

Convex Sets without Diametral Pairs

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Abstract: Let X be an infinite dimensional normed linear space. It is not difficult to see that arbitrarily near (in the Hausdorff metric) to the unit ball of X there exists a nonempty closed convex set whose diameter is not attained. We show that such sets are dense in the metric space of all nonempty bounded closed convex subsets of X if and only if either X is not a reflexive Banach space or X is a reflexive Banach space in which every weakly closed set contained in the unit sphere S_X has empty relative interior in S_X .

Key words: Diametral pair, bounded closed convex set, Hausdorff metric.

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Let X be a real normed linear space and $\mathcal{BCC}(X)$ denote the metric space (in the Hausdorff metric) of all nonempty bounded closed convex subsets of X . Two elements of a set $C \in \mathcal{BCC}(X)$, whose distance realizes the diameter of C , are called a *diametral pair* for C . Obviously, if X has a finite dimension, then each element of $\mathcal{BCC}(X)$ admits a diametral pair. For an infinite dimensional X , it is not difficult to see that the sets admitting a diametral pair are dense in $\mathcal{BCC}(X)$ (Observation 2.2), and, on the other hand, the unit ball of X can be approximated by elements of $\mathcal{BCC}(X)$ having no diametral pair (Proposition 2.1). (This latter proposition provides also an answer to a question from [3] – see Remark 2.10.)

The aim of the present paper is to investigate the normed linear spaces X in which the sets without diametral pairs are dense in $\mathcal{BCC}(X)$. It turns out that not every infinite dimensional X has this property. In our main result, Theorem 2.8, we show that denseness of the sets without diametral pairs is equivalent to a certain geometric property (W) (see Definition 2.6) which is satisfied if and only if either X is not a reflexive Banach space or X is a reflexive Banach space whose unit sphere S_X contains no weakly closed sets with a nonempty relative interior in S_X .

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1. NOTATIONS AND PRELIMINARIES

Throughout the present paper, X will denote a real normed linear space with the closed unit ball B_X , the open unit ball $B_X^0 = \text{int } B_X$ and the unit sphere $S_X = \partial B_X$. The closed and open ball of radius $r > 0$, centered at $x \in X$, are the sets $B(x, r) = x + rB_X$ and $B_X^0(x, r) = x + rB_X^0$.

Given a nonempty set $A \subset X$, its *diameter* is defined as

$$\delta(A) = \text{diam } A = \sup\{\|x - y\| : x, y \in A\}.$$

It is easy to see that one always has $\delta(A) = \delta(B)$ whenever $A \subset B \subset \overline{\text{conv}}A$. We say that two points $x, y \in X$ form a *diametral pair* for A if $x, y \in A$ and $\|x - y\| = \delta(A)$. Observe that x, y form a diametral pair for A if and only if $x - y$ is a farthest point from the origin for the set $A - A$.

By $\mathcal{BCC}(X)$ we mean the family of all nonempty, bounded, closed and convex subsets of X , equipped with the Hausdorff metric

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\}.$$

Let us start with the following easy and well known properties of convex hulls, which we shall use several times.

FACT 1.1. (a) If C_1, \dots, C_m are convex sets in X , then

$$\text{conv} \left(\bigcup_{i=1}^m C_i \right) = \left\{ \sum_{i=1}^m \lambda_i c_i : c_i \in C_i, \lambda_i \geq 0, \sum_1^m \lambda_i = 1 \right\}.$$

(b) $\text{conv}(C \cup F)$ is closed whenever $C \in \mathcal{BCC}(X)$ and $F \subset X$ is finite.

We shall also need the following two natural lemmas about diametral pairs.

LEMMA 1.2. Assume that a bounded set $A \subset X$ and a point $z \in X$ are such that $\delta(A) < \delta(D)$ where $D = \overline{\text{conv}}(A \cup \{z\})$. Then every diametral pair for D consists of z and a point of $\overline{\text{conv}}A$.

