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On Carnot's Theorem in Time Dependent Impulsive Mechanics

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1. Introduction

The results and techniques of Differential Geometry, and in particular of the theory of jet–bundles of the fiber bundle $t : \mathcal{V} \to \mathbb{R}$ with 1–dimensional base usually known as the Classical Space–Time bundle, are by this time known as powerful instruments to investigate and rationalize several aspects of time–dependent Classical Mechanics.

Nevertheless, possibly due to its intrinsic non–smooth nature, Impulsive Mechanics and system subject to impulsive constraints seemed to be not inclined to be framed in a differential geometric setup, and till few years ago only few works ([4], [1]) could be found in literature on the argument. The results became even fewer restricting our attention to time–dependent description of impulsive problems ([2]). Only recently ([6]) a proposal of a formal geometric environment for frame–independent description of time–dependent Impulsive Mechanics was presented. The new context allows a fruitful rationalization of the basic concepts of Impulsive Mechanics, such as the definition of active impulse itself, and a detailed analysis of the motion of impulsive systems in presence of bilateral constraints, independently on their positional or kinetic nature, and on their permanent or instantaneous one ([6], [7]).

The classical formulation of Carnot's theorem on impulsive constraints (see, e.g. [5], [9]) should then find a natural collocation in the new framework. Unfortunately, an intrinsic frame–independent description of Classical Mechanics, and in particular the description of Impulsive Mechanics presented in [6], [7], implies a structural loss of meaning of the kinetic energy as a mechanical quantity associated to the system. A well posed definition of kinetic energy is, of course, indissolubly related to the assignment of a frame of reference for the system ([8]). Then Carnot's theorem can be presented following two possible lines. One consists in fixing *a priori* a frame of reference for the system in a more or less explicit way. Consequently, a diffeomorphism $\mathcal{V} \simeq \mathbb{R} \times Q$ is determined by the frame of reference, with Q denoting the usual configuration manifold of the system, and the kinetic energy of the system is then a well defined quadratic function defined on $\mathbb{R} \times T(Q)$. This line of presentation has been pursued in [3], even in the more general case of one–sided impulsive constraint, and Carnot's theorem can be deduced essentially by algebraic properties of quadratic forms. The other line of presentation consists in working in the general frame–independent description of the system and investigating the class of frames of reference where the Carnot's theorem holds. This approach and its results are the arguments of this paper. It will be proved that Carnot's theorem holds only in a restricted class of frames, and that, once the constitutive characterization of the constraints is chosen, this class is determined by the constraints itself.

The paper is divided into three main sections. In Section 2, also in order to fix notation, we briefly describe the usual geometric environment generated by the Space–Time bundle $t : \mathcal{V} \to \mathbb{R}$ of Classical Mechanics and its jet–extensions, and we recall the concepts of frame of reference H, of kinetic energy related to H, of impulse acting on the system and of kinetic impulsive constraint. In Section 3 we discuss the statement of Carnot's theorem for inert impulsive constraints in the considered geometric context, showing that the theorem holds only for a restricted class of frames, whose wideness is determined by the codimension of the constraint. In Section 4 we exhibit a simple mechanical system admitting two frames of reference, one in the class where Carnot's theorem holds, the other outside the class. Moreover we show with a simple example that some other classical results regarding the kinetic energy balance in Impulsive Mechanics hold in the new geometric context independently of the frame of reference, and then have an "absolute" value.

2. Preliminaries

In Classical Mechanics, the configuration space–time of a mechanical system with a finite number of degrees of freedom is a fiber bundle $\pi : \mathcal{V} \to \mathbb{E}_1$, where $\mathcal{V}$ is an $(n+1)$–dimensional differentiable manifold and $\mathbb{E}_1$ is the 1–dimensional euclidean line. If t is the global cartesian coordinate on $\mathbb{E}_1$, the projection $t \circ \pi : \mathcal{V} \to \mathbb{R}$, that, with a slight abuse of notation, will also be denoted by t, is the embodiment of the Absolute Time Axiom in this context. The projection t selects both the family of *admissible coordinates* $(t, q^1, \ldots, q^n)$ having t as first coordinate, and the subgroup

$$\begin{cases} \bar{t} & = & t + \text{const.} \\ \bar{q}^k & = & \bar{q}^k(t, q^i) \end{cases}$$

of the group of coordinate transformations leaving dt and the fibres invariant. Moreover it determines also the n–dimensional distribution $\mathcal{V}ert$ of space–like vectors, given by those vectors which are tangent to the fibres, and the vertical vector bundle $\pi : V(\mathcal{V}) \to \mathcal{V}$.

