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Some Invariant Subspaces for A-Contractions and Applications

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1. Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the Banach algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$ then T^* stands for the adjoint operator of T, while $\mathcal{R}(T)$ and $\mathcal{N}(T)$ denote the range and the null-space of T, respectively.

A *contraction* on $\mathcal{H}$ is an operator $T \in \mathcal{B}(\mathcal{H})$ satisfying $T^*T \leq I$, where $I = I_{\mathcal{H}}$ is the identity operator. If $T^*T < I$ then T is called a *proper contraction*. The class of contractions is one of the most studied and well-understood class of operators (see for instance [2], [3], [6], [11]) and the investigations concerning different other classes in $\mathcal{B}(\mathcal{H})$ have a starting point the theory of contractions. We refer below to a class of operators which generalize the contractions.

Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $A \neq 0$. An operator $T \in \mathcal{B}(\mathcal{H})$ satisfying the inequality

$$T^*AT \leq A \tag{1.1}$$

is called an *A-contraction* on $\mathcal{H}$. If the equality in (1.1) one occurs then T is called an *A-isometry* on $\mathcal{H}$. Such operators appear in different contexts in [1], [2], [3], [5], [7]–[10], [11], and other papers. By contrast to the class of contractions (that is, of I-contractions), the class of A-contractions is not

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invariant for the adjoint mapping $T \to T^*$ in $\mathcal{B}(\mathcal{H})$, in general (see Example 4.1 [8]).

It is clear from (1.1) that $\mathcal{N}(A)$ is an invariant subspace for T (and obviously, for A). So, if A is not injective then $\mathcal{N}(A)$ is a nontrivial invariant subspace for T. In general, for an A-contraction T it is possible to get other invariant subspaces for T which contain $\mathcal{N}(A)$. For instance, we proved in [8] that the subspace

$$\mathcal{N} := \mathcal{N}(A - AT) = \mathcal{N}(A^{1/2} - A^{1/2}T) = \mathcal{N}(A - T^*A) \tag{1.2}$$

is invariant for T, where $A^{1/2}$ is the square root of A. Clearly one has $\mathcal{N}(A) \subset \mathcal{N}$, hence $\mathcal{N} = \{0\}$ implies A injective. On the other hand, $\mathcal{N} = \mathcal{H}$ means $T^*A = A$, that is $T^*|_{\overline{\mathcal{R}(A)}} = I_{\overline{\mathcal{R}(A)}}$. Thus, if A is not injective and T^* is not the identity on $\overline{\mathcal{R}(A)}$, then $\mathcal{N}(A)$ and $\mathcal{N}$ are nontrivial invariant subspaces for T.

Now, we infer from (1.1) that there exists a unique contraction $\hat{T}$ on $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A^{1/2})}$, which satisfies $\hat{T}A^{1/2}h = A^{1/2}Th$ for any $h \in \mathcal{H}$. Then it follows immediately that $\overline{\mathcal{R}(I - \hat{T})} = \overline{\mathcal{R}(A^{1/2} - A^{1/2}T)}$, hence having in view the decomposition $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(I - \hat{T})} \oplus \mathcal{N}(I - \hat{T})$ we have

$$\mathcal{N}_* := \mathcal{N}(A^{1/2} - T^*A^{1/2}) = \mathcal{N}(I - \hat{T}) \oplus \mathcal{N}(A). \tag{1.3}$$

We know (from [9] and [10]) that $\mathcal{N} = \mathcal{N}_*$ if and only if $\mathcal{N}$ (equivalently, $\mathcal{N}_*$) reduces A, and this fact has a pure ergodic character (see Theorem 2.1 [9] and Theorem 2.4 [10]). According to [9] we say that an A-contraction T on $\mathcal{H}$ is *ergodic* if $\mathcal{N} = \mathcal{N}_*$. In this case, the subspace $\mathcal{N}$ is invariant for A and T but, by contrast with the case when $A = I$, $\mathcal{N}$ is not invariant for T^*, in general (see Example 2.8 [10]).

An A-contraction T on $\mathcal{H}$ is called *regular* if it satisfies the condition $AT = A^{1/2}TA^{1/2}$. Equivalently, this means that $A^{1/2}\hat{T}A^{1/2}h = \hat{T}Ah$ for $h \in \mathcal{H}$, which implies $A^{1/2}\hat{T}k = \hat{T}A^{1/2}k$ for $k \in \overline{\mathcal{R}(A)}$. So, an A-contraction T is regular if and only if $\hat{T}$ and $A|_{\overline{\mathcal{R}(A)}}$ commute. In this case, one has $(I - \hat{T})A^{1/2}k = A^{1/2}(I - \hat{T})k$ for $k \in \overline{\mathcal{R}(A)}$, which gives that $\mathcal{N}(I - \hat{T})$ is invariant for $A^{1/2}|_{\overline{\mathcal{R}(A)}}$ and from (1.3) one obtains that $\mathcal{N}_*$ is invariant for A. Hence, *any regular A-contraction is an ergodic A-contraction.*

It is clear that any contraction T on $\mathcal{H}$ is an ergodic T^*T-contraction (being also a $(T^*T)^{1/2}$-contraction). In addition, if T is *hyponormal* that is $TT^* \leq T^*T$, then T^* is an ergodic TT^*-contraction, and also an ergodic T^*T-contraction. This happens in particular when T is *quasinormal*, that is if T and T^*T commute (see [2], [3], [12]). But in this last case T and T^* are regular T^*T-contractions.

In general, an operator $T \in \mathcal{B}(\mathcal{H})$ which is a T^*T-contraction is called a *quasi-contraction*, and if T is a T^*T-isometry then T is called a *quasi-isometry*. In this last case, one has $||T|| \geq 1$, and it was proved in [5] that $||T|| = 1$ if and only if T is hyponormal.

In this paper we deal with some invariant subspaces in the context of A-contractions. So, in Section 2 we discuss the largest invariant subspace on which a given A-contraction actions as an A-isometry. Especially, the regular case is considered here. As applications, in Section 3 we analyze in detail the quasinormal contractions seen as quasi-contractions. We obtain the concrete forms for the unitary part and for the quasi-isometric part of a quasinormal contraction, and also some facts concerning such operators. In Section 4 we obtain an asymptotic form of the largest invariant A-isometric subspace from Section 2, using the operator limit of the sequence $\{T^{*n}AT^n;\ n \geq 1\}$. We study this subspace in connection to other subspaces which appear in the general context of A-contractions. But, more precisely results are derived in the case of a regular A-contraction, or when the range of A is closed. As applications in this section, we reobtain some facts concerning the asymptotic behaviour of a quasinormal operator (see [2], [3]), by direct investigations using the context of regular A-contractions.

2. The invariant A-isometric part

As we remarked in the previous section, the null-spaces $\mathcal{N}(A)$ and $\mathcal{N} = \mathcal{N}(A - AT)$ play an important role in the study of an A-contraction T on $\mathcal{H}$, by being invariants for T. Other remarkable subspaces associated to an A-contraction T are

$$\mathcal{N}_0 := \mathcal{N}(A - T^*AT), \quad \mathcal{N}_\infty := \bigcap_{n=1}^{\infty} \mathcal{N}(A - T^{*n}AT^n).$$

We have $\mathcal{N} \subset \mathcal{N}_\infty \subset \mathcal{N}_0$, and $\mathcal{N} = \mathcal{N}_0$ if and only if $A^{1/2}T\mathcal{N}_0 \subset \mathcal{N}_*$.

By contrast with $\mathcal{N}$, $\mathcal{N}_0$ is not invariant for T even in some ergodic cases. This fact easily follows from Example 4.3 [8], where one has $\{0\} \neq \mathcal{N}(A) = \mathcal{N} = \mathcal{N}_\infty \subsetneqq \mathcal{N}_0 \neq \mathcal{H}$. But, in general, the inclusion $\mathcal{N}(A) \subset \mathcal{N} \subset \mathcal{N}_0$ can be strict even if $\mathcal{N}_0$ is invariant for T. For instance, when T is a T^*T-isometry on $\mathcal{H}$ with $||T|| = 1$, we have from Remark 2.7 [10] that $\mathcal{N} = \mathcal{N}(T) \oplus \mathcal{N}(I - T)$. Thus, if T is not an orthogonal projection and T has non zero invariant vectors in $\mathcal{H}$, then $\{0\} \neq \mathcal{N}(T) \subsetneqq \mathcal{N} \subsetneqq \mathcal{N}_\infty = \mathcal{N}_0 = \mathcal{H}$.

Concerning the subspaces $\mathcal{N}_0$ and $\mathcal{N}_\infty$, we firstly have

PROPOSITION 2.1. *The following conditions are equivalent for an A-contraction T on $\mathcal{H}$:*

(i) $T\mathcal{N}_0 \subset \mathcal{N}_0$;

(ii) $\mathcal{N}_0 = \mathcal{N}(A - T^{*2}AT^2)$;

(iii) $\mathcal{N}_0 = \mathcal{N}_\infty$.

Furthermore $T\mathcal{N}_\infty \subset \mathcal{N}_\infty$, and if $A\mathcal{N}_\infty \subset \mathcal{N}_\infty$ then $\mathcal{N}_\infty$ is the largest invariant subspace (in $\mathcal{H}$) for A and T on which T is an A-isometry.

Proof. Let T be an A-contraction on $\mathcal{H}$. Then T^n is also an A-contraction for any integer $n \geq 2$, and since the sequence $\{T^{*n}AT^n\}$ is decreasing we have

$$\mathcal{N}(A - T^{*m}AT^m) \subset \mathcal{N}(A - T^{*n}AT^n) \subset \mathcal{N}_0 \quad (m, n \geq 2)$$

This shows that (iii) implies (ii), and the equivalence of (ii) with (i) is based on the following relation (for $n = 1$)

$$||(A - T^{*n}AT^n)^{1/2}Th||^2 = \langle T^*ATh, h\rangle - \langle T^{*(n+1)}AT^{n+1}h, h\rangle, \tag{2.1}$$

where $h \in \mathcal{H}$ and $n \geq 1$.

Next, if we assume the condition (i), then for $h \in \mathcal{N}_0$ and $n \geq 1$ we have $AT^nh = T^*AT^{n+1}h$, hence $T^{*n}AT^nh = T^{*(n+1)}AT^{n+1}h$. This leads to $\mathcal{N}_0 = \mathcal{N}_\infty$, that is the condition (iii).

Now we infer from (2.1) that

$$T\mathcal{N}_\infty \subset \mathcal{N}(A - T^{*n}AT^n) \quad (n \geq 1),$$

whence $T\mathcal{N}_\infty \subset \mathcal{N}_\infty$.