Proof. We may assume that A is closed and convex, and $z = 0$. In this case, $D = \text{conv}(A \cup \{0\})$ by Fact 1.1(b). Let x, y form a diametral pair for D . By Fact 1.1(a), we can write $x = \lambda a$, $y = \mu b$ where $\lambda, \mu \in [0, 1]$ and $a, b \in A$. Let, for example, $\lambda \geq \mu$. Then

$$\begin{aligned} \delta(D) &= \|\lambda a - \mu b\| \leq \|\lambda a - \mu a\| + \|\mu a - \mu b\| \\ &\leq (\lambda - \mu)\delta(D) + \mu\delta(A) \leq \lambda\delta(D) \leq \delta(D). \end{aligned}$$

Thus all inequalities are, in fact, equalities. But this is possible only if $\lambda = 1$ and $\mu = 0$, which implies $x = a \in A = \overline{\text{conv}}A$ and $y = 0 = z$. ■

LEMMA 1.3. Assume that bounded sets $A, B \subset X$ and a point $z \in X$ are such that $\delta(B \cup \{z\}) < \delta(D)$ where $D = \overline{\text{conv}}(A \cup B \cup \{z\})$. If $x \in D$ is such that $\|x - z\| = \delta(D)$, then $x \in \overline{\text{conv}}A$.

Proof. We may assume that the sets A, B are closed and convex since this leaves unchanged the set D and the involved diameters. Since $x \in D$, we can write (by Fact 1.1(a))

$$x = \lambda_n a_n + \mu_n b_n + \nu_n z + p_n,$$

where $\lambda_n, \mu_n, \nu_n \geq 0$, $\lambda_n + \mu_n + \nu_n = 1$, $a_n \in A$, $b_n \in B$, and $p_n \rightarrow 0$ as $n \rightarrow \infty$. Then

$$\begin{aligned} \delta(D) &= \|x - z\| \leq \lambda_n \|a_n - z\| + \mu_n \|b_n - z\| + \|p_n\| \\ &\leq \lambda_n \delta(D) + \mu_n \delta(B \cup \{z\}) + \|p_n\|. \end{aligned}$$

Consequently, $\mu_n [\delta(D) - \delta(B \cup \{z\})] + \nu_n \delta(D) \leq \|p_n\|$. Since $\|p_n\| \rightarrow 0$, we must have $\mu_n \rightarrow 0$ and $\nu_n \rightarrow 0$, which implies that $\lambda_n \rightarrow 1$. It follows that $x = \lim \lambda_n a_n = \lim a_n \in A$. ■

The last lemma of this section is a variant of [1, Lemma 3.5.1].

LEMMA 1.4. Let $A, B \subset X$ be bounded convex sets and $\varphi: X \rightarrow \mathbb{R}$ a continuous convex function. If $x \in \overline{\text{conv}}(A \cup B)$ is such that

$$\varphi(x) > \sup \varphi(A), \quad \varphi(x) \geq \sup \varphi(B),$$

then $x \in \overline{B}$.

Proof. Put $m = \sup \varphi(A)$. Let $\{x_n\} \subset \text{conv}(A \cup B)$ be a sequence tending to x . By Fact 1.1(a), we can write $x_n = \lambda_n a_n + (1 - \lambda_n) b_n$, where $a_n \in A$, $b_n \in B$ and $\lambda_n \in [0, 1]$. Since φ is convex,

$$\varphi(x_n) \leq \lambda_n \varphi(a_n) + (1 - \lambda_n) \varphi(b_n) \leq \lambda_n m + (1 - \lambda_n) \varphi(x).$$

It follows that $\lambda_n (\varphi(x_n) - m) \leq (1 - \lambda_n) (\varphi(x) - \varphi(x_n))$. Thus, for any

sufficiently large n ,

$$\lambda_n \leq \frac{\varphi(x) - \varphi(x_n)}{\varphi(x_n) - m},$$

which implies that $\lambda_n \rightarrow 0$. Now, $x \in \overline{B}$ since $x - b_n = (x - x_n) + \lambda_n(a_n - b_n) \rightarrow 0$. ■

