note that from the inequality (2.5), we can write that there exists $\varepsilon_1 > 0$ such that for all $u_\varepsilon < \varepsilon_1$,

$$\overline{P}^{\gamma-t,\alpha t}_{\mu,u_\varepsilon}(E_m) < \infty.$$

Then from the inequality (2.8) it comes that for all $m \in M$,

$$\triangle(E_m) \le \gamma\lambda.$$

Therefore, from the inequality (1.3), we obtain that

$$\mathrm{Dim}(E_m) \le \gamma\lambda, \ \ m \in M.$$

And from the equalities (2.4) and (1.4), we deduce that

$$\mathrm{Dim}(E^\alpha) \le \gamma\lambda.$$

So for $\frac{s}{\alpha} < t < 0$,

$$\mathrm{Dim}(E^\alpha) \le L(E^\alpha)\inf\left\{\psi(\alpha t) + t, \ \frac{s}{\alpha} < t < 0\right\}.$$

Finally, according to the first step, it results that for all $\alpha > 0$,

$$\mathrm{Dim}(E^\alpha) \le \frac{L(E^\alpha)}{\alpha}\inf_{q\ge 1}\big(\alpha q + B_\mu(q)\big).$$

∎

Let us now present our main theorem. Thereafter, for $\eta > \alpha$ and $p \in \mathbb{N}\setminus\{0\}$, write

$$X_\alpha(\eta,p) = \left\{x \in \overline{X}^\alpha, \ \text{ if } \ 2r < \frac{1}{p} \ \text{ then } \ (2r)^\eta \le \mu\big(B(x,r)\big)\right\}.$$

It is clear that $X_\alpha(\eta,p) \subset X_\alpha(\eta,p+1)$. Furthermore, considering the equality

$$\overline{X}^\alpha = \left\{x \in \operatorname{supp}\mu, \ \limsup_{r\to 0}\frac{\log\mu\big(B(x,r)\big)}{\log 2r} \le \alpha\right\},$$

it occurs that

$$\overline{X}^\alpha = \bigcup_p X_\alpha(\eta,p). \tag{2.9}$$

Theorem 3. *Assume* $s := \inf_q B_\mu(q) < 0$. *For* $\alpha > 0$, *write*

$$T_\mu(\alpha,\eta,p) = \sup_{E\subset X_\alpha(\eta,p)} L(E),$$

$$T_\mu(\alpha,\eta) = \lim_{p\to+\infty} T_\mu(\alpha,\eta,p),$$

$$T_\mu(\alpha) = \lim_{\eta\to\alpha^+} T_\mu(\alpha,\eta),$$

then,

$$\mathrm{Dim}(\overline{X}^\alpha) \le \frac{T_\mu(\alpha)}{\alpha} \inf_{q\ge 1}\big(\alpha q + B_\mu(q)\big).$$

Remark 2. The limits $T_\mu(\alpha,\eta)$ and $T_\mu(\alpha)$ are well defined, indeed the sequence $\big(T_\mu(\alpha,\eta,p)\big)_{p\ge1}$ is increasing, since $X_\alpha(\eta,p) \subset X_\alpha(\eta,p+1)$ and for all $\eta < \eta'$, $X_\alpha(\eta,p) \subset X_\alpha(\eta',p)$, what involves that the quantity $T_\mu(\alpha,\eta)$ is decreasing when $\eta \to \alpha$. Note that, according to the Proposition 2, the inequality $\frac{L(E^\alpha)}{\alpha} \le 1$ is true, as well as the inequality $\frac{T_\mu(\alpha)}{\alpha} \le 1$. Moreover, when the equality

$$\inf_{q\ge1}\big(\alpha q + B_\mu(q)\big) = \inf_{q\ge0}\big(\alpha q + B_\mu(q)\big), \tag{2.10}$$

occurs, the hypothesis ($s := \inf_q B_\mu(q) < 0$) of Theorem 2 and Theorem 3 is satisfied and the inequality given in Theorem 3 is better than the one established by L. Olsen in Theorem 1.