Suppose that $A\mathcal{N}_\infty \subset \mathcal{N}_\infty$. Therefore $\mathcal{N}_\infty$ is invariant for A and T, and $T|_{\mathcal{N}_\infty}$ is an $A|_{\mathcal{N}_\infty}$-isometry because $\mathcal{N}_\infty \subset \mathcal{N}_0$. Let $\mathcal{M} \subset \mathcal{H}$ be another invariant subspace for A and T such that $T|_\mathcal{M}$ is an $A|_\mathcal{M}$-isometry. Then $T^n|_\mathcal{M}$ is also an $A|_\mathcal{M}$-isometry, that is $(T^n|_\mathcal{M})^*AT^nh = Ah$ for $h \in \mathcal{M}$, $n \geq 1$. Equivalently, one has $||A^{1/2}T^nh|| = ||A^{1/2}h||$ which implies $\mathcal{M} \subset \mathcal{N}(A - T^{*n}AT^n)$ for any $n \geq 1$, and so $\mathcal{M} \subset \mathcal{N}_\infty$. ∎

Remark 2.2. If the A-contraction T is not an A-isometry on $\mathcal{H}$ and the operator A is not injective, then $\mathcal{N}(A)$, $\mathcal{N}$ and $\mathcal{N}_\infty$ are nontrivial invariant subspaces for T. Furthermore, if the A-contraction T is not ergodic then $\mathcal{N}(A) \neq \mathcal{N}$. In general $\mathcal{N} \neq \mathcal{N}_\infty$ because one has

$$\mathcal{N} \subset \mathcal{N}_\infty \cap \mathcal{N}(AT - T^*A) = \mathcal{N}_\infty \cap \mathcal{N}(A - AT^2) \subset \mathcal{N}_\infty, \tag{2.2}$$

where the subspace between $\mathcal{N}$ and $\mathcal{N}_\infty$ is also invariant for T.

In particular, if T is an idempotent, that is $T^2 = T$, then $AT = T^*A$ (see [8]) and so $T^*AT = AT$, whence we infer that $\mathcal{N} = \mathcal{N}_\infty = \mathcal{N}_0$. On the other hand, when T is 2-nilpotent, that is $T^2 = 0$, then immediately follows that $\mathcal{N}_\infty \subset \mathcal{N}(A)$, consequently $\mathcal{N}(A) = \mathcal{N} = \mathcal{N}_\infty$. Such a case appears in Example 4.3 [8] quoted above, where $\mathcal{N}_\infty \neq \mathcal{N}_0$; here $\mathcal{N}_0$ and $\mathcal{N}_\infty$ are invariant for A. In all these cases, T is an ergodic A-contraction.

In general, neither $\mathcal{N}_0$ nor $\mathcal{N}_\infty$ are invariant for A, as can be seen in Example 4.4 [8] where the A-contraction T is not ergodic and $\{0\} \neq \mathcal{N} = \mathcal{N}_\infty = \mathcal{N}_0 \neq \mathcal{H}$. Finally, the Example 2.8 [10] gives an ergodic A-contraction T for which $\{0\} \neq \mathcal{N}(A) = \mathcal{N} = \mathcal{N}_\infty \subsetneqq \mathcal{N}_0 \neq \mathcal{H}$, such that $\mathcal{N}_\infty$ is invariant for A, but $\mathcal{N}_0$ is not invariant for A or T.

The above remarks lead to conclusion that the properties of subspaces $\mathcal{N}_0$ and $\mathcal{N}_\infty$ depend not essentially of the ergodic character of the A-contractions. However, certain facts about $\mathcal{N}_0$ and $\mathcal{N}_\infty$ may be obtained when T is a regular A-contraction.

Proposition 2.3. *Let T be an A-contraction on $\mathcal{H}$ and $\mathcal{M} \subset \mathcal{N}_0$ be an invariant subspace for A and $A^{1/2}T$. Then $A^{1/2}T|_{\mathcal{M}}$ is a quasinormal operator in $\mathcal{B}(\mathcal{M})$ if and only if $ATh = A^{1/2}TA^{1/2}h$, $h \in \mathcal{M}$.*

Furthermore, if T is a regular A-contraction, then $\mathcal{N}_0$ and $\mathcal{N}_\infty$ are invariant for A, and $\mathcal{N}_\infty$ is the largest subspace into $\mathcal{N}_0$ which is invariant for A and $A^{1/2}T$.

Proof. Let $\mathcal{M} \subset \mathcal{N}_0$ be a closed subspace such that $A\mathcal{M} \subset \mathcal{M}$ and $A^{1/2}T\mathcal{M} \subset \mathcal{M}$. Then $(A^{1/2}T|_{\mathcal{M}})^* = P_{\mathcal{M}}(A^{1/2}T)^*|_{\mathcal{M}}$, $P_{\mathcal{M}}$ being the orthogonal projection onto $\mathcal{M}$, and for $h \in \mathcal{M}$ we obtain (because $h \in \mathcal{N}_0$)

$$Ah = T^*ATh = P_{\mathcal{M}}(A^{1/2}T)^*A^{1/2}Th = (A^{1/2}T|_{\mathcal{M}})^*A^{1/2}Th.$$

This firstly implies

$$A^{1/2}TAh = (A^{1/2}T|_{\mathcal{M}})(A^{1/2}T|_{\mathcal{M}})^*A^{1/2}Th$$

and later on (because $A^{1/2}Th \in \mathcal{M}$)

$$A^{3/2}Th = (A^{1/2}T|_{\mathcal{M}})^*(A^{1/2}T|_{\mathcal{M}})^2h.$$

Finally, the two relations show that the operator $A^{1/2}T|_{\mathcal{M}}$ is quasinormal in $\mathcal{B}(\mathcal{M})$ if and only if $A^{3/2}Th = A^{1/2}TAh$ for any $h \in \mathcal{M}$. Since this condition just means that the operators $A|_{\mathcal{M}}$ and $A^{1/2}T|_{\mathcal{M}}$ commute, it is equivalent to the fact that $A^{1/2}|_{\mathcal{M}}$ commutes with $A^{1/2}T|_{\mathcal{M}}$ ($\mathcal{M}$ being a reducing subspace for A), that is $ATh = A^{1/2}TA^{1/2}h$, for $h \in \mathcal{M}$.

Now we suppose that $AT = A^{1/2}TA^{1/2}$ on $\mathcal{H}$. For $n \geq 2$, T^n is also an A-contraction, while the condition $AT^n = A^{1/2}T^nA^{1/2}$ can be easily obtained by induction and using the fact that the operator $A^{1/2}$ is injective on his range. Thus, for $n \geq 1$ one obtains

$$(A - T^{*n}AT^n)A = A^2 - T^{*n}A^2T^n = A(A - T^{*n}AT^n),$$

which yields $A\mathcal{N}_0 \subset \mathcal{N}_0$ and $A\mathcal{N}_\infty \subset \mathcal{N}(A - T^{*n}AT^n)$ for $n \geq 1$, and later $A\mathcal{N}_\infty \subset \mathcal{N}_\infty$. So $\mathcal{N}_0$ and $\mathcal{N}_\infty$ are invariant subspaces for A, $\mathcal{N}_\infty$ being also invariant for T, consequently $\mathcal{N}_\infty$ is invariant for $A^{1/2}T$.

Next, let $\mathcal{M} \subset \mathcal{N}_0$ be as above. Using the condition from hypothesis, we get for $h \in \mathcal{M}$

$$AT^*A^{1/2}T^2h = T^*ATA^{1/2}Th = A^{3/2}Th,$$

whence $ATh = A^{1/2}T^*A^{1/2}T^2h = T^*AT^2h$. This gives $Ah = T^*ATh = T^{*2}AT^2h$, and repeating the same argument we will obtain by induction that $Ah = T^{*n}AT^nh$, for $h \in \mathcal{M}$ and $n \geq 2$. Thus we have $\mathcal{M} \subset \mathcal{N}(A - T^{*n}AT^n)$, for $n \geq 1$, and finally $\mathcal{M} \subset \mathcal{N}_\infty$. Hence $\mathcal{N}_\infty$ is the largest subspace into $\mathcal{N}_0$ which is invariant for A and $A^{1/2}T$. ∎

Corollary 2.4. *Let T be an A-contraction on $\mathcal{H}$ such that $\mathcal{N}_\infty$ is invariant for A. Then T is a regular A-isometry on $\mathcal{N}_\infty$ if and only if the operator $A^{1/2}T|_{\mathcal{N}_\infty}$ is quasinormal in $\mathcal{B}(\mathcal{N}_\infty)$.*

Proof. By hypothesis and Proposition 2.1 we have that $\mathcal{N}_\infty$ is invariant for A and T, and $T|_{\mathcal{N}_\infty}$ is an $A|_{\mathcal{N}_\infty}$-isometry. The conclusion follows from Proposition 2.3. ∎

Corollary 2.5. *An A-isometry T on $\mathcal{H}$ is regular if and only if the operator $A^{1/2}T$ is quasinormal in $\mathcal{H}$.*

Concerning the subspace $\mathcal{N}_0$ we have now

THEOREM 2.6. *Let T be a regular A-contraction on $\mathcal{H}$. One has:*
(i) *T is a regular A^n-contraction and a regular $A^{1/2^n}$-contraction, and furthermore we have*

$$\mathcal{N}_0 = \mathcal{N}(A^n - T^* A^n T) = \mathcal{N}(A^{1/2^n} - T^* A^{1/2^n} T) \quad (n \geq 1). \tag{2.3}$$

(ii) *The subspace $\mathcal{N}_0$ is invariant for T if and only if $\mathcal{N}_0$ is invariant for $A^{n/2}T$, for some (equivalently, all) integers $n \geq 1$.*

Proof. (i) The fact that T is a regular A^n-contraction can be proved by induction. For the first equality in (2.3) we use the identity

$$T^* A^n T = (A^{1/2})^{2(n-1)} A^{1/2} T^* A^{1/2} T \quad (n \geq 2),$$

which clearly follows from the condition $AT = A^{1/2}TA^{1/2}$. Thus, for $h \in \mathcal{H}$ we have $T^* A^n Th = A^n h$ if and only if

$$(A^{1/2})^{2(n-1)} Ah = (A^{1/2})^{2(n-1)} A^{1/2} T^* A^{1/2} Th,$$

or equivalently (since $A^{1/2}$ is injective on $A^{1/2}\mathcal{H}$), $Ah = A^{1/2}T^*A^{1/2}Th$ $= T^*ATh$. This gives the first equality in (2.3).

Now, we show that T is an $A^{1/2}$-contraction on $\mathcal{H}$. Recall that the operator $A^{1/2}$ can be obtained as the strong limit of a sequence $\{p_n(A)\}_{n \geq 1}$ of polynomials in A with positive coefficients and $p_n(0) = 0$ (see [6], pg. 261). As T is an A^j-contraction for $j \geq 1$, we obtain

$$\langle T^* p_n(A) Th, h \rangle \leq \langle p_n(A) h, h \rangle,$$

for any $h \in \mathcal{H}$ and $n \geq 1$. So, by passing to limit when $n \to \infty$, we get $T^* A^{1/2} T \leq A^{1/2}$. Hence T is an $A^{1/2}$-contraction on $\mathcal{H}$.