COROLLARY 1.5. *Let $\{C_k\}$ be a sequence of convex sets in X and $\varphi: X \rightarrow \mathbb{R}$ a continuous convex function. Denote $D_n = \overline{\text{conv}} \bigcup_{k \geq n} C_k$ ($n \geq 1$). If D_1 is bounded and $x \in D_1$ is such that $\varphi(x) > \sup \varphi(C_k)$ ($k \in \mathbb{N}$), then $x \in \bigcap_{n \geq 1} D_n$.*

Proof. Let $n \geq 2$. We have $x \in D_1 = \overline{\text{conv}}(A \cup B)$ where

$$A = \text{conv} \left(\bigcup_{1 \leq k < n} C_k \right) \quad \text{and} \quad B = \text{conv} \left(\bigcup_{k \geq n} C_k \right).$$

Put $m = \max_{1 \leq k < n} \{\sup \varphi(C_k)\}$ and observe that $m < \varphi(x)$. Moreover, convexity of φ easily implies $\sup \varphi(A) = m$ and $\sup \varphi(B) \leq \varphi(x)$. By Lemma 1.4, $x \in \overline{B} = D_n$. ■

2. MAIN RESULTS

The proof of the following proposition is somewhat similar to the one in [4].

PROPOSITION 2.1. *Let X be an infinite dimensional normed linear space and $\varepsilon \in (0, 1)$. There exists a symmetric $C \in \mathcal{BCC}(X)$ without diametral pairs, such that $(1 - \varepsilon)B_X \subset C \subset B_X$.*

Proof. Since X is infinite dimensional, there exists a basic sequence $\{e_n\} \subset S_X$ (see, e.g., [2, Theorem 4.1.30 and Proposition 4.3.4] applied to the completion of X). Fix an arbitrary sequence $\{t_n\} \subset (0, 1)$ such that $t_n \rightarrow 1$. We claim that the set

$$C = \overline{\text{conv}}[(1 - \varepsilon)B_X \cup \{\pm t_n e_n\}_{n \geq 1}]$$

has the desired properties. Note that $\delta(C) = 2$ since $C \subset B_X$ and C contains the pairs $\pm t_n e_n$ ($n \in \mathbb{N}$). To show that C has no diametral pairs, it suffices to prove that $C \subset B_X^0$. If this is not the case, take $x \in C \cap S_X$ and $f \in S_{X^*}$ such that $f(x) = 1$. By Corollary 1.5, applied to the sets

$$(1 - \varepsilon)B_X, \{t_1 e_1\}, \{-t_1 e_1\}, \{t_2 e_2\}, \{-t_2 e_2\}, \dots,$$

we must have $x \in \bigcap_{n \geq 1} \overline{\text{conv}}\{\pm t_k e_k\}_{k \geq n}$. But the last set is contained in $\bigcap_{n \geq 1} \overline{\text{span}}\{e_k\}_{k \geq n} = \{0\}$, which is a contradiction since $\|x\| = 1$. ■

On the other hand, the following easy result holds true.

OBSERVATION 2.2. *For every normed linear space X , the elements of $\mathcal{BCC}(X)$ that do admit a diametral pair are dense in $\mathcal{BCC}(X)$.*

Proof. Let $C \in \mathcal{BCC}(X)$ have no diametral pairs. We may suppose that $\delta(C) = 1$. Fix an arbitrary $\varepsilon > 0$ and choose $\alpha > 0$ so small that $(1 + 2\varepsilon)(1 - \alpha) > 1 + \varepsilon$. Choose $u, v \in C$ so that $\|u - v\| > 1 - \alpha$. We may suppose that $u = 0$; hence $1 - \alpha < \|v\| < 1$. Denote

$$D_0 = C \cup \{-\varepsilon v\} \cup \{(1 + \varepsilon)v\}, \quad D = \text{conv } D_0.$$

