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The Kalton-Peck space is the complexification of the real Kalton-Peck space ☆

Jesús M.F. Castillo [a,*], Yolanda Moreno [b]

[a] Universidad de Extremadura, Instituto de Matemáticas Imuex, E-06011 Badajoz, Spain
[b] Universidad de Extremadura, Instituto de Matemáticas Imuex, E-07001 Cáceres, Spain

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ABSTRACT

The Kalton-Peck Z_2 space is the derived space obtained from the scale of ℓ_p spaces by complex interpolation at $1/2$. If we denote by Z_2^{real} the derived space obtained from the scale of ℓ_p spaces by real interpolation at $(1/2, 1/2)$, we show that Z_2 is the complexification of Z_2^{real}. We also show that Z_2^{real} shares the most important properties of Z_2: it is isomorphic to its dual, it is singular and contains no complemented copies of ℓ_2.

This paper outgrowths from [4], in which, in the words of its authors, *it is shown that Rochberg's generalized interpolation spaces* $\mathfrak{X}^{(n)}$ *arising from analytic families of Banach spaces form exact sequences* $0 \to \mathfrak{X}^{(n)} \to \mathfrak{X}^{(n+k)} \to \mathfrak{X}^{(k)} \to 0$ *and that nontriviality, having strictly singular quotient map, or having strictly cosingular embedding depend only on the basic case* $n = k = 1$. Our attempt has been to bring down to sound earth its Open end 6.3: *It is the feeling of the authors that most of the work done in this paper could be reproduced for real interpolation by either the* K *or* J *methods with a careful analysis of the work done in [Carro et al. 1995]. It would be interesting to know to what extent the same occurs for other interpolation methods.*

That is what we will do in this first part of the paper in the following way: It is a fact well known to all those who know it that most interpolation methods generate exact sequences of the interpolated spaces; say, if $[X_0, X_1]_\mu$ is the interpolated space obtained from the pair (X_0, X_1) with parameters set at μ, then there exists a natural exact sequence $0 \longrightarrow [X_0, X_1]_\mu \longrightarrow X \longrightarrow [X_0, X_1]_\mu \longrightarrow 0$. In this representation, $[X_0, X_1]_\mu$ would be the first Rochberg space and the twisted sum space X the

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* Corresponding author.
E-mail addresses: castillo@unex.es (J.M.F. Castillo), ymoreno@unex.es (Y. Moreno).

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second Rochberg space. After that, higher order Rochberg spaces [21] can be rather naturally generated when the method provides a sequence of interpolators (see [10,11]). In this way, if $\mathfrak{R}^{(n)}$ denotes the n^{th}-Rochberg space with $\mathfrak{R}^{(1)} = [X_9, X_1]_\mu$ and $\mathfrak{R}^{(2)} = X$, these Rochberg spaces form natural exact sequences $0 \longrightarrow \mathfrak{R}^{(n)} \longrightarrow \mathfrak{R}^{(n+m)} \longrightarrow \mathfrak{R}^{(m)} \longrightarrow 0$ (see [4,3]). Once again, most –but not all– standard interpolation methods fit into this schema, as it is more or less implicit in the papers of Cwikel et al. [14], Carro et al. [6] and Rochberg [21] and made explicit when Cwikel, Kalton, Milman and Rochberg introduce their unifying method [15], from now on called the CKMR method.

On the other hand, the theory created by Kalton [17] establishes that exact sequences of quasi Banach spaces are in correspondence with a special type of nonlinear maps called quasilinear maps. Given an exact sequence $0 \longrightarrow X \longrightarrow X_\Omega \longrightarrow X \longrightarrow 0$ with associated quasilinear map Ω and another sequence $0 \longrightarrow X \longrightarrow X_\Phi \longrightarrow X \longrightarrow 0$ with associated quasilinear map Φ, the two exact sequences are called projectively equivalent [19,2] if there is an scalar λ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "\lambda"'] & X_\Omega \arrow[r] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X \arrow[r] & X_\Phi \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

is commutative, which means that there is a linear map L such that $\lambda\Omega - \Phi$ is the sum of a bounded plus a linear map, both $X \to X$.

The same occurs for the exact sequences of Rochberg spaces. In that case, the quasilinear maps have a special form and additional properties (see below) [19,10] and are called differentials [18,8,9], which justifies that we call the process of obtaining the associated differential out from an interpolation method as derivation. The derivation process for the CKRM method is described and studied in detail in [15]. We make a rapid survey of what the reader needs to know about the CKRM method to ease the reading of this paper. In the same way, we assume from the reader a certain acquaintance with exact sequences, quasilinear maps and interpolation methods (that can be obtained from, respectively, [16,17,1] or, all at once, from [2]).