Next, we prove that the $A^{1/2}$-contraction T is regular, too. We remark that the inequality $T^*AT \leq A$ implies that there is an operator $C \in \mathcal{B}(\mathcal{H})$ such that $A^{1/2}T = CA^{1/2}$. Since T is a regular A-contraction, we get

$$(A^{1/2})^2 T = AT = A^{1/2} T A^{1/2} = C(A^{1/2})^2$$

and by induction we obtain $(A^{1/2})^n T = C(A^{1/2})^n$ for any $n \geq 1$. This leads to $p(A^{1/2})T = Cp(A^{1/2})$ for any polynomial p with scalar coefficients. Then considering a sequence of approximation polynomials (as above) for the square root $A^{1/4}$ of $A^{1/2}$, we deduce that $A^{1/4}T = CA^{1/4}$. This implies $A^{1/4}TA^{1/4} = CA^{1/2} = A^{1/2}T$, which just means that T is a regular $A^{1/2}$-contraction. Also, it follows by induction on $n \geq 1$ that T is a regular $A^{1/2^n}$-contraction, and clearly, the first equality in (2.2) gives $\mathcal{N}_0 = \mathcal{N}(A^{1/2^n} - T^* A^{1/2^n} T)$, for any $n \geq 1$.

(ii) If $\mathcal{N}_0$ is invariant for T then, being also invariant for $A^{1/2}$, $\mathcal{N}_0$ will be invariant for $A^{n/2}T$, for any $n \geq 1$. Conversely, we suppose that $\mathcal{N}_0$ is invariant for $A^{n/2}T$, for some $n \geq 1$. We have for $h \in \mathcal{N}_0$

$$T^* A^n T^2 h = T^* A^{n/2} T A^{n/2} T h = A^{n/2} A^{n/2} T h = A^n T h,$$

where we used from assertion (i) the fact that T is a regular A^n-contraction and a regular $A^{1/2}$-contraction, while in the second equality one has in view that $A^{n/2}Th \in \mathcal{N}_0$. Finally, from (2.3) we obtain that $T\mathcal{N}_0 \subset \mathcal{N}_0$. ∎

Corollary 2.7. *If T is a regular A-contraction on $\mathcal{H}$ such that $A^{1/2}T$ is a quasinormal operator on $\mathcal{H}$, then $\mathcal{N}_0 = \mathcal{N}_\infty$ and this subspace reduces $A^{1/2}T$.*

Proof. From hypothesis we infer for $h \in \mathcal{N}_0$,

$$T^* A T A^{1/2} T h = A^{1/2} T T^* A T h = A^{1/2} T A h = A A^{1/2} T h,$$

and respectively,

$$T^* A T T^* A^{1/2} h = T^* A^{1/2} T^* A^{1/2} A^{1/2} T h = T^* A^{3/2} h = A T^* A^{1/2} h.$$

This means that $\mathcal{N}_0$ is a reducing subspace for the operator $A^{1/2}T$, and both Theorem 2.6 (ii) and Proposition 2.1 imply finally $\mathcal{N}_0 = \mathcal{N}_\infty$. ∎

Remark 2.8. When T is a regular A-contraction and $\mathcal{N}_0 = \mathcal{N}_\infty$, then we also have

$$\mathcal{N}_0 = \mathcal{N}(A - T^{*n} A T^n) \quad (n \geq 2), \tag{2.4}$$

which completes the relations (2.3). On the other hand, let us remark that the condition $AT = A^{1/2} T A^{1/2}$ not assures that $A^{1/2}T$ is quasinormal, in general. For instance, when T is a non-unitary coisometry and $A = I$ on $\mathcal{H}$.

3. Applications to quasinormal contractions

The above results can be applied to obtain some facts on the quasinormal contractions, as those concerning their unitary, isometric and quasi-isometric parts.

Theorem 3.1. *Let T be a quasinormal contraction on $\mathcal{H}$. One has:*
(i) *$\mathcal{N}(I - T^*T)$ is the largest subspace which reduces T to an isometry.*
(ii) *$\mathcal{N}(I - TT^*)$ is an invariant subspace for T, and T is an isometry on this subspace.*
(iii) *$\mathcal{M} := \bigcap_{n=1}^{\infty} \mathcal{N}(I - T^n T^{*n})$ is the largest subspace which reduces T to a unitary operator, and we have*

$$\mathcal{M} = \bigcap_{n=0}^{\infty} T^n \mathcal{N}(I - TT^*). \tag{3.1}$$

Proof. The assertion (i) follows immediately from Proposition 2.1 and Corollary 2.7 when $A = I$.

(ii) It is easy to see (T being quasinormal) that $TT^* \le T^*T$ and

$$\mathcal{N}(I - TT^*) \subset \mathcal{N}(I - T^*T). \tag{3.2}$$

Thus, for $h \in \mathcal{N}(I - TT^*)$ we have $h = T^*Th$ and $Th = TT^*Th$, hence $Th \in \mathcal{N}(I - TT^*)$. Therefore $\mathcal{N}(I - TT^*)$ is invariant for T.

(iii) By Proposition 2.1 the subspace $\mathcal{M}$ from (iii) is the largest invariant subspace for T^* on which T^* is an isometry. Let us prove that $T\mathcal{M} \subset \mathcal{M}$. Let $h \in \mathcal{M}$, hence $h = T^j T^{*j} h$ for $j \ge 1$. Using the fact that $\mathcal{M} \subset \mathcal{N}(I - TT^*)$ and (3.2) we have $T^*Th = h$, and we obtain for $n \ge 1$,

$$T^n T^{*n} Th = T^n T^{*(n-1)} h = T(T^{n-1} T^{*(n-1)} h) = Th.$$

Thus $T\mathcal{M} \subset \mathcal{N}(I - T^n T^{*n})$ for $n \ge 1$, whence it follows that $T\mathcal{M} \subset \mathcal{M}$. Consequently, $\mathcal{M}$ reduces T to a unitary operator (by (3.2)), being even the largest subspace with this property, because $\mathcal{M}$ is the largest invariant subspace for T^* on which T^* is an isometry. The subspace $\mathcal{M}$ can be also expressed as in (3.1) by Theorem 2.4 [7]. ∎

Recall that W. Mlak proved in [4], using the unitary dilation, that for any hyponormal contraction T the largest subspace which reduces T to a unitary operator has the form (3.1). But in [7], this fact was shown for the quasinormal contractions without the use of unitary dilation.

Corollary 3.2. *Let T be a quasinormal contraction on $\mathcal{H}$. Then the subspace $\mathcal{N}(I-TT^*)$ is reducing for T if and only if T is a unitary operator on $\mathcal{N}(I-TT^*)$. In this case, $\mathcal{N}(I-TT^*)$ is the largest subspace which reduces T to a unitary operator.*

Proof. Suppose that T is unitary on $\mathcal{N}(I-TT^*)$. Then we have $T^*\mathcal{N}(I-TT^*) = T^*T\mathcal{N}(I-TT^*) = \mathcal{N}(I-TT^*)$, because T is an isometry on $\mathcal{N}(I-TT^*)$. So $\mathcal{N}(I-TT^*)$ reduces T to a unitary operator, being the largest subspace with this property, by Theorem 3.1 (*iii*). The converse part of the corollary is immediate. ∎

Remark 3.3. Since any contraction T on $\mathcal{H}$ is also a quasi-contraction, one has

$$\bigcap_{n=1}^{\infty} \mathcal{N}(I - T^{*n}T^n) \subset \bigcap_{n=2}^{\infty} \mathcal{N}(T^*T - T^{*n}T^n), \tag{3.3}$$

where in the left side and the right side we have the largest invariant subspace for T on which T is an isometry, and respectively, a quasi-isometry. When T is quasinormal we can obtain more complete facts in the following

Theorem 3.4. *Let T be a quasinormal contraction on $\mathcal{H}$. One has:*
(i) *$\mathcal{N}(T^*T - T^{*2}T^2)$ is the largest subspace which reduces T to a quasi-isometry.*
(ii) *$\mathcal{N}(T^*T - TT^*TT^*)$ is an invariant subspace for T and T^*T, and T is a quasi-isometry on this subspace. Furthermore we have*

$$\begin{aligned} \mathcal{N}(T^*T - TT^*TT^*) &= \mathcal{N}(I - TT^*) \oplus \mathcal{N}(T) \\ &\subset \mathcal{N}(T^*T - TT^*) \cap \mathcal{N}(T^*T - T^{*2}T^2). \end{aligned} \tag{3.4}$$

(iiii) *$\tilde{\mathcal{M}} := \bigcap_{n=1}^{\infty} \mathcal{N}(T^*T - T^nT^*TT^{*n})$ is the largest subspace which reduces T, on which T and T^* are T^*T-isometries. Moreover, we have*

$$\tilde{\mathcal{M}} = \bigcap_{n=0}^{\infty} T^n\mathcal{N}(I - T^nT^{*n}) \oplus \mathcal{N}(T). \tag{3.5}$$

Proof. The assumption that T is a quasinormal contraction assures that T is also a regular T^*T-contraction. Then both Proposition 2.1 and Corollary 2.7 imply the assertion (i).

Also we remark (T being a quasinormal contraction) that

$$TT^*TT^* \leq TT^* \leq T^*T,$$

which shows on one hand that T^* is a T^*T-contraction on $\mathcal{H}$, and on the other hand we infer the inclusion

$$\mathcal{N}(T^*T - TT^*TT^*) \subset \mathcal{N}(T^*T - TT^*). \tag{3.6}$$

Next, if $h \in \mathcal{N}(T^*T - TT^*TT^*)$, then using (3.6) and the fact that T is quasinormal we obtain

$$T^*Th = TT^*TT^*h = T^*T^2T^*h = T^*TT^*Th = T^{*2}T^2h.$$

This leads to the inclusion

$$\mathcal{N}(T^*T - TT^*TT^*) \subset \mathcal{N}(T^*T - T^{*2}T^2), \tag{3.7}$$

and both (3.6) and (3.7) give the inclusion from (3.4).