The Absolute Space Axiom is introduced in this context by assigning a riemannian metric Φ on each fiber of $t : \mathcal{V} \to \mathbb{R}$, i.e. a differentiable positive definite scalar product on $\mathcal{V}ert$. We denote with

$$g_{ij}(p) = \Phi\left(\left(\frac{\partial}{\partial q^i}\right)_p, \left(\frac{\partial}{\partial q^j}\right)_p\right), \qquad p \in \mathcal{V}$$

with the functions g_{ij} taking intrinsically into account, as usual, the massive properties of the system.

A (global) frame of reference in $\mathcal{V}$ consists of a (complete) vector field H of the form

$$H = \frac{\partial}{\partial t} + H^i(t, q)\frac{\partial}{\partial q^i}, \quad i = 1, \ldots, n,$$

or, that is the same, a (global) section of the affine bundle $J_1(\mathcal{V})$, the first jet–extension of the bundle $\mathcal{V}$. The affine structure of $J_1(\mathcal{V})$, modelled on the vertical vector bundle $V(\mathcal{V})$, can be pointed out by introducing *admissible coordinates* $(t, q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$ in $J_1(\mathcal{V})$, subject to the transformation rules

$$\begin{cases} \bar{t} & = & t + \text{const.} \\ \bar{q}^k & = & \bar{q}^k(t, q^i) \\ \dot{\bar{q}}^k & = & \dfrac{\partial \bar{q}^k}{\partial q^i}\, \dot{q}^i + \dfrac{\partial \bar{q}^k}{\partial t} \end{cases} \tag{1}$$

The affine linearity of the transformation rules of the $\dot{q}$ variables will play a crucial role in the following, distinguishing the analysis of the problem based on jet–bundles by the one based on the product manifolds $\mathbb{R} \times Q$ and $\mathbb{R} \times T(Q)$.

In fact, in this second case, one can introduce in $\mathbb{R}\times T(Q)$ the so called *bundle coordinates* $(t, x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n)$ that are subject to transformation rules

$$\begin{cases} \bar{t} & = & t + \text{const.} \\ \overline{x}^k & = & \overline{x}^k(x^i) \\ \dot{\overline{x}}^k & = & \dfrac{\partial \overline{x}^k}{\partial x^i}\,\dot{x}^i \end{cases} \tag{2}$$

The outcome of the difference between the transformation rules (1) and (2) is cleared by the following

DEFINITION 2.1. Let H be an assigned frame of reference. Then:

i) the velocity of $p = (t, q^1, \ldots, q^n, \dot{q}^{\,1}, \ldots, \dot{q}^{\,n}) \in J_1(\mathcal{V})$ with respect to the frame H is the vertical vector $\mathbf{v} \in V(\mathcal{V})$ such that

$$\mathbf{v} = p - H(\pi(p)) = \left(\dot{q}^{\,i} - H^i(t, q^k)\right) \frac{\partial}{\partial q^{\,i}}\,;$$

ii) the kinetic energy of the system with respect to H is the function $\mathcal{T} : J_1(\mathcal{V}) \to \mathbb{R}$ such that

$$\mathcal{T}(t, q^{\,k}, \dot{q}^{\,k}) = \frac{1}{2}\Phi(\mathbf{v}, \mathbf{v}) = \frac{1}{2} g_{ij}(t, q^{\,k})\left(\dot{q}^{\,i} - H^i(t, q^{\,k})\right)\left(\dot{q}^{\,j} - H^j(t, q^{\,k})\right).$$

A straightforward calculation shows that the function $\mathcal{T}$ has the correct invariance behaviour with respect to (1). Moreover, the definition gives back the natural interpretation of kinetic energy of the system as a mechanical quantity related to an assigned frame of reference. The same invariance properties and the same mechanical meaning can be given to the quadratic form

$$\mathcal{K}(t, x^k, \dot{x}^k) = \frac{1}{2}\, g_{ij}(t, x^k)\dot{x}^i \dot{x}^j$$

only by considering the transformation rules (2) in $\mathbb{R} \times T(Q)$ and loosing the frame independence of the analysis ([8]).