Then $D \in \mathcal{BCC}(X)$ (Fact 1.1(b)) and $d_H(C, D) = d_H(C, D_0) \leq \varepsilon$. Since $\|(1 + \varepsilon)v + \varepsilon v\| > (1 + 2\varepsilon)(1 - \alpha)$ and, for each $x \in C$,

$$\begin{aligned} \|x + \varepsilon v\| &\leq \|x\| + \varepsilon < 1 + \varepsilon, \\ \|x - (1 + \varepsilon)v\| &\leq \|x - v\| + \varepsilon < 1 + \varepsilon, \end{aligned}$$

the points $-\varepsilon v$ and $(1 + \varepsilon)v$ form a diametral pair for D_0 , and hence also for D . ■

A natural question arises: In which normed spaces X the sets without diametral pairs are dense in $\mathcal{BCC}(X)$? The rest of the present paper is devoted to a complete answer to this question.

DEFINITION 2.3. For a sequence $\{x_n\} \subset X$, we denote

$$\Lambda(\{x_n\}) = \bigcap_{n \geq 1} \overline{\text{conv}}\{x_k\}_{k \geq n}.$$

LEMMA 2.4. *Let $\{x_n\}, \{y_n\}$ be sequences in X .*

- (a) $\Lambda(\{x_n\}) = \Lambda(\{y_n\})$ whenever $x_n - y_n \xrightarrow{\text{weak}} 0$.
- (b) $\Lambda(\{x_n\}) = \{x\}$ whenever $x_n \xrightarrow{\text{weak}} x$.
- (c) $\Lambda(\{x_{n_k}\}) \subset \Lambda(\{x_n\})$ whenever $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Proof. (a) If $x \notin \Lambda(\{x_n\})$, then there is an index n_0 such that $x \notin \overline{\text{conv}\{x_k\}_{k \geq n_0}}$. By the Hahn-Banach separation theorem, there exists $f \in X^* \setminus \{0\}$ such that $f(x) > \sup_{k \geq n_0} f(x_k) =: \sigma$. Fix an arbitrary $\alpha \in (\sigma, f(x))$. There exists an index $n_1 \geq n_0$ such that $|f(x_k - y_k)| < \alpha - \sigma$ whenever $k \geq n_1$. Then we have $f(y_k) = f(x_k) + f(y_k - x_k) < \sigma + (\alpha - \sigma) = \alpha$ whenever $k \geq n_1$. Consequently, $f(x) > \alpha \geq \sup f(\overline{\text{conv}\{y_k\}_{k \geq n_1}})$, which implies that $x \notin \Lambda(\{y_n\})$. We have proved that $\Lambda(\{y_n\}) \subset \Lambda(\{x_n\})$; the other inclusion follows by symmetry.

The proof of (b) follows from (a) with $y_n = x$ ($n \in \mathbb{N}$), and (c) is obvious. ■

LEMMA 2.5. *A normed linear space X is not a reflexive Banach space if and only if there exists a sequence $\{x_n\} \subset B_X^0$ with $\Lambda(\{x_n\}) = \emptyset$.*

Proof. If X is not a reflexive Banach space, there exists a decreasing sequence $\{C_n\} \subset \mathcal{BCC}(X)$ with empty intersection. (Indeed, this is a well known characterization of non-reflexivity for Banach spaces, and if X is incomplete, we can put $C_n = (\hat{y} + \frac{1}{n}B_{\hat{X}}) \cap X$ where \hat{X} is the completion of X and $\hat{y} \in \hat{X} \setminus X$.) We may assume that $C_1 \subset B_X^0$. Now, it suffices to choose arbitrarily $x_n \in C_n$ ($n \in \mathbb{N}$) to get the desired sequence. Conversely, if X is a reflexive Banach space, every sequence $\{x_n\} \subset B_X^0$ has a weakly convergent subsequence and hence $\Lambda(\{x_n\}) \neq \emptyset$ by Lemma 2.4 (c)-(b). ■

DEFINITION 2.6. We shall say that a normed space X has the *property (W)* if for each $v \in S_X$ and each $\varepsilon > 0$ there exists a sequence $\{v_n\} \subset S_X$ such that

$$\|v_n - v\| < \varepsilon \quad (n \in \mathbb{N}) \quad \text{and} \quad \Lambda(\{v_n\}) \subset B_X^0.$$

The following standard continuity properties are true in arbitrary normed linear spaces, except for the fact that, in general, the set $A - A$ need not be closed for $A \in \mathcal{BCC}(X)$ (this is the only reason for the reflexivity assumption).