Now denote $\mathcal{N}_n := \mathcal{N}(T^*T - T^nT^*TT^{*n})$ for $n \geq 1$. Clearly, $\mathcal{N}_n$ is the corresponding subspace $\mathcal{N}_0$ for the regular T^*T-contraction T^{*n}, therefore by Proposition 2.3, $\mathcal{N}_n$ is invariant for T^*T. Also, since T is quasinormal we have $T^*TT^nT^{*n} = T^nT^{*n}T^*T$, whence

$$\mathcal{N}_n = \mathcal{N}[T^*T(I - T^nT^{*n})] = \mathcal{N}[(I - T^nT^{*n})T^*T].$$

Thus we infer that $\mathcal{N}(T) = \mathcal{N}(T^*T) \subset \mathcal{N}_n$ and $\mathcal{N}(I - T^nT^{*n}) \subset \mathcal{N}_n$, and furthermore, $T^*T\mathcal{N}_n \subset \mathcal{N}(I - T^nT^{*n})$. As $\mathcal{N}_n$ reduces T^*T, we can define the operator $P_n := T^*T|_{\mathcal{N}_n}$ in $\mathcal{B}(\mathcal{N}_n)$. Then using (3.7) and the fact that $\mathcal{N}_n \subset \mathcal{N}_1$ for $n \geq 2$, we obtain that $P_n^2 = P_n$, and since $P_n \geq 0$, P_n will be an orthogonal projection in $\mathcal{B}(\mathcal{N}_n)$. But we have

$$\mathcal{N}(P_n) = \mathcal{N}_n \cap \mathcal{N}(T^*T) = \mathcal{N}(T),$$

and on the other hand,

$$\begin{aligned}\mathcal{R}(P_n) &= \{h \in \mathcal{N}_n : h = T^*Th\} = \{h \in \mathcal{H} : T^*Th = T^nT^{*n}T^*Th\} \\ &= \{h \in \mathcal{H} : h = T^*Th = T^nT^{*n}h\} \\ &= \mathcal{N}(I - T^*T) \cap \mathcal{N}(I - T^nT^{*n}) = \mathcal{N}(I - T^nT^{*n}),\end{aligned}$$

because T being a quasinormal contraction, one has for $n \geq 2$,

$$\mathcal{N}(I - T^nT^{*n}) \subset \mathcal{N}(I - TT^*) \subset \mathcal{N}(I - T^*T).$$

Thus, it follows that the Hilbert space $\mathcal{N}_n$ admits the orthogonal decomposition

$$\mathcal{N}_n = \mathcal{N}(I - T^nT^{*n}) \oplus \mathcal{N}(T) \quad (n \geq 1). \tag{3.8}$$

In particular, for $n = 1$ this just gives the decomposition from (3.4) of the subspace $\mathcal{N}_1$, whence we also infer that $\mathcal{N}_1$ is an invariant subspace for T, because $\mathcal{N}(I - TT^*)$ and $\mathcal{N}(T)$ are such subspaces. Furthermore, from (3.7) we have that T is a quasi-isometry, or equivalently a T^*T-isometry, on $\mathcal{N}_1$. All assertions from (ii) are proved.

Next, if $\tilde{\mathcal{M}} := \bigcap_{n=1}^{\infty} \mathcal{N}_n$, then from (3.8) we obtain for $n \geq 1$ that

$$\tilde{\mathcal{M}} \ominus \mathcal{N}(T) \subset \mathcal{N}(I - T^nT^{*n}),$$

therefore if $h \in \tilde{\mathcal{M}} \ominus \mathcal{N}(T)$ then $h = T^nT^{*n}h$. But $\tilde{\mathcal{M}}$ is just the corresponding subspace $\mathcal{N}_\infty$ for the T^*T-contraction T^*, hence $\tilde{\mathcal{M}}$ is the largest invariant subspace for T^* on which T^* is a T^*T-isometry. So, for h as above one has $T^{*n}h \in \tilde{\mathcal{M}} \subset \mathcal{N}_1$, hence $h \in T^n\mathcal{N}_1 \subset T^n\mathcal{N}(I - TT^*)$ by (3.8), for $n \geq 1$. Thus we obtain

$$\tilde{\mathcal{M}} \ominus \mathcal{N}(T) \subset \bigcap_{n=1}^{\infty} T^n\mathcal{N}(I - TT^*) = \bigcap_{n=0}^{\infty} T^n\mathcal{N}(I - TT^*) := \mathcal{M}$$

which yields the inclusion $\tilde{\mathcal{M}} \subset \mathcal{M} \oplus \mathcal{N}(T)$. But by Theorem 3.1, the subspace $\mathcal{M}$ defined above reduces the operator T to a unitary operator and particularly, T^* is a T^*T-isometry on $\mathcal{M}$. Having in view the maximality property quoted above for the subspace $\tilde{\mathcal{M}}$, we infer that $\mathcal{M} \subset \tilde{\mathcal{M}}$, and also $\mathcal{M} \oplus \mathcal{N}(T) \subset \tilde{\mathcal{M}}$. Consequently, $\tilde{\mathcal{M}} = \mathcal{M} \oplus \mathcal{N}(T)$ which means the equality (3.5). But $\mathcal{N}(T)$ is a reducing subspace for T, because T is quasinormal. It follows that $\tilde{\mathcal{M}}$ is also invariant for T, and as $\tilde{\mathcal{M}} \subset \mathcal{N}_1$, by (3.7) we have that T is a quasi-isometry, or equivalently, a T^*T-isometry, on $\tilde{\mathcal{M}}$. In fact, $T|_{\tilde{\mathcal{M}}}$ is the orthogonal sum between a unitary operator and zero, relative to the decomposition $\tilde{\mathcal{M}} = \mathcal{M} \oplus \mathcal{N}(T)$. Therefore, $\tilde{\mathcal{M}}$ has the required properties in the statement (iii), and the proof is finished. ∎

COROLLARY 3.5. *Let T be a quasinormal contraction on $\mathcal{H}$. Then $\tilde{\mathcal{M}}$ is the largest subspace which reduces T to a normal quasi-isometry. Moreover, T is injective if and only if T is a unitary operator on $\tilde{\mathcal{M}}$, or equivalently,*

$$\tilde{\mathcal{M}} = \bigcap_{n=0}^{\infty} T^n\mathcal{N}(I - TT^*). \tag{3.9}$$

Proof. As we quoted in the previous proof, by the decomposition (3.5) we have $T|_{\tilde{\mathcal{M}}} = U \oplus 0$, U being a unitary operator on the subspace $\mathcal{M}$ given by the right side in (3.9). This shows that T is a normal quasi-isometry on $\tilde{\mathcal{M}}$ and furthermore, $T|_{\tilde{\mathcal{M}}}$ one reduces to a unitary operator if and only if $\mathcal{N}(T) = \{0\}$, or equivalently $\tilde{\mathcal{M}} = \mathcal{M}$. Now, if $\mathcal{L}$ is another subspace which reduces T to a normal quasi-isometry, then easily follows that T^* is also a T^*T-isometry on $\mathcal{L}$, so $\mathcal{L} \subset \tilde{\mathcal{M}}$ having in view the property of $\tilde{\mathcal{M}}$ in Theorem 3.4 (iii). Hence $\tilde{\mathcal{M}}$ has the required property in Corollary 3.5. □

Now, preserving the above notations, we immediately obtain the following corollary which completes Corollary 3.2.

COROLLARY 3.6. *For a quasinormal contraction T on $\mathcal{H}$, the following are equivalent:*

(i) $\tilde{\mathcal{M}} = \mathcal{N}(T^*T - TT^*TT^*)$;
(ii) $\mathcal{N}(I - TT^*) = \bigcap_{n=0}^{\infty} T^n \mathcal{N}(I - TT^*)$;
(iii) $\mathcal{N}(I - TT^*) = T\mathcal{N}(T^*T - TT^*TT^*)$;
(iv) $\mathcal{N}(I - TT^*)$ *is an invariant subspace for* T^*;
(v) $\mathcal{N}(T^*T - TT^*TT^*)$ *is an invariant subspace for* T^*.

Proof. Clearly, (i) implies (ii) by the relations (3.5) and (3.8). Assuming (ii) we get (by (3.8) for $n = 1$), $\mathcal{N}(I - TT^*) = T\mathcal{N}(I - TT^*) = T\mathcal{N}_1$, so (ii) implies (iii). Now, the equality from (iii) means that for any $h \in \mathcal{N}(I - TT^*)$ there exists $h_1 \in \mathcal{N}_1$ such that $Th_1 = h$. Then we have by (3.8),

$$T^*h = T^*Th_1 \in T^*T\mathcal{N}(I - TT^*) = \mathcal{N}(I - TT^*),$$

because $\mathcal{N}(I - TT^*) \subset \mathcal{N}(I - T^*T)$. This shows that $\mathcal{N}(I - TT^*)$ is an invariant subspace for T^*, and so (iii) implies (iv). Next, the assertions (iv) and (v) are even equivalent, by the relation (3.8) for $n = 1$, because $\mathcal{N}(T)$ reduces T. Finally, (v) implies (i) by Theorem 3.4 (the assertions (ii) and (iii)). ∎

Remark 3.7. Corollary 3.6 shows that, for a quasinormal contraction T, the subspace $\mathcal{N}_0$ corresponding to the regular I-contraction T^*, and the one for the regular T^*T-contraction T^*, respectively $\mathcal{N}_0 = \mathcal{N}(I - TT^*)$ and $\mathcal{N}_0 = \mathcal{N}(T^*T - TT^*TT^*)$, are not invariant for T^*, in general. But they are always invariant for T and T^*T.

Now, from Theorem 3.1 and Theorem 3.4 we infer the following

Corollary 3.8. *Let T be a quasinormal contraction on $\mathcal{H}$. Then $\mathcal{H}$ admits the orthogonal decomposition*

$$\mathcal{H} = \overline{\mathcal{R}(T^*T - T^{*2}T^2)} \oplus \mathcal{N}(I - T^*T) \oplus \mathcal{N}(T), \tag{3.10}$$

where $\mathcal{N}(I - T^*T)$ is the largest subspace which reduces T to an isometry, and $\overline{\mathcal{R}(T^*T - T^{*2}T^2)}$ is the largest subspace which reduces T to an injective proper quasinormal contraction.

Proof. Since T is quasinormal one has $T^{*2}T^2 = (T^*T)^2$. Then by Proposition 3.3 [3] we have

$$\mathcal{N}(T^*T - T^{*2}T^2) = \mathcal{N}(I - T^*T) \oplus \mathcal{N}(T), \tag{3.11}$$

this being the largest subspace which reduces T to a quasi-isometry (by Theorem 3.4(i)). But by Theorem 3.1(i), $\mathcal{N}(I - T^*T)$ is the largest subspace which reduces T to an isometry. Thus, we conclude that the range subspace from (3.10) reduces T to an injective and completely non isometric contraction, being the largest subspace with this property. Clearly, if $0 \neq h \in \overline{\mathcal{R}(T^*T - T^{*2}T^2)}$ then one has $h \notin \mathcal{N}(I - T^*T)$ that is $||Th|| < ||h||$, hence $\overline{\mathcal{R}(T^*T - T^{*2}T^2)}$ reduces T to a proper contraction. Conversely, if $\mathcal{M} \subset \mathcal{H}$ is a subspace which reduces T to an injective proper contraction, then T is also a non isometric contraction on $\mathcal{M}$, hence $\mathcal{M} \subset \overline{\mathcal{R}(T^*T - T^{*2}T^2)}$, thus this range has the required property. ∎

Having in view (3.11) we also have the following fact which was obtained in [5] in a different way.