The geometric context of Impulsive Mechanics is based on the introduction of two affine fiber bundles $\pi : L_1(\mathcal{V}) \to \mathcal{V}$ and $\pi : R_1(\mathcal{V}) \to \mathcal{V}$, respectively the left and right jet bundle of $\mathcal{V}$, both diffeomorphic to $J_1(\mathcal{V})$, representing the possible left and right velocities of the system. Admissible coordinates $(t, q^{\,i}, \dot{q}^{\,i}_L)$ and $((t, q^{\,i}, \dot{q}^{\,i}_R)$, whose transformation rules are identical to (1), will be systematically assumed to describe the bundles $L_1(\mathcal{V}), R_1(\mathcal{V})$ respectively. In analogy with Def. (2.1), we denote with $\mathbf{v}_L, \mathbf{v}_R$ the vertical vectors representing the left and right velocities of elements in $L_1(\mathcal{V}), R_1(\mathcal{V})$ with respect to an assigned frame of reference.

A free impulse acting on the system is the assignment of a pair $(\mathcal{A}, I)$, where $\mathcal{A}$ is a suitable fibered submanifold of $J_1(\mathcal{V})$ and $I : \mathcal{A} \to V(\mathcal{V})$ is a fibered map assigning to each kinetic state $p \in \mathcal{A}$ a vertical vector $I(p)$ representing the jump of velocities of the system ([6]). Since I is a fibered map, the condition $\pi(p) = \pi(p + I(p))$ holds for all $p \in \mathcal{A}$, showing that impulses affect the velocities of the system without changing its position.

A kinetic impulsive constraint acting on the system is the assignment of a suitable pair $(\mathcal{A}_1, \mathcal{A}_2)$ of fibered submanifolds, with $\mathcal{A}_1 \subset L_1(\mathcal{V})$, $\mathcal{A}_2 \subset R_1(\mathcal{V})$. The properties of the submanifolds $\mathcal{A}_1, \mathcal{A}_2$ reflect the nature of the constraint, distinguishing among kinetic impulsive constraints imposed on the system depending on its position, its velocity or at an assigned instant. For example, a cogwheel sliding on a line and impacting in a rack placed in a fixed position of the line can be modelled with a submanifold $\xi : \mathcal{S} \to \mathcal{V}$, that represents the fixed position of the rack, and a submanifold $\mathcal{A}_2 \subset R_1(\mathcal{V})$, that represents the kinetic constraint, obeying the condition $\pi(\mathcal{A}_2) = \mathcal{S}$. This example will be described in details in Section 4. Note however that, depending on the dimension and the nature of $\mathcal{S}$, the rack can consist of a single tooth, so that the corresponding kinetic impulsive constraint can be thought of as an instantaneous constraint, or the rack can have a strictly positive length, so that the corresponding kinetic impulsive constraint can become a (thenceforth) permanent kinetic constraint (see [7] for complete survey and classification).

Definition 2.2. An inert constraint is a kinetic impulsive constraint such that $\mathcal{A}_1 \equiv L_1(\mathcal{V})$ and $\mathcal{A}_2$ admits a linear affine description

$$\sum_{i=1}^{n} a_{\mu i}\, \dot{q}^{\,i}_{R} + b_\mu = 0, \quad \mu = 1, \ldots, n-r. \tag{3}$$

where the matrix $a_{\mu i}(t, q)$ is supposed, here and from now on, to be everywhere of maximum rank. Once again, a straightforward calculation shows that the representation (3) has the correct invariance behaviour with respect to (1). Differently, a linear description

$$\sum_{i=1}^{n} a_{\mu i}\, \dot{q}^{\,i} = 0, \quad \mu = 1, \ldots, n-r$$

is not invariant with respect to (1) but it is invariant using bundle coordinates in a fixed frame. Then Def. (2.2) is the most natural adaptation to the frame independent context of the classical definition of inert constraint ([5]).