FACT 2.7. *Let X be a reflexive Banach space. It is elementary to prove that:*

- the function $A \mapsto \varrho(A) := \sup_{x \in A} \|x\|$ is continuous on $\mathcal{BCC}(X)$;
- $A \mapsto A - A$ is a continuous mapping of $\mathcal{BCC}(X)$ into $\mathcal{BCC}(X)$;
- $(t, A) \mapsto tA$ is a continuous mapping of $\mathbb{R} \times \mathcal{BCC}(X)$ into $\mathcal{BCC}(X)$.

THEOREM 2.8. *For a normed linear space X , the following assertions are equivalent.*

- (i) *The elements of $\mathcal{BCC}(X)$ with no diametral pairs are dense in $\mathcal{BCC}(X)$.*
- (ii) *X has the property (W) (see Definition 2.6).*
- (iii) *Either X is not a reflexive Banach space, or X is a reflexive Banach space such that every weakly closed set contained in S_X has empty relative interior in S_X .*

Proof. (i) \Rightarrow (iii) Suppose (iii) is false, i.e., X is a reflexive Banach space and S_X contains a weakly compact set K with a nonempty relative interior in S_X . Let $v \in K$ and $r > 0$ be such that $B(v, r) \cap S_X \subset K$. Define $C = [-v, v]$ and fix an arbitrary $\varepsilon \in (0, \frac{r}{3})$. Since the mapping $\Phi(D) = \frac{D-D}{\varrho(D)}$ (where ϱ is as in Fact 2.7) is continuous on the set $\{D \in \mathcal{BCC}(X) : \varrho(D) > 0\}$, and $\Phi(C) = C$, there exists $\delta \in (0, 1)$ such that $d_H(\Phi(D), C) < \varepsilon$ whenever $d_H(D, C) < \delta$.

Let $D \in \mathcal{BCC}(X)$ satisfy $d_H(D, C) < \delta$. Denote $\hat{D} = \Phi(D)$. Observe that $\varrho(\hat{D}) = 1$ and the following equivalences hold: D has a diametral pair iff $D - D$ has a farthest point from 0 iff \hat{D} has a farthest point from 0 iff $\hat{D} \cap S_X \neq \emptyset$. Let us prove that \hat{D} intersects S_X .

Since $\varrho(\hat{D}) = 1$, there exists a sequence $\{x_n\} \subset \hat{D}$ such that $\|x_n\| \rightarrow 1$. Since $d_H(\hat{D}, C) < \varepsilon$, there exists a sequence $\{t_n\} \subset [-1, 1]$ such that $\|x_n - t_nv\| < \varepsilon$ ($n \in \mathbb{N}$). Since \hat{D} is symmetric, we can suppose that $t_n \in [0, 1]$ for each n . Since X is a reflexive Banach space, we can suppose (by passing to a subsequence) that $\{x_n\}$ weakly converges to some $x \in \hat{D}$. Denote $\hat{x}_n = \frac{x_n}{\|x_n\|}$ and observe that $\|x_n - \hat{x}_n\| \rightarrow 0$ and

$$1 - t_n = \|x_n\| - \|t_nv\| \leq \|x_n - t_nv\| < \varepsilon.$$

Consequently, $\{\hat{x}_n\}$ weakly converges to x and, for any sufficiently large n ,

$$\begin{aligned} \|\hat{x}_n - v\| &\leq \|\hat{x}_n - x_n\| + \|x_n - t_nv\| + \|t_nv - v\| \\ &< \varepsilon + \varepsilon + (1 - t_n) < 3\varepsilon < r. \end{aligned}$$

It follows that $\hat{x}_n \in K$ for each sufficiently large n , and hence $x \in K \subset S_X$. This proves that D has a diametral pair whenever $d_H(D, C) < \delta$; hence (i) is false.