Corollary 3.9. *A quasi-isometry T on $\mathcal{H}$ with $||T|| = 1$ is quasinormal if and only if T is a partial isometry.*

Finally, we infer from Corollary 3.8 the following

Corollary 3.10. *An injective quasinormal contraction is completely non isometric if and only if it is a proper contraction.*

4. Asymptotic form of the invariant A-isometric part

Let T be an A-contraction on $\mathcal{H}$. Since $\{T^{*n}AT^n;\ n \geq 1\}$ is a bounded decreasing sequence of positive operators it converges strongly to an operator $A_T \in \mathcal{B}(\mathcal{H})$. If T is a contraction (i.e., $A = I$) we will denote by S_T the strong limit of $\{T^{*n}T^n;\ n \geq 1\}$. So, if $\hat{T}$ is the contraction on $\overline{\mathcal{R}(A)}$ associated to the A-contraction T as in Section 1, then $S_{\hat{T}}$ will be the strong limit in $\mathcal{B}(\overline{\mathcal{R}(A)})$ of the sequence $\{\hat{T}^{*n}\hat{T}^n;\ n \geq 1\}$. Since $A^{1/2}T^n h = \hat{T}^n A^{1/2} h$ and $T^{*n}A^{1/2}k = A^{1/2}\hat{T}^{*n}k$ for $h \in \mathcal{H}$, $k \in \overline{\mathcal{R}(A)}$ and $n \geq 1$, one has

$$A_T h = A^{1/2} S_{\hat{T}} A^{1/2} h \quad (h \in \mathcal{H}). \tag{4.1}$$

This gives $A_T \leq A$ because $S_{\hat{T}} \leq I$. We also have $S_{\hat{T}} = \hat{T}^* S_{\hat{T}} \hat{T}$ and $A_T = T^* A_T T$. We can use the operators A_T and $S_{\hat{T}}$ in order to obtain more informations on the subspace $\mathcal{N}_\infty$ defined in Section 2.

THEOREM 4.1. *Let T be an A-contraction on $\mathcal{H}$. Then we have*

$$\mathcal{N}_\infty = \mathcal{N}(A - A_T) = (A^{1/2})^{-1}\mathcal{N}(I - S_{\hat{T}}). \tag{4.2}$$

Furthermore, if $||A|| \leq 1$ then

$$\mathcal{N}_\infty \cap \mathcal{N}(A - A^2) = \mathcal{N}(A) \oplus \mathcal{N}(I - A_T) = \mathcal{N}_\infty \cap \mathcal{N}(A_T - A_T^2) \tag{4.3}$$

and

$$\mathcal{N}(I - A_T) = \mathcal{N}(I - A) \cap \mathcal{N}(I - S_{\hat{T}}) = \mathcal{N}(I - A) \cap \mathcal{N}_\infty. \tag{4.4}$$

Proof. If $h \in \mathcal{N}_\infty$ then $Ah = T^{*n}AT^n h$ for any $n \geq 1$ and taking $n \to \infty$ one obtains $Ah = A_T h$, that is $h \in \mathcal{N}(A - A_T)$. So $\mathcal{N}_\infty \subset \mathcal{N}(A - A_T)$. Next, if $h \in \mathcal{N}(A - A_T)$ then using (4.1) and the fact that $A^{1/2}$ is injective on $\overline{\mathcal{R}(A)}$ we obtain $(I - S_{\hat{T}})A^{1/2}h = 0$, which yields $h \in (A^{1/2})^{-1}\mathcal{N}(I - S_{\hat{T}})$. Thus we have $\mathcal{N}(A - A_T) \subset (A^{1/2})^{-1}\mathcal{N}(I - S_{\hat{T}})$. Finally, if $h \in (A^{1/2})^{-1}\mathcal{N}(I - S_{\hat{T}})$, or equivalently $A^{1/2}h \in \mathcal{N}(I - S_{\hat{T}})$, then since $\hat{T}$ and $S_{\hat{T}}$ are contraction and $\hat{T}$ is also a $S_{\hat{T}}$-isometry on $\overline{\mathcal{R}(A)}$ it follows (see Proposition 3.1 (j) from [2]) that

$$||A^{1/2}T^n h|| = ||\hat{T}^n A^{1/2} h|| = ||A^{1/2} h|| \quad (n \geq 1).$$

This gives $(A - T^{*n}AT^n)h = 0$, that is $h \in \mathcal{N}(A - T^{*n}AT^n)$, for $n \geq 1$, therefore $h \in \mathcal{N}_\infty$. Thus, $(A^{1/2})^{-1}\mathcal{N}(I - S_{\hat{T}}) \subset \mathcal{N}_\infty$, and consequently the two equalities in (4.2) hold.

Now we suppose that $||A|| \leq 1$ that is $A \leq I$ (since $A \geq 0$). As $A_T \leq A$ implies $0 \leq I - A \leq I - A_T$, one obtains that

$$\mathcal{N}(I - A_T) = \mathcal{N}(I - A) \cap \mathcal{N}(A - A_T) = \mathcal{N}(I - A) \cap \mathcal{N}_\infty.$$

This gives one relation in (4.4) and also (by Proposition 3.3 [2])

$$\mathcal{N}_\infty \cap \mathcal{N}(A - A^2) = \mathcal{N}_\infty \cap (\mathcal{N}(A) \oplus \mathcal{N}(I - A)) = \mathcal{N}(A) \oplus \mathcal{N}(I - A_T),$$

that is the first relation in (4.3). Next if $h \in \mathcal{N}_\infty \cap \mathcal{N}(A_T - A_T^2)$, then $Ah = A_T h$ and $h = h_0 + h_1$ with $h_0 \in \mathcal{N}(A_T)$ and $h_1 \in \mathcal{N}(I - A_T)$. Hence $Ah = A_T h_1 = h_1$ and also $Ah = Ah_0 + Ah_1$, whence we get $Ah_0 = (I - A)h_1 = 0$ because $\mathcal{N}(I - A_T) \subset \mathcal{N}(I - A)$ by the previous remark. Thus $h_0 \in \mathcal{N}(A)$ and then $h \in \mathcal{N}(A) \oplus \mathcal{N}(I - A_T)$. Consequently

$$\mathcal{N}_\infty \cap \mathcal{N}(A_T - A_T^2) \subset \mathcal{N}(A) \oplus \mathcal{N}(I - A_T)$$

and as the converse inclusion is obvious, we obtain the second relation in (4.3).

For the first equality in (4.4) we remarked above that $\mathcal{N}(I - A_T) \subset \mathcal{N}(I - A)$. So, if $h \in \mathcal{N}(I - A_T)$ we have $h = A_T h = Ah = A^{1/2}h$, and also by the second equality in (4.2) we obtain $h = A^{1/2}h \in \mathcal{N}(I - S_{\hat{T}})$. Hence

$$\mathcal{N}(I - A_T) \subset \mathcal{N}(I - A) \cap \mathcal{N}(I - S_{\hat{T}}).$$

Conversely, if $h \in \mathcal{N}(I - A) \cap \mathcal{N}(I - S_{\hat{T}})$ we have $h = A^{1/2}h \in \mathcal{N}(I - S_{\hat{T}})$ which means (by (4.2)) $h \in \mathcal{N}(A - A_T)$. Thus $h = Ah = A_T h$, hence $h \in \mathcal{N}(I - A_T)$ and we obtained the inclusion

$$\mathcal{N}(I - A) \cap \mathcal{N}(I - S_{\hat{T}}) \subset \mathcal{N}(I - A_T).$$

We conclude that the former equality (4.4) holds and the proof is finished. ∎

COROLLARY 4.2. *If T is an A-contraction on $\mathcal{H}$ such that $||A|| \leq 1$ then*

$$\mathcal{N}(A) = \mathcal{N}_\infty \cap \mathcal{N}(A_T). \tag{4.5}$$

Furthermore, the following assertions are equivalent:

(i) $\mathcal{N}_\infty = \mathcal{N}(A)$;
(ii) $\mathcal{R}(A^{1/2}) \cap \mathcal{N}(I - S_{\hat{T}}) = \{0\}$;
(iii) $||S_{\hat{T}}^{1/2}k|| < ||k||$ *for every* $k \in \mathcal{R}(A^{1/2})$, $k \neq 0$;
(iv) $||A_T^{1/2}h|| < ||A^{1/2}h||$ *for every* $h \notin \mathcal{N}(A)$.

Proof. If $h \in \mathcal{N}_\infty \cap \mathcal{N}(A_T)$ then by (4.3) one has $h = h_0 + h_1$ with $h_0 \in \mathcal{N}(A)$ and $h_1 \in \mathcal{N}(I - A_T)$, hence $h - h_0 = h_1 = 0$ because $h - h_0 \in \mathcal{N}(A_T)$. So, $h = h_0 \in \mathcal{N}(A)$ and we have the inclusion $\mathcal{N}_\infty \cap \mathcal{N}(A_T) \subset \mathcal{N}(A)$, the converse being trivial.

Now we suppose that $\mathcal{N}_\infty = \mathcal{N}(A)$ and let $k = A^{1/2}h \in \mathcal{R}(A^{1/2}) \cap \mathcal{N}(I - S_{\hat{T}})$. Then $A^{1/2}h = S_{\hat{T}}A^{1/2}h$, whence by (4.1) one has $Ah = A_T h$. Hence by (4.2) we have $h \in \mathcal{N}_\infty = \mathcal{N}(A)$ which gives $k = A^{1/2}h = 0$. This means that $\mathcal{R}(A^{1/2}) \cap \mathcal{N}(I - S_{\hat{T}}) = \{0\}$ and so we obtained the implication (i) $\Rightarrow$ (ii). Next, the assumption (ii) ensures that for $0 \neq k \in \mathcal{R}(A^{1/2})$ one has $(I - S_{\hat{T}})k \neq 0$, or equivalently

$$||k||^2 - ||S_{\hat{T}}^{1/2}k||^2 = ||(I - S_{\hat{T}})^{1/2}k||^2 > 0,$$

which provides the implication (ii) $\Rightarrow$ (iii). Similarly, we infer from (iii) that $(I - S_{\hat{T}})A^{1/2}h \neq 0$ for $h \notin \mathcal{N}(A)$, which also gives $A^{1/2}(I - S_{\hat{T}})A^{1/2}h \neq 0$ because $(I - S_{\hat{T}})A^{1/2}h \in \overline{\mathcal{R}(A)}$. Hence $(A - A_T)h \neq 0$, or equivalently $\langle (A - A_T)h, h\rangle > 0$, that is the inequality from (iv). Finally, the implication (iv) $\Rightarrow$ (i) is trivial, having in view the first relation in (4.2). ∎

We can also describe $\mathcal{N}(A_T)$ as follows

COROLLARY 4.3. *If T is an A-contraction on $\mathcal{H}$ and $A_0 = A|_{\overline{\mathcal{R}(A)}}$ then*

$$\mathcal{N}(A_T) = (A^{1/2})^{-1}\mathcal{N}(S_{\hat{T}}) = \mathcal{N}(A) \oplus \mathcal{N}(S_{\hat{T}}A_0^{1/2}). \tag{4.6}$$

Furthermore, the following assertions are equivalent:

(i) $\mathcal{N}(A_T) = \mathcal{N}(A)$;
(ii) $\mathcal{R}(A^{1/2}) \cap \mathcal{N}(S_{\hat{T}}) = \{0\}$;
(iii) $A^{1/2}T^n h \not\to 0$ *for every* $h \notin \mathcal{N}(A)$, $h \neq 0$.