A constitutive characterization of a kinetic impulsive constraint $(\mathcal{A}_1, \mathcal{A}_2)$ is a rule assigning to each $p_1 \in \mathcal{A}_1$ an element of the set of vertical vectors

$$V(\mathcal{A}_1, \mathcal{A}_2) = \{V = p_2 - p_1, \text{ s.t. } p_1 \in \mathcal{A}_1,\, p_2 \in \mathcal{A}_2\} \subset V(\mathcal{V}),$$

where the use of the natural diffeomorphism between $L_1(\mathcal{V}), R_1(\mathcal{V})$ and $J_1(\mathcal{V})$ is implicitly understood. The resulting map $I : \mathcal{A}_1 \to V(\mathcal{V})$ represents of course the reactive impulse given by the constraint. The Gauss's minimality principle allows the definitions of a very natural constitutive characterization and an ideality criterion for a wide class of impulsive constraints ([7]): later on we systematically assume that impulsive constraints obey this ideality criterion. Note that, independently of the choice of a frame of reference and of a constitutive characterization, every impulse obeys the identity

$$I = \mathbf{v}_R - \mathbf{v}_L\,. \tag{4}$$

3. Carnot's Theorem

The imposition of an impulsive constraint $(\mathcal{A}_1, \mathcal{A}_2)$ on a mechanical system determines a reactive impulse $I : \mathcal{A}_1 \subset L_1(\mathcal{V}) \to V(\mathcal{V})$ and a corresponding quadratic form

$$\mathcal{T}_\Delta : \mathcal{A}_1 \subset L_1(\mathcal{V}) \to \mathbb{R} \quad \text{s.t.} \quad \mathcal{T}_\Delta = \frac{1}{2}\Phi(I, I) = \frac{1}{2}\, g_{hk} I^h I^k$$

The assignment of a frame of reference H allows the definitions of left and right kinetic energy relative to H of the system

$$\mathcal{T}_L : L_1(\mathcal{V}) \to \mathbb{R} \quad \text{s.t.} \quad \mathcal{T}_L = \frac{1}{2}\Phi(\mathbf{v}_L, \mathbf{v}_L) = \frac{1}{2}\, g_{ij}\, (\dot{q}^{\,i}_{\,L} - H^i)(\dot{q}^{\,j}_{\,L} - H^j)$$

$$\mathcal{T}_R : R_1(\mathcal{V}) \to \mathbb{R} \quad \text{s.t.} \quad \mathcal{T}_R = \frac{1}{2}\Phi(\mathbf{v}_R, \mathbf{v}_R) = \frac{1}{2}\, g_{ij}\, (\dot{q}^{\,i}_{\,R} - H^i)(\dot{q}^{\,j}_{\,R} - H^j)$$

With the usual mild abuse of notation based on the natural diffeomorphism between $L_1(\mathcal{V}), R_1(\mathcal{V})$ and $J_1(\mathcal{V})$, the three functions can be thought of as defined on $J_1(\mathcal{V})$.

An obvious calculation shows that, independently on the nature of the kinetic constraint and on the constitutive characterization chosen, the difference $\Delta\mathcal{T} = \mathcal{T}_R - \mathcal{T}_L$ is intrinsically dependent on the frame H, and different frames give different $\Delta\mathcal{T}$. On the contrary, the quadratic form $\mathcal{T}_\Delta$ is independent of the frame. Then relations of the form $\Delta\mathcal{T} = -\mathcal{T}_\Delta$ (see [5]) or $\Delta\mathcal{T} = \alpha\mathcal{T}_\Delta$ with $\alpha \in \mathbb{R}$ (see [3]) can be interpreted only as a property of the frame of reference describing the system and not as an intrinsic property of the system itself.