Proof. We stand in the interesting case where $\overline{X}^\alpha \ne \varnothing$. From the Theorem 2 it follows that for all $\eta > \alpha$,

$$\mathrm{Dim}\,\big(X_\alpha(\eta,p)\big) \le \frac{L\big(X_\alpha(\eta,p)\big)}{\eta} \inf_{q\ge1}\big(\eta q + B_\mu(q)\big).$$

Thus

$$\mathrm{Dim}\,\big(X_\alpha(\eta,p)\big) \le \frac{T_\mu(\alpha,\eta,p)}{\eta} \inf_{q\ge1}\big(\eta q + B_\mu(q)\big).$$

From the equalities (2.9) and (1.4), letting $p \to +\infty$, we obtain that

$$\mathrm{Dim}(\overline{X}^\alpha) \le \frac{T_\mu(\alpha,\eta)}{\eta} \inf_{q\ge1}\big(\eta q + B_\mu(q)\big).$$

Finally, letting $\eta \to \alpha$, it follows that

$$\mathrm{Dim}(\overline{X}^\alpha) \le \frac{T_\mu(\alpha)}{\alpha} \inf_{q\ge1}\big(\alpha q + B_\mu(q)\big).$$

∎

3. Example

In this paragraph, we intend to construct a class of coin-tossing measures μ verifying the conditions

$$\frac{T_\mu(\alpha)}{\alpha} < 1 \quad \text{and} \quad \inf_{q\geq 1}\big(\alpha q + B_\mu(q)\big) = \inf_{q\geq 0}\big(\alpha q + B_\mu(q)\big). \tag{3.1}$$

that will lead us, thanks to the Theorem 3, to establish that

$$\operatorname{Dim}(\overline{X}^\alpha) \leq \frac{T_\mu(\alpha)}{\alpha}\inf_{q\geq 1}\big(\alpha q + B_\mu(q)\big) < \inf_{q\geq 0}\big(\alpha q + B_\mu(q)\big).$$

Put $\mathcal{A}$ the set of the words constructed with $\{0,1\}$ as alphabet. The length of a word j is denoted by $|j|$. For all $j \in \mathcal{A}$, put $N_0(j)$ the number of times the letter 0 appears in j. If j, $j' \in \mathcal{A}$, write jj' the word starting by j and gotten while putting j' after j. For all $j \in \mathcal{A}$ such that $j = j_1 j_2 \cdots j_n$, put I_j the dyadic interval of order n defined by

$$I_j = \left[\sum_{k=1}^{n}\frac{j_k}{2^n}, \sum_{k=1}^{n}\frac{j_k}{2^n} + \frac{1}{2^n}\right[.$$

We denote by $\mathcal{F}_n$ the family of all the dyadic intervals of order n and for all $x \in [0,1[$ we call $I_n(x)$ the element of $\mathcal{F}_n$ containing x. Let $0 < p_0 \leq p_1$ such that $p_0 + p_1 = 1$ and $\mathcal{L} \subset (\cup_n \mathcal{F}_n)$ we associate the following measure μ on $\mathbb{R}$ such that

$$\mu\big(\mathbb{R}\setminus[0,1[\big) = 0$$

and for all $I_j \in \mathcal{F}_n$ and $l \in \{0,1\}$,

$$\mu\left(I_{jl}\right) = \begin{cases} p_l\mu(I_j) & \text{if } I_j \text{ contains an interval of } \mathcal{L}, \\ \dfrac{\mu(I_j)}{2} & \text{otherwise.} \end{cases}$$

It is clear that $\operatorname{supp}\mu = [0,1]$. For the construction of the example satisfying the conditions (3.1), we choose the part $\mathcal{L}$ as follows. Let β_1, β_2, γ_1 and γ_2 be real numbers such that

$$\frac{1}{2} < \beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \frac{1}{3}.$$

We say that an interval $I_j \in \mathcal{F}_n$ is of

$$\text{type } T_1 \text{ if } \ \beta_1 < \frac{N_0(j)}{n} < \gamma_1, \quad \text{and of} \quad \text{type } T_2 \text{ if } \ \beta_2 < \frac{N_0(j)}{n} < \gamma_2.$$