Proof. From (4.1) one infers that $h \in \mathcal{N}(A_T)$ if and only if $A^{1/2}h \in \mathcal{N}(S_{\hat{T}})$, or equivalently $h \in (A^{1/2})^{-1}\mathcal{N}(S_{\hat{T}})$, what gives the first equality in (4.6). On the other hand, since $0 \leq A_T \leq A$ it follows that $\mathcal{N}(A) \subset \mathcal{N}(A_T)$, hence

$$\mathcal{N}(A_T) = \mathcal{N}(A) \oplus (\overline{\mathcal{R}(A)} \cap \mathcal{N}(A_T)).$$

But $k \in \overline{\mathcal{R}(A)} \cap \mathcal{N}(A_T)$ if and only if $A_0^{1/2}k \in \mathcal{N}(S_{\hat{T}})$, or equivalently $k \in \mathcal{N}(S_{\hat{T}}A_0^{1/2})$. Thus one obtains the other equality in (4.6).

Clearly, $\mathcal{N}(A_T) = \mathcal{N}(A)$ if and only if $\mathcal{N}(S_{\hat{T}} A_0^{1/2}) = \{0\}$. Now we suppose that $\mathcal{N}(A_T) = \mathcal{N}(A)$. Then for $A^{1/2}h \in \mathcal{N}(S_{\hat{T}})$ we have

$$S_{\hat{T}} A^{1/2} h = 0 = S_{\hat{T}} A^{1/2} h_1$$

where $h = h_1 + h_0$ with $h_1 \in \overline{\mathcal{R}(A)}$ and $h_0 \in \mathcal{N}(A)$. So $h_1 \in \mathcal{N}(S_{\hat{T}} A_0^{1/2})$ which means $h_1 = 0$ by our assumption, hence $h = h_0$ and $A^{1/2}h = 0$. Thus we have the implication (i) $\Rightarrow$ (ii). Next, using (ii), we obtain for $0 \neq h \notin \mathcal{N}(A)$ that $A^{1/2}h \notin \mathcal{N}(S_{\hat{T}})$ that is $\hat{T}^n A^{1/2} h \not\to 0$, or equivalently $A^{1/2}T^n \not\to 0$. Therefore (ii) implies (iii), and obviously (iii) ensures (i). ∎

Remark 4.4. The second relation in (4.2) shows that $A^{1/2}\mathcal{N}_\infty$ is contained in $\mathcal{N}(I - S_{\hat{T}})$, but $\mathcal{N}(I - S_{\hat{T}})$ is not contained in $\mathcal{N}_\infty$, and also $\mathcal{N}_\infty$ and $\mathcal{N}(I - S_{\hat{T}})$ are not invariant for A, in general. However, if $||A|| \leq 1$ then $\mathcal{N}(I - A_T)$ and hence $\mathcal{N}_\infty \cap \mathcal{N}(A - A^2)$ are invariant (in fact, reducing) subspaces for A. Now we can describe the case when $\mathcal{N}_\infty$ is invariant for A (completing Proposition 2.1).

PROPOSITION 4.5. *The following are equivalent for an A-contraction T on $\mathcal{H}$:*

(i) $\mathcal{N}_\infty$ *is invariant for* A;
(ii) $\mathcal{N}_\infty$ *is invariant for* A_T;
(iii) $\mathcal{N}_\infty \subset \mathcal{N}(AA_T - A_T A)$;
(iv) $\mathcal{N}_\infty \subset \mathcal{N}(A^2 - A_T^2)$.

Furthermore, in this case we have

$$\mathcal{N}_\infty = (A + A_T)^{-1}\mathcal{N}_\infty = \mathcal{N}(AA_T - A_T A) \cap \mathcal{N}(A^2 - A_T^2). \tag{4.7}$$

Proof. The statements (i) and (ii) are obviously equivalent, having in view the first relation in (4.2). Now the assumption (i) ensures for $h \in \mathcal{N}_\infty$ that $AA_T h = A^2 h = A_T A h$, that is $h \in \mathcal{N}(AA_T - A_T A)$. Hence (i) implies (iii). Since for $h \in \mathcal{N}(AA_T - A_T A)$ one has

$$(A^2 - A_T^2)h = (A + A_T)(A - A_T)h,$$

the implication (iii) $\Rightarrow$ (iv) is immediate. Finally, supposing (iv), we have for $h \in \mathcal{N}_\infty$ that $Ah = A_T h$ and so

$$(A - A_T)Ah = (A^2 - A_T^2)h = 0,$$

that is $Ah \in \mathcal{N}_\infty$. Hence (iv) implies (i).

Now let $h \in \mathcal{N}(AA_T - A_TA) \cap \mathcal{N}(A^2 - A_T^2)$ so that $AA_Th = A_TAh$ and $A^2h = A_T^2h$. Then one obtains $A(A+A_T)h = A_T(A+A_T)h$, or $(A-A_T)(A+A_T)h = 0$. This gives that $(A+A_T)h \in \mathcal{N}_\infty$, therefore $h \in (A+A_T)^{-1}\mathcal{N}_\infty$. Conversely, if $k \in (A+A_T)^{-1}\mathcal{N}_\infty$ which means that $(A+A_T)k \in \mathcal{N}_\infty$, then $A(A+A_T)k = A_T(A+A_T)k$, or equivalently, $(A+A_T)(A-A_T)k = 0$. This shows that $(A-A_T)k \in \mathcal{N}(A+A_T)$, and as $0 \le A \le A+A_T$ one has $\mathcal{N}(A+A_T) \subset \mathcal{N}(A)$, hence $A(A-A_T)k = 0$. Since $\mathcal{R}(A_T) \subset \mathcal{R}(A^{1/2})$ (by (4.1)) it follows that $(A-A_T)k \in \mathcal{R}(A^{1/2})$, and by previous remark $(A-A_T)k \in \mathcal{N}(A)$, therefore $(A-A_T)k = 0$, that is $k \in \mathcal{N}_\infty$. Thus we proved the inclusions

$$\mathcal{N}(AA_T - A_TA) \cap \mathcal{N}(A^2 - A_T^2) \subset (A+A_T)^{-1}\mathcal{N}_\infty \subset \mathcal{N}_\infty. \tag{4.8}$$

In the case when $\mathcal{N}_\infty$ is invariant for A, these inclusions become the equalities (4.7), having in view the conditions (iii) and (iv) of above. This ends the proof. ∎

Now we present two cases in which Proposition 4.5 can be applied, where the subpace $\mathcal{N}_\infty$ has a special form.

THEOREM 4.6. *Let T be an A-contraction on $\mathcal{H}$ such that either the range $\mathcal{R}(A)$ is closed, or the A-contraction T is regular. Then one has*

$$\mathcal{N}_\infty = \mathcal{N}(A) \oplus \mathcal{N}(I - S_{\hat{T}}), \tag{4.9}$$

while $\mathcal{N}_\infty$ and $\mathcal{N}(I - S_{\hat{T}})$ are invariant subspaces for A.

Moreover, in the regular case we have $AS_{\hat{T}}k = S_{\hat{T}}Ak$ for $k \in \overline{\mathcal{R}(A)}$, $A_Th = S_{\hat{T}}Ah$ for $h \in \mathcal{H}$ and

$$\mathcal{N}(A_T) = \mathcal{N}(A) \oplus \mathcal{N}(S_{\hat{T}}), \quad \overline{\mathcal{R}(A_T)} = \overline{\mathcal{R}(S_{\hat{T}})}. \tag{4.10}$$

Proof. Firstly we suppose that the range $\mathcal{R}(A)$ is closed. Then $\mathcal{R}(A) = \mathcal{R}(A^{1/2})$ and having in mind the definition of $S_{\hat{T}}$ we have $0 \le S_{\hat{T}} \le \hat{T}^{*n}\hat{T}^n \le I$ for any $n \ge 1$, hence

$$\begin{aligned} \mathcal{N}(I - S_{\hat{T}}) &= \{A^{1/2}h \in \mathcal{R}(A) : \|\hat{T}^n A^{1/2}h\| = \|A^{1/2}h\|,\ n \ge 1\} \\ &\subset \{k \in \mathcal{H} : \|A^{1/2}T^n k\| = \|A^{1/2}k\|,\ n \ge 1\} \\ &= \{k \in \mathcal{H} : T^{*n}AT^n k = Ak,\ n \ge 1\} = \mathcal{N}_\infty. \end{aligned}$$

From (4.2) we infer that $A^{1/2}\mathcal{N}_\infty \subset \mathcal{N}(I - S_{\hat{T}}) \subset \mathcal{N}_\infty$, and we also have $A^{1/2}\mathcal{N}(I-S_{\hat{T}}) \subset A^{1/2}\mathcal{N}_\infty \subset \mathcal{N}(I-S_{\hat{T}})$, which means that $\mathcal{N}_\infty$ and $\mathcal{N}(I-S_{\hat{T}})$ are invariant subspaces for A.

Now it is clear that $\mathcal{N}(A) \oplus \mathcal{N}(I - S_{\hat{T}}) \subset \mathcal{N}_\infty$, and to prove the equality let $h \in \mathcal{N}_\infty$ such that h is orthogonal to $\mathcal{N}(A) \oplus \mathcal{N}(I - S_{\hat{T}})$. Then $h \in \mathcal{R}(A)$ and h is orthogonal to $\mathcal{N}(I - S_{\hat{T}})$ which implies that h is also orthogonal to $A\mathcal{N}_\infty$ as a subspace of $\mathcal{N}(I-S_{\hat{T}})$. In particular $\langle h, Ah\rangle = 0$ that is $A^{1/2}h = 0$, and since $h \in \mathcal{R}(A)$ we conclude that $h = 0$. Thus the equality (4.9) holds if $\mathcal{R}(A)$ is closed.

Next we suppose that T is a regular A-contraction, that is one has $AT = A^{1/2}TA^{1/2}$. Then $A^{1/2}\hat{T}A^{1/2}h = \hat{T}Ah$, for $h \in \mathcal{H}$ which means that $A^{1/2}\hat{T} = \hat{T}A^{1/2}$ on $\overline{\mathcal{R}(A)}$. Using this relation one obtains immediately that $S_{\hat{T}}Ah = A^{1/2}S_{\hat{T}}A^{1/2}h = A_T h$ for every $h \in \mathcal{H}$, and also $S_{\hat{T}}A^{1/2} = A^{1/2}S_{\hat{T}}$, or equivalently $S_{\hat{T}}A = AS_{\hat{T}}$, on $\overline{\mathcal{R}(A)}$. This relation later on implies that $\mathcal{N}(S_{\hat{T}})$ and $\mathcal{N}(I - S_{\hat{T}})$ are invariant subspaces for A. But using (4.1) one infers that $\mathcal{N}(I - S_{\hat{T}}) \subset \mathcal{N}_\infty$ and also $A^{1/2}\mathcal{N}_\infty \subset \mathcal{N}(I - S_{\hat{T}}) \subset \mathcal{N}_\infty$ so that $\mathcal{N}_\infty$ is invariant for A, too. Clearly, we have $\mathcal{N}(A) \oplus \mathcal{N}(I - S_{\hat{T}}) \subset \mathcal{N}_\infty$. To prove here the equality, let $h \in \mathcal{N}_\infty$ such that h is orthogonal to $\mathcal{N}(A) \oplus \mathcal{N}(I - S_{\hat{T}})$. Since $Ah \in \mathcal{N}(I - S_{\hat{T}})$ we have $\langle h, Ah\rangle = 0$ so that $A^{1/2}h = 0$, and as $h \in \overline{\mathcal{R}(A)}$ one has $h = 0$. Hence the equality (4.9) holds if the A-contraction T is regular.