LEMMA 3.1. *For every assigned frame of reference, the relation*

$$\Delta\mathcal{T} = \mathcal{T}_R - \mathcal{T}_L = -\mathcal{T}_\Delta + \Phi(\mathbf{v}_R, I)$$

holds.

Proof.

$$\Delta\mathcal{T} = \frac{1}{2}\Phi(\mathbf{v}_R, \mathbf{v}_R) - \frac{1}{2}\Phi(\mathbf{v}_L, \mathbf{v}_L) = \frac{1}{2}\Phi(\mathbf{v}_R, \mathbf{v}_R) - \frac{1}{2}\Phi(\mathbf{v}_R - I,\, \mathbf{v}_R - I)$$

$$= -\frac{1}{2}\Phi(I, I) + \Phi(\mathbf{v}_R, I) = -\mathcal{T}_\Delta + \Phi(\mathbf{v}_R, I).$$

∎

The lemma shows that, independently of the nature of the constraint and of the constitutive characterization, the class of frames of reference where an analogous of the Carnot's theorem holds is selected by the condition

$$\Phi(\mathbf{v}_R, I) = g_{ij}\,(\dot{q}^{\,i}_R - H^i)\, I^j = 0. \tag{5}$$

Since both the kinetic status of the system and the reactive impulse due to the impulsive constraint are independent of the frame of reference, condition (5) has a clear geometric interpretation: the Carnot's theorem holds for a frame of reference H if and only if the corresponding right velocity $\mathbf{v}_R$ of the system turns out to be orthogonal to the reactive impulse.

We focus now our attention to the classical formulation of Carnot's theorem (see, e.g. [5]). Let the system be subject to an inert constraint, and let the constitutive characterization of the constraint be given by the Gauss's minimality principle. Then $\mathcal{A}_2$ can be described by coordinates $(t, q^1, \ldots, q^n, z^1, \ldots, z^r)$ such that

$$\varphi : \mathcal{A}_2 \to R_1(\mathcal{V}) \quad \dot{q}^{\,i}_R = h^i_\alpha z^\alpha + k^i. \tag{6}$$

Let $V(\mathcal{A}_2, \mathcal{A}_2)$ be defined as

$$\begin{aligned} V(\mathcal{A}_2, \mathcal{A}_2) &= \{p_2 - \overline{p}_2 \mid p_2, \overline{p}_2 \in \mathcal{A}_2,\, \pi(p_2) = \pi(\overline{p}_2)\} \\ &= \left\{ V \in V(\mathcal{V}) \mid V = h^i_\alpha v^\alpha \frac{\partial}{\partial q^{\,i}} \right\} \end{aligned} \tag{7}$$

Due to the affine linear description of $\mathcal{A}_2$, the space $V(\mathcal{A}_2, \mathcal{A}_2)$ is a vector subbundle of $V(\mathcal{V})$ with fiber dimension r. We have then the following

THEOREM 3.1. *The reactive impulse I_{Gauss} determined by the inert constraint is orthogonal to the vector subbundle $V(\mathcal{A}_2, \mathcal{A}_2)$.*

Proof. Taking into account (6), the reactive impulse is a vertical vector field of the form

$$I = (h^i_\alpha z^\alpha + k^i - \dot{q}^{\,i}_{\,L}) \frac{\partial}{\partial q^{\,i}}$$

Denoting with Γ the restriction of the scalar product Φ to the subbundle $V(\mathcal{A}_2, \mathcal{A}_2)$ and with $\gamma_{\alpha\beta} = g_{ij} h^i_\alpha h^j_\beta$ the corresponding non–singular matrix with inverse $\gamma^{\alpha\beta}$, the minimum of $\Phi(I, I)$ is attained for the values $z^\beta = \gamma^{\alpha\beta} g_{ij} h^i_\alpha (\dot{q}^{\,j}_{\,L} - k^j)$ so that the corresponding reactive impulse is

$$I_{Gauss} = \big(h^m_\beta\, \gamma^{\alpha\beta} g_{ij}\, h^i_\alpha (\dot{q}^{\,j}_{\,L} - k^j) + k^m - \dot{q}^{\,m}_{\,L}\big) \frac{\partial}{\partial q^{\,m}}$$