Let $I \in \mathcal{F}_n$ be of type T_1 (respectively of type T_2), put $\tilde{I}$ the set of intervals of order $n+6$ contained in I and of the same type that I. Let $n_0 \in \mathbb{N}$ be a multiple of 6 and (n_p) be the sequence of integers defined by:

$$n_0, \quad n_{3i+1} = 2^{n_{3i}} n_0, \quad n_{3i+2} = 2n_{3i+1} \quad \text{and} \quad n_{3i+3} = 2n_{3i+2}.$$

Remark that

$$n_p = n_0 + 6k, \ k \in \mathbb{N}.$$

For $k \in \mathbb{N}$ we construct the family $\mathcal{G}_k$ of disjoined dyadic intervals of order $n_0 + 6k$ such that:

- any element I_j of $\mathcal{G}_k$ verifies the relation

$$\beta_1 < \frac{N_0(j)}{n} < \gamma_2,$$

- $\mathcal{G}_0$ contains two intervals $I^1_{n_0}$ and $I^2_{n_0}$ respectively of type T_1 and T_2,
- any element of $\mathcal{G}_{k+1}$ is contained in an element of $\mathcal{G}_k$ that we call his father,
- all the elements of $\mathcal{G}_k$ give birth to the same number of son in $\mathcal{G}_{k+1}$, to pass from $\mathcal{G}_k$ to $\mathcal{G}_{k+1}$ we distinguish the three following cases:

  1^{st} *case*: If $n_{3i} \leq n_0 + 6k < n_{3i+1}$, then for each $I \in \mathcal{G}_k$ we select two intervals in $\widetilde{I}$. So $\mathcal{G}_{k+1}$ is the union of all these selected intervals.

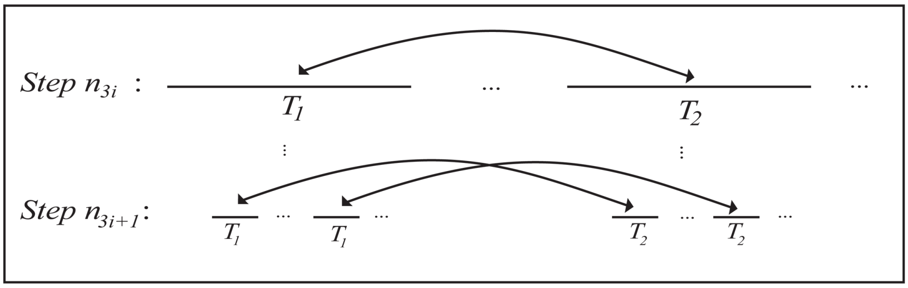
  2^{nd} *case*: If $n_{3i+1} \leq n_0 + 6k < n_{3i+2}$, then for each $I \in \mathcal{G}_k$ of type T_1 we select an interval in $\widetilde{I}$, and for each $I \in \mathcal{G}_k$ of type T_2 we select an interval I_j of order $n_0 + 6(k+1)$ containing a selected interval of order n_{3i+2} of type T_1. So $\mathcal{G}_{k+1}$ is the union of all these selected intervals. Note that all the intervals in $\mathcal{G}_{n_{3i+1}}$ are of type T_1.

  3^{rd} *case*: If $n_{3i+2} \leq n_0 + 6k < n_{3i+3}$, then for each $I \in \mathcal{G}_k$ having an ancestor of order n_{3i+1} and of type T_1, we select an interval I_j of order $n_0 + 6(k+1)$ containing a selected interval of order n_{3i+3} of type T_2, and for each $I \in \mathcal{G}_k$ remaining of type T_1 we select an interval in $\widetilde{I}$. So $\mathcal{G}_{k+1}$ is the union of all these selected intervals.