Finally, since $A_T\mathcal{H} = S_{\hat{T}}A\mathcal{H}$ it follows that $\overline{\mathcal{R}(A_T)} = \overline{\mathcal{R}(S_{\hat{T}})}$ that is the second relation in (4.10), and which also implies the former relation in (4.10). ∎

Corollary 4.7. *Let T be a regular A-contraction on $\mathcal{H}$. Then $\mathcal{N}(A_T)$, $\mathcal{N}(S_{\hat{T}})$ and $\mathcal{N}(S_{\hat{T}} - S_{\hat{T}}^2)$ are invariant subspaces for A, and one has*

$$\mathcal{N}(S_{\hat{T}}) = \mathcal{N}(S_{\hat{T}}A_0^{1/2}). \tag{4.11}$$

Moreover, if $||A|| \leq 1$ then $\mathcal{N}(A_T - A_T^2)$ is an invariant subspace for A, and if $A = A^2$ then we have

$$\begin{aligned} \mathcal{N}(A_T - A_T^2) &= (A)^{-1}\mathcal{N}(S_{\hat{T}} - S_{\hat{T}}^2) \\ &= \mathcal{N}(A) \oplus \mathcal{N}(S_{\hat{T}} - S_{\hat{T}}^2) = \mathcal{N}(S_{\hat{T}}) \oplus \mathcal{N}_\infty. \end{aligned} \tag{4.12}$$

In the last case, one has $A_T = A_T^2$ if and only if $S_{\hat{T}} = S_{\hat{T}^2}$.

Proof. It was seen in the previous proof that $\mathcal{N}(S_{\hat{T}})$ is an invariant subspace for A and the first relation in (4.10) gives that $\mathcal{N}(A_T)$ is also invariant for A. In addition, both this relation from (4.10) and the second relation from (4.6) lead to (4.11).

Since $\mathcal{N}(S_{\hat{T}} - S_{\hat{T}}^2) = \mathcal{N}(S_{\hat{T}}) \oplus \mathcal{N}(I - S_{\hat{T}})$, this subspace will be invariant for A, such as the two contained subspaces.

In the case when $||A|| \leq 1$ one has $\mathcal{N}(A_T - A_T^2) = \mathcal{N}(A_T) \oplus \mathcal{N}(I - A_T)$ and it follows that $\mathcal{N}(A_T - A_T^2)$ is invariant for A (by the above remark and Remark 4.4).

Now we assume that $A = A^2$. Then for $h \in \mathcal{H}$ we have

$$(A_T - A_T^2)h = S_{\hat{T}}Ah - S_{\hat{T}}^2 A^2 h = (S_{\hat{T}} - S_{\hat{T}}^2)Ah,$$

hence $h \in \mathcal{N}(A_T - A_T^2)$ if and only if $Ah \in \mathcal{N}(S_{\hat{T}} - S_{\hat{T}}^2)$, or equivalently $h \in (A)^{-1}\mathcal{N}(S_{\hat{T}} - S_{\hat{T}}^2)$. This gives the first relation in (4.12). On the other hand, since $A = A^2$ one has $\mathcal{R}(A) = \mathcal{N}(I - A)$, and from (4.4) one obtains $\mathcal{N}(I - A_T) = \mathcal{N}(I - S_{\hat{T}})$. Thus we have

$$\mathcal{N}(A_T - A_T^2) = \mathcal{N}(A) \oplus \mathcal{N}(S_{\hat{T}}) \oplus \mathcal{N}(I - S_{\hat{T}}) = \mathcal{N}(A) \oplus \mathcal{N}(S_{\hat{T}} - S_{\hat{T}}^2),$$

that is the second relation in (4.12). Clearly, from this relation it follows that $A_T = A_T^2$ if and only if $S_{\hat{T}} = S_{\hat{T}}^2$. Also, we infer from the previous relation and (4.9) that

$$\mathcal{N}(A_T - A_T^2) = \mathcal{N}(S_{\hat{T}}) \oplus \mathcal{N}_\infty$$

which is the last relation in (4.12). The proof is finished. ∎

COROLLARY 4.8. *Let T be a regular A-contraction such that $||A|| \leq 1$ and $A_T = A_T^2$. Then $S_{\hat{T}} = S_{\hat{T}}^2$ and furthermore, if $\mathcal{N}(A) = \mathcal{N}(A_T)$ one has $A = A_T$.*

Proof. From the relation (4.4) and (4.10) we have

$$\mathcal{H} = \mathcal{N}(A_T - A_T^2) = \mathcal{N}(A) \oplus \mathcal{N}(S_{\hat{T}}) \oplus \mathcal{N}(I - A) \cap \mathcal{N}(I - S_{\hat{T}}),$$

whence it follows

$$\overline{\mathcal{R}(A)} = \mathcal{N}(S_{\hat{T}}) \oplus \mathcal{N}(I - A) \cap \mathcal{N}(I - S_{\hat{T}}) = \mathcal{N}(S_{\hat{T}} - S_{\hat{T}}^2),$$

that is $S_{\hat{T}} = S_{\hat{T}}^2$. Now if $\mathcal{N}(A) = \mathcal{N}(A_T)$, or equivalently $\mathcal{N}(S_{\hat{T}}) = \{0\}$, then we have $\mathcal{N}(I - S_{\hat{T}}) = \overline{\mathcal{R}(A)}$ that is $S_{\hat{T}} = I$. Hence $A_T = S_{\hat{T}}A = A$. ∎

Remark 4.9. Since T is an A_T-isometry on $\mathcal{H}$ there exists a (unique) isometry V on $\overline{\mathcal{R}(A_T)}$ such that $VA_T^{1/2}h = A_T^{1/2}Th$, $h \in \mathcal{H}$. On the other hand, because $\hat{T}$ is a $S_{\hat{T}}$-isometry on $\overline{\mathcal{R}(A)}$ there exists a (unique) isometry $\hat{V}$ on $\overline{\mathcal{R}(S_{\hat{T}})}$ satisfying $\hat{V}S_{\hat{T}}^{1/2}k = S_{\hat{T}}^{1/2}\hat{T}k$, $k \in \overline{\mathcal{R}(A)}$. But in the regular case one has $V = \hat{V}$, since $\overline{\mathcal{R}(A_T)} = \overline{\mathcal{R}(S_{\hat{T}})}$ (by (4.10)) and

$$VA_T^{1/2}h = A_T^{1/2}Th = S_{\hat{T}}^{1/2}A^{1/2}Th,$$
$$S_{\hat{T}}^{1/2}\hat{T}A^{1/2}h = \hat{V}S_{\hat{T}}^{1/2}A^{1/2}h = \hat{V}A_T^{1/2}h$$

for $h \in \mathcal{H}$. Here we used the fact that $A_T^{1/2} = S_{\hat{T}}^{1/2}A^{1/2}$ which follows from Theorem 4.6. In this case, $\mathcal{N}(I - S_{\hat{T}})$ is the largest invariant subspace for $\hat{T}$ on which $\hat{T}$ is an isometry and we even have

$$\hat{T}|_{\mathcal{N}(I-S_{\hat{T}})} = V|_{\mathcal{N}(I-S_{\hat{T}})}$$

because $\mathcal{N}(I - S_{\hat{T}})$ is also invariant for V and for $h \in \mathcal{N}(I - S_{\hat{T}})$ one has

$$\hat{T}h = S_{\hat{T}}\hat{T}h = S_{\hat{T}}^{1/2}\hat{T}h = VS_{\hat{T}}^{1/2}h = Vh.$$

In addition, if $S_{\hat{T}}$ is a projection then $\mathcal{N}(I - S_{\hat{T}}) = \mathcal{R}(S_{\hat{T}})$ is the largest subspace which reduces $\hat{T}$ to an isometry, so that V is the isometric part of $\hat{T}$.

As an application to quasinormal contractions, we can obtain the following facts, partially known from [2], [3], which complete ones from Section 3.

Proposition 4.10. *For a quasinormal contraction T on $\mathcal{H}$ we have:*
(i) $S_T = S_T^2$ *and the largest subspace which reduces T to an isometry is*

$$\mathcal{N}(I - S_T) = \mathcal{N}(I - T^*T) = \mathcal{N}(I - S_{\hat{T}}) \tag{4.13}$$

where $\hat{T} = T|_{\overline{\mathcal{R}(T^*)}}$.
(ii) *The largest subspace which reduces T to a quasi-isometry, or equivalently to a partial isometry, is*

$$\mathcal{N}(T^*T - S_T) = \mathcal{N}(T) \oplus \mathcal{N}(I - S_T) = \mathcal{H} \ominus \mathcal{N}(S_{\hat{T}}). \tag{4.14}$$

(iii) *The largest subspace which reduces T to a strongly stable contraction, or equivalently to a proper contraction, is*

$$\mathcal{N}(S_T) = \mathcal{N}(T) \oplus \mathcal{N}(S_{\hat{T}}). \tag{4.15}$$

Furthermore, $\mathcal{N}(S_{\hat{T}})$ reduces T and there is no nonzero subspace of $\mathcal{N}(S_{\hat{T}})$ which reduces T to a quasi-isometry.

Proof. One considers T a quasinormal contraction. Since T and T^*T commute it follows that $S_T = T^*TS_T$, S_T being the strong limit of the sequence $\{T^{*n}T^n;\ n \geq 1\}$. Thus we have $(I - T^*T)S_T = 0$, whence $\overline{\mathcal{R}(S_T)} \subset \mathcal{N}(I - T^*T)$. Let now $h \in \mathcal{N}(I - T^*T) \cap \mathcal{N}(S_T)$. Then $S_T h = 0$ which means that $T^n h \to 0$ $(n \to \infty)$. Since $\mathcal{N}(I - T^*T)$ reduces T to an isometry, one has $||T^n h|| = ||h||$ for $n \geq 1$, hence $h = 0$. So, one has $\mathcal{N}(I - T^*T) = \overline{\mathcal{R}(S_T)}$, therefore $\mathcal{N}(I - T^*T)$ and $\mathcal{N}(S_T)$ are orthogonal subspaces. Next, if $h \in \mathcal{H}$ is orthogonal to $\mathcal{N}(S_T) \oplus \mathcal{N}(I - T^*T)$ then $h \in \overline{\mathcal{R}(S_T)} \cap \mathcal{N}(S_T)$ by the previous remark, and so $h = 0$. Hence we have

$$\mathcal{H} = \mathcal{N}(S_T) \oplus \mathcal{N}(I - T^*T).$$

But it is clear that $\mathcal{N}(I - T^*T) \subset \mathcal{N}(I - S_T)$ and so we obtain

$$\mathcal{H} = \mathcal{N}(S_T) \oplus \mathcal{N}(I - S_T) = \mathcal{N}(S_T - S_T^2),$$

and consequently $S_T = S_T^2$. Also one has $\mathcal{N}(I - T^*T) = \mathcal{N}(I - S_T)$, this being the largest subspace which reduces T to an isometry. In addition, since $S_{\hat{T}} = S_T|_{\overline{\mathcal{R}(T^*)}}$ and as $\mathcal{N}(I - S_T) \subset \overline{\mathcal{R}(T^*)}$, it follows that $\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{\hat{T}})$. Thus the assertion (i) is proved.