(iii) \Rightarrow (ii) Let $v \in S_X$ and $0 < \varepsilon < 1$. First, suppose X is not a reflexive Banach space. By Lemma 2.5, there exists a sequence $\{x_n\} \subset \varepsilon B_X^0$ such that

$\Lambda(\{x_n\}) = \emptyset$. Define $y_n = v + x_n$ and $v_n = \frac{y_n}{\|y_n\|}$. Passing to a subsequence if necessary, we can suppose that $\|y_n\| \rightarrow \alpha \in \mathbb{R}$. Note that $1 - \varepsilon \leq \alpha \leq 1 + \varepsilon$ since $1 - \varepsilon < \|y_n\| < 1 + \varepsilon$ ($n \in \mathbb{N}$). Then we have

$$\|v_n - v\| \leq \|v_n - y_n\| + \|y_n - v\| \leq |1 - \|y_n\|| + \|x_n\| < 2\varepsilon$$

and, by Lemma 2.4,

$$\Lambda(\{v_n\}) = \Lambda\left(\left\{\frac{y_n}{\alpha}\right\}\right) = \frac{1}{\alpha}\Lambda(\{y_n\}) = \frac{1}{\alpha}(v + \Lambda(\{x_n\})) = \emptyset.$$

Now, let X be a reflexive Banach space such that S_X contains no weakly compact set with a nonempty interior (relative to S_X). The set $A = S_X \cap B(v, \varepsilon)$ is not weakly compact. Reflexivity of X and the Eberlein-Šmul'yan theorem imply that A contains a sequence $\{v_n\}$ that weakly converges to some $y \in X \setminus A$. Now, by Lemma 2.4,

$$\Lambda(\{v_n\}) = \{y\} \subset (B_X \cap B(v, \varepsilon)) \setminus A \subset B_X^0.$$

(ii) \Rightarrow (i) Assume (ii). Then an elementary compactness argument and Lemma 2.4(c) imply that X cannot be finite dimensional. We have to prove that arbitrarily near to any $C \in \mathcal{BCC}(X)$ there exists $D \in \mathcal{BCC}(X)$ with no diametral pairs. If C is a singleton, we can suppose that $C = \{0\}$ and obtain the assertion easily from Proposition 2.1.

Now, let C have a positive diameter. We may assume that $\delta(C) = 1$, and that the origin 0 and some point $v \in S_X$ form a diametral pair for C . Fix an arbitrary $\varepsilon \in (0, \frac{1}{4})$. By (ii), there exists a sequence $\{u_n\} \subset (1 + 2\varepsilon)S_X$ such that $\|u_n - (1 + 2\varepsilon)v\| < \frac{\varepsilon}{3}$ and $\Lambda(\{u_n\}) \subset (1 + 2\varepsilon)B_X^0$. Fix a sequence $\{\sigma_n\} \subset (1 - \frac{\varepsilon}{3}, 1)$ such that $\sigma_n \rightarrow 1$. Define

$$v_n = \sigma_n u_n - \varepsilon v, \quad D_0 = C \cup \{-\varepsilon v\} \cup \{v_n\}_{n \geq 1}, \quad D = \overline{\text{conv} D_0}.$$

Observe that $D \supset C$, $\text{dist}(-\varepsilon v, C) \leq \|-\varepsilon v - 0\| = \varepsilon$ and

$$\begin{aligned} \text{dist}(v_n, C) &\leq \|v_n - v\| = \|\sigma_n u_n - (1 + \varepsilon)v\| \\ &\leq \|\sigma_n u_n - u_n\| + \|u_n - (1 + 2\varepsilon)v\| + \|\varepsilon v\| \\ &< (1 - \sigma_n) + \frac{\varepsilon}{3} + \varepsilon < \frac{5}{3}\varepsilon. \end{aligned}$$