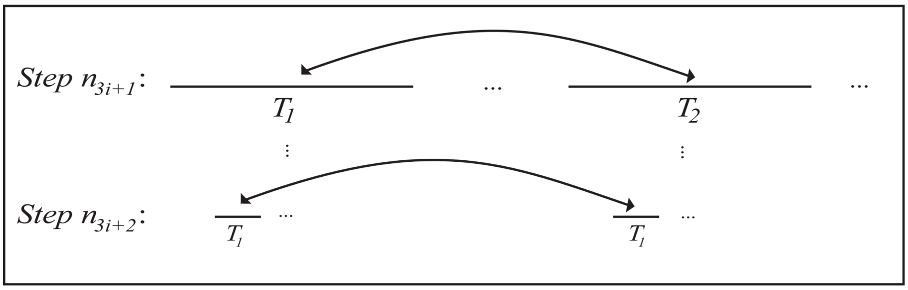
An elementary calculus of counting assures that the construction of the family $(\cup_{k\geq 0} \mathcal{G}_k)$ is possible for any n_0 big enough, and allows us furthermore to impose the following separation condition: for all $k \geq 0$, if $I, J \in \mathcal{G}_k$ are of order n, then the distance between I and J is bigger than $\frac{1}{2^{n-1}}$. Besides for all $k \geq 1$, if $I \in \mathcal{G}_k$ is of order n, then the distances between I and his father's endpoints are bigger than $\frac{1}{2^n}$. We associate the following relation on $(\cup_{k\geq 0} \mathcal{G}_k)$: the two elements of $\mathcal{G}_0$ are in relation and two elements of $\mathcal{G}_{k+1}$ are in relation if their fathers, elements of $\mathcal{G}_k$, are in relation. Thereafter we write $\mathcal{L} = (\cup_{k\geq 0} \mathcal{G}_k)$ and we call selected interval any interval $I \in \mathcal{L}$.

The diagram below summarizes this construction, note that the arrows indicate the intervals of the same generation that are in relation.

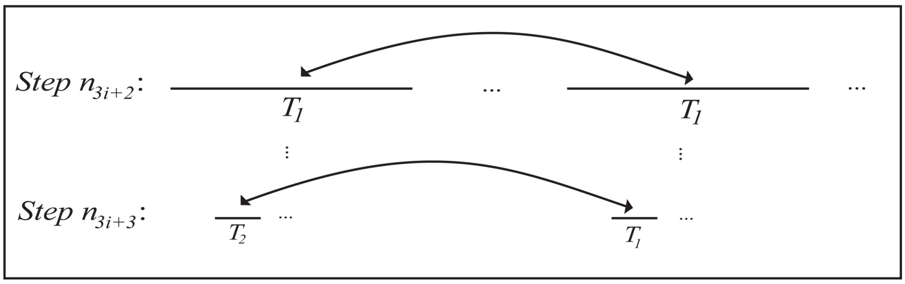
1st case



2nd case



3rd case



Proposition 3.

$$\lim_{q\to+\infty} B_\mu(q) = -\infty.$$

Proof. First, let us remark that for all $I_j \in \mathcal{F}_n$,

$$p_0^n \le \mu\left(I_j\right) \le p_1^n. \tag{3.2}$$

Let $\big(B(x_i,r_i)\big)_{i\in I}$ be a centered ε–packing of $\operatorname{supp}\mu$. For all $i \in I$, let us consider the largest interval $I_n(x_i)$ included in $B(x_i,r_i)$. It results that $B(x_i,r_i)$ is covered by at the more two contiguous intervals of $\mathcal{F}_{n-1}$. It follows that

$$\frac{1}{2^n} \le 2r_i \le \frac{1}{2^{n-2}} \tag{3.3}$$

and according to (3.2), we obtain

$$p_0^n \le \mu\big(B(x_i,r_i)\big) \le 2p_1^{n-1}. \tag{3.4}$$

From (3.3), we deduce that for all $t \in \mathbb{R}$, there exist c_1, $c_2 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$\frac{c_1}{2^{nt}} \le (2r_i)^t \le \frac{c_2}{2^{nt}}, \tag{3.5}$$

and from (3.4), it follows that for all $q > 0$,

$$p_0^{nq} \le \mu\big(B(x_i,r_i)\big)^q \le 2^q p_1^{(n-1)q}. \tag{3.6}$$

Then, considering (3.5) and (3.6), there exists $c_3 \in \mathbb{R}$ such that

$$\mu\big(B(x_i,r_i)\big)^q (2r_i)^t \le c_3 2^q p_1^{(n-1)q} 2^{-nt}. \tag{3.7}$$