Then, given any element $V \in V(\mathcal{A}_2, \mathcal{A}_2)$ and taking into account the expression (7), we have

$$\begin{aligned} \Phi(I_{Gauss}, V) &= g_{ms}\, I^m V^s \\ &= g_{ms}\, (h^m_\beta\, \gamma^{\alpha\beta} g_{ij}\, h^i_\alpha (\dot{q}^{\,j}_{\,L} - k^j) + k^m - \dot{q}^{\,m}_{\,L})(h^s_\lambda v^\lambda) \\ &= g_{ms}\, h^m_\beta h^s_\lambda\, \gamma^{\alpha\beta} g_{ij}\, h^i_\alpha (\dot{q}^{\,j}_{\,L} - k^j) v^\lambda - g_{ms}\, h^s_\lambda (\dot{q}^{\,m}_{\,L} - k^m) v^\lambda \\ &= g_{ij}\, h^i_\lambda (\dot{q}^{\,j}_{\,L} - k^j) v^\lambda - g_{ms}\, h^s_\lambda (\dot{q}^{\,m}_{\,L} - k^m) v^\lambda = 0 \end{aligned}$$

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Note that, although foreseeable, the result expresses in the unique meaningful way for Impulsive Mechanics the fact that the Gauss's minimality principle determines an orthogonal projector in the vertical vector bundle $V(\mathcal{V})$ and it strictly depends on the affine linear nature of $\mathcal{A}_2$. In fact, both the orthogonality between the impulse and the constraint, and the orthogonality between the impulse and the tangent space of the constraint, relating vectors belonging to different vector spaces, are meaningless.

COROLLARY 3.1. *Let $(L_1(\mathcal{V}), \mathcal{A}_2)$ be an inert constraints locally described by eqs.* (3) *and let $H = \frac{\partial}{\partial t} + H^k(t,q)\frac{\partial}{\partial q^k}$ be a frame of reference obeying the conditions*

$$\sum_{i=1}^{n} a_{\mu i}\ H^i + b_\mu = 0, \quad \mu = 1, \ldots, n-r.$$

Then the Carnot's theorem holds for the frame H. Moreover, if $\operatorname{codim}(\mathcal{A}_2) = n-1$, *then H is essentially unique.*

Proof. The statement easily follows by condition (5), definition (7) and theorem (3.1).

4. EXAMPLES

4.1. EXAMPLE 1: THE BEHAVIOUR OF DIFFERENT FRAMES FOR A SINGLE SYSTEM.

In the following simple example, we exhibit a mechanical system subject to an inert constraint and two different frames having a different behaviour regarding the Carnot's theorem.

A cogwheel of assigned mass m, radius r and inertia momentum $A = \frac{1}{2}mr^2$ moving in a vertical plane slides along an horizontal straight guide and impacts in a rack placed on the guide in a fixed position. Let the position x of the center of mass and the orientation θ of the cogwheel be the lagrangian coordinates describing the configuration of the system. The fixed position of the rack can be represented by the submanifold $\mathcal{S} = \{x \geq \overline{x}\}$, if we have an extended rack, or by the submanifold $\mathcal{S} = \{x = \overline{x}\}$, if we have a single tooth. Independently of the nature of $\mathcal{S}$, we have that (see [7]):

- the vertical scalar product is locally given by the constant matrix $g_{ij} = diag(m, A)$;
- the space $L_1(\mathcal{V})$ is described by coordinates $(t, x, \theta, \dot{x}_L, \dot{\theta}_L)$ and its elements can be represented in vector form as $V = \frac{\partial}{\partial t} + \dot{x}_L\frac{\partial}{\partial x} + \dot{\theta}_L\frac{\partial}{\partial \theta}$;
- the manifold $\mathcal{A}_2 \subset R_1(\mathcal{V})$ is defined by the condition $\mathcal{A}_2 = \{\dot{x}_R - r\dot{\theta}_R = 0\}$ and its elements can be represented in vector form as $V = \frac{\partial}{\partial t} + r\dot{\theta}_R\frac{\partial}{\partial x} + \dot{\theta}_R\frac{\partial}{\partial \theta}$.