Consequently, $d_H(D, C) \leq \frac{5}{3}\varepsilon$. Let us compute $\delta(D)$. For $x \in C$, $n, k \in \mathbb{N}$, $n \neq k$, we have

$$\begin{aligned} \|x + \varepsilon v\| &\leq \|x\| + \|\varepsilon v\| \leq 1 + \varepsilon, \\ \|x - v_n\| &\leq \|x - v\| + \|v - v_n\| < 1 + \frac{5}{3}\varepsilon, \\ \|v_n - v_k\| &\leq \|v_n - v\| + \|v - v_k\| < \frac{5}{3}\varepsilon + \frac{5}{3}\varepsilon < 1, \\ \|v_n + \varepsilon v\| &= \sigma_n \|u_n\| = \sigma_n(1 + 2\varepsilon) \nearrow 1 + 2\varepsilon \quad (\text{as } n \rightarrow \infty). \end{aligned}$$

Thus $\delta(D) = \delta(D_0) = 1 + 2\varepsilon$. It remains to show that D has no diametral pairs.

Suppose that $x, y \in C$ are such that $\|x - y\| = \delta(C)$. By Lemma 1.2, applied to $z = -\varepsilon v$ and $A = C \cup \{v_n\}_{n \geq 1}$, one of the two points, say y , equals $-\varepsilon v$. By Lemma 1.3, applied to $A = \{v_n\}_{n \geq 0}$, $B = C$ and $z = -\varepsilon v$, we must have $x \in \overline{\text{conv}}\{v_n\}_{n \geq 1}$. Hence

$$x - y \in \overline{\text{conv}}\{v_n + \varepsilon v\}_{n \geq 1} = \overline{\text{conv}}\{\sigma_n u_n\}_{n \geq 1},$$

$\|x - y\| = 1 + 2\varepsilon$ and $\|\sigma_n u_n\| = \sigma_n(1 + 2\varepsilon) < 1 + 2\varepsilon$ ($n \in \mathbb{N}$). By Corollary 1.5 (with $\varphi = \|\cdot\|$ and C_n being the singleton $\{\sigma_n u_n\}$ for each n) and Lemma 2.4, we have

$$x - y \in \Lambda(\{\sigma_n u_n\}) = \Lambda(\{u_n\}) \subset (1 + 2\varepsilon)B_X^0$$

which is a contradiction that completes the proof. ■

Let us conclude with two simple remarks. Recall that X is said to have the *Kadets-Klee property* if the topological spaces (S_X, weak) and (S_X, norm) have the same convergent sequences.

- Remarks 2.9.* (a) Every finite dimensional normed linear space fails the property (W).
 (b) Every normed linear space that is not a reflexive Banach space has the property (W).
 (c) Every infinite dimensional reflexive Banach space with the Kadets-Klee property has the property (W).

(To see this, fix $v \in S_X$ and $\varepsilon > 0$. Since $S_X \cap B(v, \varepsilon)$ is not norm compact, it contains a sequence having no convergent subsequence. This sequence admits a subsequence $\{v_n\}$ that weakly converges to some $y \in B_X \cap B(v, \varepsilon)$. By the Kadets-Klee property, we must have $y \in B_X^0$. By Lemma 2.4 (b), $\Lambda(\{v_n\}) = \{y\} \subset B_X^0$.)

- (d) To get an example of an infinite dimensional reflexive Banach space without the property (W), start with an arbitrary infinite dimensional reflexive Banach space $(X, \|\cdot\|)$, fix an $f \in S_{X^*}$, and define an equivalent norm by $\|x\| = \max\{\|x\|, 2|f(x)|\}$. Then $(X, \|\cdot\|)$ fails (W) since the corresponding unit sphere contains the closed convex set $B_{(X, \|\cdot\|)} \cap f^{-1}(\frac{1}{2})$ with a nonempty relative interior.

Remark 2.10. Note that the symmetric set C constructed in Proposition 2.1 has no farthest points from the origin. It follows that a normed linear space X is finite dimensional iff every symmetric element of $\mathcal{BCC}(X)$ has a farthest point from the origin iff every element of $\mathcal{BCC}(X)$ is remotal. This answers a question from [3] (see Remark 2.8 therein) where a similar result was proved under the additional assumption that X is a reflexive Banach space.

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