Otherwise, for all $n \in \mathbb{N}\setminus\{0\}$, any interval of $\mathcal{F}_{n-1}$, meets to the more two balls of $\big(B(x_i,r_i)\big)_{i\in I}$ verifying the relation $\frac{1}{2^n} \le 2r_i \le \frac{1}{2^{n-2}}$, so according to (3.7), there exists a constant C that only depends on q and t such that

$$\sum_{\frac{1}{2^n}\le 2r_i\le\frac{1}{2^{n-2}}} \mu\big(B(x_i,r_i)\big)^q (2r_i)^t \le C(2\,p_1^q\,2^{-t})^n. \tag{3.8}$$

For $\varepsilon > 0$ small enough, while writing,

$$\sum_{i\in I} \mu\big(B(x_i,r_i)\big)^q (2r_i)^t = \sum_{n\ge1} \sum_{\frac{1}{2^n}\le 2r_i\le\frac{1}{2^{n-2}}} \mu\big(B(x_i,r_i)\big)^q (2r_i)^t,$$

it comes from the inequality (3.8) that

$$\sum_{i\in I}\mu\big(B(x_i,r_i)\big)^q(2r_i)^t<\infty,\quad t>1+q\,\frac{\log p_1}{\log 2}.$$

We deduce that

$$\Lambda_\mu(q)\le 1+q\,\frac{\log p_1}{\log 2},\quad q>0.$$

Then, according to the Proposition 1

$$B_\mu(q)\le 1+q\,\frac{\log p_1}{\log 2},\quad q>0. \tag{3.9}$$

Finally

$$\lim_{q\to+\infty}B_\mu(q)=-\infty.$$

∎

Proposition 4. *Put $B'_{\mu-}(1)$ the left derivative number of B_μ at 1. Then*

$$B'_{\mu-}(1)\le -1.$$

Proof. Let us recall that $B_\mu(1)=0$ and B_μ is convex. So to prove that $B'_{\mu-}(1)\le -1$, it is sufficient to establish that for all $q<1$,

$$B_\mu(q)\ge 1-q,$$

what comes back to show that, according to (1.2), if $(\cup_i E_i)$ is a partition of $\operatorname{supp}\mu$, then $\sum_{i\in I}\overline{P}^{q,t}_\mu(E_i)=\infty$. Let us consider the case where for all $i\in I$, $\overline{P}^{q,t}_\mu(E_i)<\infty$, the contrary case is obvious. Put $0<\varepsilon<\frac{1}{2^{n_0}}$. For all $i\in I$, choose $\delta_i<\varepsilon$ such that

$$\overline{P}^{q,t}_{\mu,\delta_i}(E_i)\le\overline{P}^{q,t}_\mu(E_i)+\frac{1}{2^i}. \tag{3.10}$$

According to the Besicovitch covering theorem [7], there exists an integer ζ (that only depends on $\mathbb{R}$) such that each E_i is covered by $\cup_{u=1}^{\zeta}\big(\cup_j B(x_{ij},\delta_i)\big)$ and for all $1\le u\le\zeta$, $\big(B(x_{ij},\delta_i)\big)_j$ is a packing. Considering (3.10), it follows that

$$\sum_{u=1}^{\zeta}\sum_j\mu\big(B(x_{ij},\delta_i)\big)^q(2\delta_i)^t\le\zeta\left(\overline{P}^{q,t}_\mu(E_i)+\frac{1}{2^i}\right).$$

Then,

$$\sum_i \left(\sum_{u=1}^{\zeta} \sum_j \mu\big(B(x_{ij}, \delta_i)\big)^q (2\delta_i)^t \right) \leq \zeta \sum_i \overline{P}_\mu^{q,t}(E_i) + \zeta. \tag{3.11}$$

Let us consider the sum

$$\sum_i \left(\sum_{u=1}^{\zeta} {\sum_j}' \mu\big(B(x_{ij}, \delta_i)\big)^q (2\delta_i)^t \right), \tag{3.12}$$

where $\sum_j'$ is taken on all j such that the distance between x_{ij} and $I_{n_0}^1$ (respectively $I_{n_0}^2$) is bigger than $\frac{1}{2^{n_0}}$. In this case, there exists $C \in \mathbb{R}$ that only depends on n_0 such that

$$\mu\big(B(x_{ij}, \delta_i)\big) \leq C\, m\big(B(x_{ij}, \delta_i)\big),$$

where m is the Lebesgue measure. We deduce that

$$C^{q-1}(2\delta_i)^{q-1+t} \leq \mu\big(B(x_{ij}, \delta_i)\big)^{q-1}(2\delta_i)^t \mu\big(B(x_{ij}, \delta_i)\big). \tag{3.13}$$