A straightforward calculation shows that the minimality criterion applied to the quadratic form $\Phi(I, I)$ gives $\dot{\theta}_{R_{min}} = \frac{2\dot{x}_L + r\dot{\theta}_L}{3r}$, and then $I_{gauss} = -\frac{1}{3}(\dot{x}_L -$

$r\dot\theta_L)\frac{\partial}{\partial x} + \frac{2}{3r}(\dot x_L - r\dot\theta_L)\frac{\partial}{\partial\theta}$. It follows that, with the usual abuse of notation,

$$\mathcal{T}_\Delta = \frac{1}{6}\, m\, \dot x^2 - \frac{1}{3}\, mr\, \dot x\, \dot\theta + \frac{1}{6}\, mr^2\, \dot\theta^2.$$

Let now $H_1 = \frac{\partial}{\partial t}$ be the frame of rest of the rack; with obvious notation we have

$$\mathcal{T}_L^{H^1} = \frac{1}{2}\, m\, \dot x^2 + \frac{1}{4}\, mr^2\, \dot\theta^2\,;$$

$$\mathcal{T}_R^{H^1} = \frac{1}{2}\, m \left(\frac{2\dot x + r\dot\theta}{3r}\right)^2 + \frac{1}{4}\, mr^2 \left(\frac{2\dot x + r\dot\theta}{3r}\right)^2$$

$$\mathcal{T}_R^{H^1} - \mathcal{T}_L^{H^1} = -\left(\frac{1}{6}\, m\, \dot x^2 - \frac{1}{3}\, mr\, \dot x\, \dot\theta + \frac{1}{6}\, mr^2\, \dot\theta^2\right) = -\mathcal{T}_\Delta\,.$$

Differently, let $H_2 = \frac{\partial}{\partial t} + \dot x_0 \frac{\partial}{\partial x}$ with $\dot x_0$ fixed, be a frame moving with respect to the rack with constant velocity; then:

$$\mathcal{T}_L^{H^2} = \frac{1}{2}\, m\, (\dot x - \dot x_0)^2 + \frac{1}{4}\, mr^2\, \dot\theta^2\,;$$

$$\mathcal{T}_R^{H^2} = \frac{1}{2}\, m \left(\frac{2\dot x + r\dot\theta}{3r} - \dot x_0\right)^2 + \frac{1}{4}\, mr^2 \left(\frac{2\dot x + r\dot\theta}{3r}\right)^2$$

$$\begin{aligned}\mathcal{T}_R^{H^2} - \mathcal{T}_L^{H^2} &= -\frac{1}{6}\, m\, \dot x^2 + \frac{1}{3}\, mr\, \dot x\, \dot\theta - \frac{1}{6}\, mr^2\, \dot\theta^2 + \frac{1}{3}\, mr\, \dot x_0\, \dot x - \frac{1}{3}\, mr\, \dot x_0\, \dot\theta \\ &= -\mathcal{T}_\Delta + \frac{1}{3}\, m\, \dot x_0\, (\dot x - r\dot\theta).\end{aligned}$$

4.2. Example 2: a classical result holding in the new context. The following example (see [5]) shows that, differently from the Carnot's theorem, there are classical results regarding the kinetic energy balance in an impulsive problem that holds independently of the frame of reference, and then have an "absolute" meaning.

Theorem 4.1. *In the free impulse problem, the increase of kinetic energy is found by taking the scalar product of the impulse with the mean of left and right velocities.*

Proof. Taking into account eq. (4), we have

$$\begin{aligned}\Delta\mathcal{T} = \mathcal{T}_R - \mathcal{T}_L &= \frac{1}{2}\,\Phi(\mathbf{v}_R,\mathbf{v}_R) - \frac{1}{2}\,\Phi(\mathbf{v}_L,\mathbf{v}_L)\\ &= \frac{1}{2}\,\Phi(\mathbf{v}_R-\mathbf{v}_L,\mathbf{v}_R+\mathbf{v}_L)\\ &= \Phi\Big(I,\frac{\mathbf{v}_R+\mathbf{v}_L}{2}\Big)\end{aligned}$$

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