To show (ii), we firstly remark that T is a regular $A = T^*T$-contraction on $\mathcal{H}$ and that $\hat{T} = T|_{\overline{\mathcal{R}(T^*)}}$ is the corresponding contraction on $\overline{\mathcal{R}(T^*)}$ which satisfies $\hat{T}|T|h = |T|Th$, $h \in \overline{\mathcal{R}(T^*)}$. In this case (i.e., $A = T^*T$) we have $A_T = S_T$ and the corresponding subspace $\mathcal{N}_\infty$ given by the relation (4.9) is

$$\mathcal{N}(T^*T - S_T) = \mathcal{N}(T) \oplus \mathcal{N}(I - S_T).$$

Since $\mathcal{N}(T)$ and $\mathcal{N}(I - S_T)$ reduce T, $\mathcal{N}(T^*T - S_T)$ is just the largest subspace which reduces T to a T^*T-isometry, that is to a quasi-isometry, or equivalently (by Corollary 3.9) to a partial isometry. This gives the assertion (ii).

For the same meaning of T, we infer from (4.10) and from the above decomposition of $\mathcal{H}$ that

$$\mathcal{N}(S_T) = \mathcal{N}(T) \oplus \mathcal{N}(S_{\hat{T}}) = \mathcal{H} \ominus \mathcal{N}(I - S_T),$$

this being the largest subspace which reduces T to a completely non isometric contraction, or equivalently to a strongly stable contraction, having in view the definition of S_T. But by Corollary 3.8, $\mathcal{N}(S_T)$ is the largest subspace which reduces T to a proper contraction. Finally, we remark that

$$\mathcal{H} = \mathcal{N}(S_{\hat{T}}) \oplus \mathcal{N}(T^*T - S_T),$$

hence the subspace $\mathcal{N}(S_{\hat{T}})$ has the required property. The assertion (iii) is proved and the proof is finished. ∎

The dual version of the preceding proposition can be also given.

Proposition 4.11. *For a quasinormal contraction T on $\mathcal{H}$ we have:*
(i) $S_{T^*} = S_{T^*}^2$ *and the largest subspace which reduces T to a unitary operator, or equivalently, on which T^* is an isometry, is*

$$\mathcal{N}(I - S_{T^*}) = \mathcal{N}(I - S_{\hat{T}^*}), \tag{4.16}$$

where $\hat{T} = T|_{\overline{\mathcal{R}(T^*)}}$.
(ii) *The largest subspace which reduces T^* to a T^*T-isometry, or equivalently, on which T is a normal partial isometry, is*

$$\mathcal{N}(T^*T - S_{T^*}) = \mathcal{N}(T) \oplus \mathcal{N}(I - S_{T^*}) = \mathcal{H} \ominus \mathcal{N}(S_{\hat{T}^*}). \tag{4.17}$$

(iii) *The largest subspace which reduces T^* to a strongly stable contraction is*

$$\mathcal{N}(S_{T^*}) = \mathcal{N}(T) \oplus \mathcal{N}(S_{\hat{T}^*}) = \mathcal{N}(S_T) \oplus \big(\mathcal{N}(I - S_T) \ominus \mathcal{N}(I - S_{T^*})\big). \tag{4.18}$$

Furthermore, one has

$$\mathcal{N}(S_{\hat{T}^*}) = \mathcal{N}(S_{\hat{T}}) \oplus \big(\mathcal{N}(I - S_T) \ominus \mathcal{N}(I - S_{T^*})\big). \tag{4.19}$$

Proof. Let T be a quasinormal contraction. Since T^* and T^*T commute, T^* is a regular T^*T-contraction on $\mathcal{H}$. In this case, the corresponding contraction A_{T^*} ($A = T^*T$) is equal to S_{T^*}. Indeed, because $TT^* \le T^*T \le I$ we have for $n \ge 1$

$$T^nT^*TT^{*n} \le T^nT^{*n} = T^{n-1}TT^*T^{*(n-1)} \le T^{n-1}T^*TT^{*(n-1)},$$

whence it follows that

$$A_{T^*} = s - \lim_n T^nT^*TT^{*n} = s - \lim_n T^nT^{*n} = S_{T^*}.$$

Since $T^nT^*TT^{*n} = T^nT^{*n}T^*T = T^*TT^nT^{*n}$, we infer that $S_{T^*} = S_{T^*}T^*T = T^*TS_{T^*}$, and also $(I-T^*T)S_{T^*} = 0$. This implies that $\mathcal{R}(S_{T^*}) \subset \mathcal{N}(I-T^*T)$, hence $\mathcal{N}(I-S_{T^*}) \subset \mathcal{R}(S_{T^*}) \subset \mathcal{N}(I-S_T)$ having in view (4.13). But S_{T^*} and T^*T commute, therefore $\overline{\mathcal{R}(T^*)}$ reduces S_{T^*}, and we have $S_{T^*}|_{\overline{\mathcal{R}(T^*)}} = S_{\hat{T}^*}$ where $\hat{T} \in \mathcal{B}(\overline{\mathcal{R}(T^*)})$ verifies $\hat{T}|T| = |T|\hat{T}$ on $\overline{\mathcal{R}(T^*)}$. Indeed, for T^* as a T^*T-contraction there is a contraction $T_* \in \mathcal{B}(\overline{\mathcal{R}(T^*)})$ satisfying $T_*|T| = |T|T^* = T^*|T|$ on $\overline{\mathcal{R}(T^*)}$, hence $T_* = T^*|_{\overline{\mathcal{R}(T^*)}}$. Since $\overline{\mathcal{R}(T^*)}$ reduces T one has $T_*^* = T|_{\overline{\mathcal{R}(T^*)}} = \hat{T}$, therefore $\hat{T}^* = T_* = T^*|_{\overline{\mathcal{R}(T^*)}}$, and the relation quoted above between S_{T^*} and $S_{\hat{T}^*}$ follows immediately.

Now, since $\mathcal{N}(I - S_{T^*}) \subset \mathcal{R}(T^*)$ we have $\mathcal{N}(I - S_{T^*}) = \mathcal{N}(I - S_{\hat{T}^*})$. But $\mathcal{N}(I - S_{T^*})$ is an invariant subspace for T, because if $h \in \mathcal{N}(I - S_{T^*})$ then using the fact that T is quasinormal we get

$$\begin{aligned} S_{T^*}Th &= \lim_n T^nT^*TT^{*n}h = \lim_n T^nT^*TT^*TT^{*(n-1)}h = \\ &= \lim_n TT^*TT^{n-1}T^*TT^{*(n-1)}h = TT^*TS_{T^*}h = TS_{T^*}h = Th, \end{aligned}$$

hence $Th \in \mathcal{N}(I-S_{T^*})$. On the other hand, $\mathcal{N}(I-S_{T^*})$ is the largest invariant subspace for T^* on which T^* is an isometry (being the corresponding subspace $\mathcal{N}_\infty$ for the regular T^*T-contraction T^*). Since $\mathcal{N}(I - S_{T^*}) \subset \mathcal{N}(I - T^*T)$, it follows that $\mathcal{N}(I - S_{T^*})$ is the largest subspace which reduces T to a unitary operator, or equivalently, on which T^* is an isometry. Since one has

$$\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{T^*}) \oplus (\mathcal{N}(I - S_T) \ominus \mathcal{N}(I - S_{T^*})),$$

T will be a shift, or equivalently T^* a strongly stable contraction, hence

$$\mathcal{N}(I - S_T) \ominus \mathcal{N}(I - S_{T^*}) = \mathcal{N}(I - S_T) \cap \mathcal{N}(S_{T^*}).$$

having in view that $\mathcal{N}(S_{T^*})$ is the largest subspace on which T^* is strongly stable. On the other hand, using the fact that $TT^* \leq T^*T$ and that T is quasinormal, one obtains that $S_{T^*} \leq S_T$, whence $\mathcal{N}(S_T) \subset \mathcal{N}(S_{T^*})$. Now, because S_T is an orthogonal projection, we infer from above relations that

$$\mathcal{H} = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T) = \mathcal{N}(I - S_{T^*}) \oplus \mathcal{N}(S_{T^*}) = \mathcal{N}(S_{T^*} - S_{T^*}^2),$$

consequently S_{T^*} is an orthogonal projection, which ends the proof of the statements (i). Also, we obtain that $\mathcal{N}(S_{T^*})$ is the largest subspace which reduces T^* to a strongly stable contraction, and clearly we have from the above remarks and (4.10),

$$\mathcal{N}(S_{T^*}) = \mathcal{N}(S_T) \oplus (\mathcal{N}(I - S_T) \ominus \mathcal{N}(I - S_{T^*})) = \mathcal{N}(T) \oplus \mathcal{N}(S_{\hat{T}^*}).$$

This leads to the assertion (iii).

Next, we remark that the subspace $\mathcal{N}_\infty$ for the regular T^*T-contraction T^* given by the relation (4.9) is

$$\mathcal{N}(T^*T - S_{T^*}) = \mathcal{N}(T) \oplus \mathcal{N}(I - S_{T^*}) = \mathcal{H} \ominus \mathcal{N}(S_{\hat{T}^*}),$$

and this is the largest subspace which reduces T^* to a T^*T-isometry, because $\mathcal{N}(I - S_{T^*})$ reduce T. Equivalently, $\mathcal{N}(T^*T - S_{T^*}) = \tilde{\mathcal{M}}$, the subspace from (3.5), hence this subspace has the required property relative to T in (ii). The assertion (ii) holds, and the proof is finished. ∎

Finally from Corollary 3.8 and (4.14) we obtain

Corollary 4.12. *If T is a quasinormal contraction on $\mathcal{H}$ and $\hat{T} = T|_{\overline{\mathcal{R}(T^*)}}$ then*

$$\overline{\mathcal{R}(T^*T - T^{*2}T^2)} = \mathcal{N}(S_{\hat{T}}), \tag{4.20}$$

hence T and T^* are strongly stable contractions on this subspace. Also, a quasinormal contraction is strongly stable if and only if it is a proper contraction.

We notice that the above facts concerning the quasinormal contractions are obtained by different methods as ones from [2], [3]. Here we only used the context of the regular A-contractions.

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