Otherwise, the union of the balls that appear in the sum (3.12) recovers $\operatorname{supp}\mu$ deprived of $I_{n_0}^1$, $I_{n_0}^2$ and the intervals of order n_0 that their are contiguous. Therefore, according to (3.13), we obtain that

$$\left(1 - \frac{6}{2^{n_0}}\right) C^{q-1}(2\varepsilon)^{q-1+t} \leq \sum_i \left(\sum_{u=1}^{\zeta} {\sum_j}' \mu\big(B(x_{ij}, \delta_i)\big)^q (2\delta_i)^t \right).$$

We deduce that, while considering (3.11),

$$\left(1 - \frac{6}{2^{n_0}}\right) C^{q-1}(2\varepsilon)^{q-1+t} \leq \zeta \sum_i \overline{P}_\mu^{q,t}(E_i) + \zeta.$$

Letting $\varepsilon \to 0$, it results that $\sum_{i \in I} \overline{P}_\mu^{q,t}(E_i) = \infty$ while $t < 1 - q$, so that

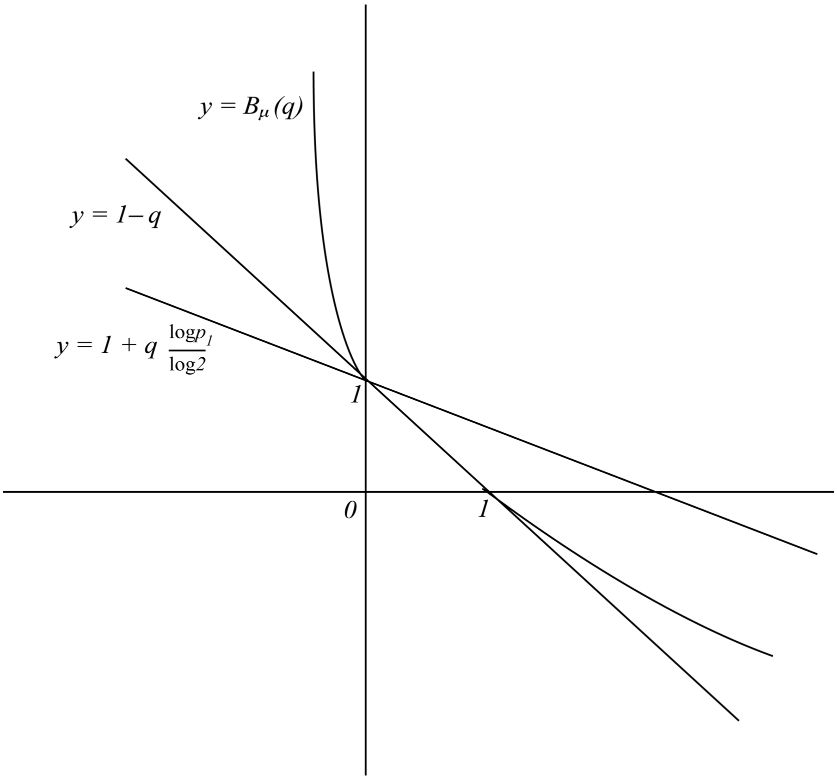
$$B_\mu(q) \geq 1 - q, \quad q < 1. \tag{3.14}$$

∎

Remark 3. From the Proposition 1, and the inequality (3.14), we can deduce that

$$B_\mu(q) = 1 - q,\ q \in [0, 1]\,.$$

Moreover while considering the inequality (3.9), we draw below the typical shape of the function B_μ.



PROPOSITION 5. *Consider the Cantor set*

$$\mathcal{C} = \bigcap_{k\geq 1}\left(\bigcup_{I_j\in\mathcal{G}_k} I_j\right)$$

and the function g defined on $[0,1]$ by

$$g(x) = -\frac{x\log\left(\frac{p_0}{p_1}\right) + \log p_1}{\log 2}.$$

(i) *If $x \notin \mathcal{C}$, then*

$$\lim_{r\to 0}\frac{\log(\mu\big(B(x,r)\big)}{\log 2r} = 1.$$

(ii) *If $x \in \mathcal{C}$, then*

$$g(\beta_1) \leq \liminf_{r\to 0}\frac{\log(\mu\big(B(x,r)\big)}{\log 2r} \leq \limsup_{r\to 0}\frac{\log(\mu\big(B(x,r)\big)}{\log 2r} \leq g(\gamma_2).$$

Proof. (i) Put $x \notin \mathcal{C}$. Thanks to the separation condition, for $r > 0$ small enough, the ball $B(x,r)$ is contained in the union of two contiguous intervals of order N, I_N^1 and I_N^2 that do not meet $\mathcal{C}$. For all interval of order n, $I_n \subset I_N^1 \cup I_N^2$ there exist c, $c' \in \mathbb{R}$ such that

$$\frac{c}{2^n} \leq \mu\big(I_n(x)\big) \leq \frac{c'}{2^n}.$$

We deduce that

$$\lim_{n\to+\infty} \frac{\log(\mu\big(I_n(x)\big)}{\log\left(\dfrac{1}{2^n}\right)} = 1. \tag{3.15}$$

Consider the largest interval $I_n(x)$ contained in the ball $B(x,r)$, it follows that $B(x,r)$ is contained in the union of two contiguous intervals of order $n-1$, $I_{n-1}(x)$ and J_{n-1}, thus

$$\frac{c}{2^n} \leq \mu\big(B(x,r)\big) \leq \frac{c'}{2^n}$$

and

$$|I_n(x)| \leq 2r \leq 2\,|I_{n-1}(x)|\,.$$

Therefore, from (3.15), we obtain

$$\lim_{r\to 0} \frac{\log(\mu\big(B(x,r)\big)}{\log 2r} = 1.$$

(ii) It is clear that if $I_j \in \mathcal{G}_k$ is of order n, then

$$\mu\left(I_j\right) = p_0^{N_0(j)} p_1^{n-N_0(j)},$$

thus

$$\mu(I_j) = |I_j|^{g\left(\frac{N_0(j)}{n}\right)}\,. \tag{3.16}$$

Otherwise, let us recall that for $I_j \in \mathcal{G}_k$,

$$\beta_1 < \frac{N_0(j)}{n} < \gamma_2.$$

Since the function g is strictly increasing, it follows that

$$g\left(\beta_1\right) < \frac{\log(\mu\big(I_j(x)\big)}{\log|I_j(x)|} < g\left(\gamma_2\right). \tag{3.17}$$

Put $x \in \mathcal{C}$ and $r < \frac{1}{2^{n_0+6}}$. Thanks to the separation condition, $B(x,r)$ is contained in one of the intervals $I^1_{n_0}$ or $I^2_{n_0}$. Consider the smallest interval $I_n(x)$ containing the ball $B(x,r)$, it follows, from the separation condition, that if $B(x,r)$ doesn't contain the selected interval $I_{n+6}(x)$, then it necessarily contains the selected interval $I_{n+12}(x)$, therefore, we can write

$$\mu\big(I_{n+12}(x)\big) \le \mu\big(B(x,r)\big) \le \mu\big(I_n(x)\big)$$

and

$$|I_{n+12}(x)| \le 2r \le |I_n(x)|\,.$$

From (3.17), it results that

$$g(\beta_1) \le \liminf_{r\to 0} \frac{\log\big(\mu(B(x,r)\big)}{\log 2r} \le \limsup_{r\to 0} \frac{\log\big(\mu(B(x,r)\big)}{\log 2r} \le g(\gamma_2).$$

∎

We stand thereafter in the case where $g(\gamma_2) < 1$, Even if we choose $p_0 > \gamma_2$. Thus, according to the Proposition 5,

$$\overline{X}^{g(\gamma_2)} = \mathcal{C}.$$

In all what follows, we choose the real number α such that

$$g(\gamma_1) < \alpha \le g(\gamma_2) \text{ and } \overline{X}^{\alpha} \neq \varnothing.$$

Proposition 6.

$$T_\mu(\alpha) \le g(\gamma_1) < \alpha.$$

Proof. Put $M \subset X_\alpha(\eta,p)$ and $\big(B(x_i,r_i)\big)$ a centered ε–packing of M. It is clear that for all $i \in I$, $x_i \in \mathcal{C}$. Then consider the largest selected interval $I_n(x_i)$ of order n, containing x_i and contained in $B(x_i,r_i)$. It follows that

$$\frac{1}{2^n} \le 2r_i.$$

Consider the partition $I_1 \cup I_2$ of I such that

$$I_1 = \{i \in I : I_n(x_i) \text{ is of type } T_1\} \text{ and } I_2 = I \setminus I_1.$$

Let us recall that, any interval $I_n(x_i)$, $i \in I_2$, is in relation with an unique selected interval of order n and of type T_1 centered in $x'_i \in M$ that is denoted by $I_n(x'_i)$. Thanks to the separation condition, $\big(B(x'_i, \frac{1}{2^n})\big)_{i\in I_2}$ is a centered

ε–packing of M. Then we consider the family $\big(B(y_i,\delta_i)\big)_{i\in I}$ indexed by I and defined by

$$B(y_i,\delta_i)=\begin{cases}B(x_i,r_i), & i\in I_1\\ B\left(x_i',\frac{1}{2^n}\right), & i\in I_2.\end{cases}$$

We verify that

$$\frac{\log\mu\big(B(y_i,\delta_i)\big)}{\log 2\delta_i}\le\frac{\log\mu\big(I_n(x_i)\big)}{\log\left(\frac{1}{2^n}\right)},\quad i\in I_1$$

and

$$\frac{\log\mu\big(B(y_i,\delta_i)\big)}{\log 2\delta_i}\le\frac{\log\mu\big(I_n(x_i')\big)}{\log\left(\frac{1}{2^n}\right)},\quad i\in I_2.$$

From (3.16) and as g is increasing, we deduce that for all $i\in I$,

$$\frac{\log\mu\big(B(y_i,\delta_i)\big)}{\log 2\delta_i}\le g(\gamma_1).$$

Thus

$$L^2_{\varepsilon,(B(x_i,r_i))_{i\in I}}(M)\le g(\gamma_1).$$

Then $L^2_\varepsilon(M)\le g(\gamma_1)$, letting $\varepsilon\to 0$, we deduce that $L^2(M)\le g(\gamma_1)$. Since the sequence $\big(L^k(M)\big)_k$ is decreasing it follows that $L(M)\le g(\gamma_1)$. Therefore, $T_\mu(\alpha)\le g(\gamma_1)$. But $g(\gamma_1)<\alpha$, and then $T_\mu(\alpha)<\alpha$. ∎

Corollary 1.

$$\operatorname{Dim}(\overline{X}^\alpha)\le\frac{T_\mu(\alpha)}{\alpha}\inf_{q\ge1}\big(\alpha q+B_\mu(q)\big)<\inf_{q\ge0}\big(\alpha q+B_\mu(q)\big).$$

Proof. From the Proposition 3 and the Theorem 3, we deduce the first inequality. Otherwise, as $\alpha<1$ and from the Proposition 4, it follows that $B'_{\mu-}(1)\le-\alpha$. Then

$$\inf_{q\ge1}\big(\alpha q+B_\mu(q)\big)=\inf_{q\ge0}\big(\alpha q+B_\mu(q)\big).$$

Therefore,

$$\frac{T_\mu(\alpha)}{\alpha}\inf_{q\ge1}\big(\alpha q+B_\mu(q)\big)=\frac{T_\mu(\alpha)}{\alpha}\inf_{q\ge0}\big(\alpha q+B_\mu(q)\big).$$

Finally, according to the Proposition 6, we deduce that

$$\frac{T_\mu(\alpha)}{\alpha}\inf_{q\ge1}\big(\alpha q+B_\mu(q)\big)<\inf_{q\ge0}\big(\alpha q+B_\mu(q)\big).$$

∎

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