1. Preliminaries: the CKRM method

Let **Ban** be the class of all complex Banach spaces. A mapping $\mathcal{X}$: **Ban** $\to$ **Ban** will be called a pseudolattice if

(i) for each $B \in$ **Ban** the space $\mathcal{X}(B)$ consists of B-valued sequences $\{b_n\}_{n\in\mathbb{Z}}$ and if
(ii) whenever A is a closed subspace of B it follows that $\mathcal{X}(A)$ is a closed subspace of $\mathcal{X}(B)$ and if
(iii) there exists a positive constant $C > 0$ such that, for all $A, B \in$ **Ban**, and all bounded linear operators $T : A \to B$ and every sequence $\{a_n\}_{n\in\mathbb{Z}} \in \mathcal{X}(A)$, the sequence $\{T(a_n)\}_{n\in\mathbb{Z}} \in \mathcal{X}(B)$ satisfies the estimate

$$\|\{T(a_n)\}_{n\in\mathbb{Z}}\|_{\mathcal{X}(B)} \le C(\mathcal{X})\|T\|_{A\to B}\|\{a_n\}_{n\in\mathbb{Z}}\|_{\mathcal{X}(A)}.$$

Fix a pair of pseudolattices $\mathbf{X} = \{\mathcal{X}_0, \mathcal{X}_1\}$. Given a compatible pair of Banach spaces $B = (B_0, B_1, \Sigma)$ —i.e., a pair of spaces B_0, B_1 considered as subspaces of another Banach space Σ— we define $\mathcal{J}(\mathbf{X}, B)$ to be the space of all $(B_0 \cap B_1)$-valued sequences $\{b_n\}_{n\in\mathbb{Z}}$ for which the sequence $\{e^{jn}b_n\}_{n\in\mathbb{Z}} \in \mathcal{X}_j(B_j)$ for $j = 0, 1$. This space is normed by

$$\|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathcal{J}(\mathbf{X},B)} = \max_{j=0,1} \|\{e^{jn}b_n\}\|_{\mathcal{X}_j(B_j)}.$$

The pseudolattice pair $\mathbf{X}$ is *nontrivial* if, for the special one-dimensional Banach pair $(\mathbb{C}, \mathbb{C})$ and each $z \in \mathbb{A} = \{z \in \mathbb{C} : 1 < |z| < e\}$ (the open annulus) there exists $\{b_n\}_{n\in\mathbb{Z}} \in \mathcal{J}(\mathbf{X}, (\mathbb{C}, \mathbb{C}))$ such that the series $\sum_{n\in\mathbb{Z}} z^n b_n$ converges to a nonzero number. The pseudolattice pair $\mathbf{X}$ is *Laureant compatible* if it is nontrivial and for every $z \in \mathbb{A}$ the Laureant series $\sum_{n\in\mathbb{Z}} z^n b_n$ converges absolutely with respect to the norm of $B_0 + B_1$. Therefore the sum of this series is an analytic function of z in $\mathbb{A}$ and can be differentiated term-by-term. The series for its derivative $\sum_{n\in\mathbb{Z}} nz^{n-1} b_n$ also converges absolutely in $B_0 + B_1$. See both claims at [15, page 248].

Given a compatible pair $B = (B_0, B_1)$ of Banach spaces and $0 < \theta < 1$, we define the *interpolation space* $B_{\mathbf{X},\theta}$ to consist of all elements of the form $b = \sum_{n\in\mathbb{Z}} e^{\theta n} b_n$ with $\{b_n\}_{n\in\mathbb{Z}} \in \mathcal{J}(\mathbf{X}, B)$, endowed with the natural quotient norm:

$$\|b\|_{B_{\mathbf{X},\theta}} = \inf \left\{ \|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathcal{J}(\mathbf{X},B)} : b = \sum_{n\in\mathbb{Z}} e^{\theta n} b_n \right\}.$$

According to the previous claims one may "think" every element $\{b_n\}_{n\in\mathbb{Z}} \in \mathcal{J}(\mathbf{X}, B)$ as the analytic map $\sum_{n\in\mathbb{Z}} z^n b_n$, where $z \in \mathbb{A}$, with all the precautions. This is, we shall informally write $\{b_n\}_{n\in\mathbb{Z}} = \sum_{n\in\mathbb{Z}} z^n b_n$. Therefore, we have the natural *evaluation map* $\delta_\theta : \mathcal{J}(\mathbf{X}, B) \longrightarrow B_{\mathbf{X},\theta}$ given by the rule $\delta_\theta(\{b_n\}_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} e^{\theta n} b_n$.

Intermission: The module structure. Banach spaces Z with a 1-unconditional basis admit an obvious structure of ℓ_∞-module given by the product $\ell_\infty \times Z \to Z$ in which $(\xi x)(n) = \xi(n)x(n)$. We are especially interested in the situation in which one deals with pairs $B = (B_0, B_1)$ of spaces with a joint 1-unconditional basis. In such case, we say a pseudolattice pair $\mathbf{X}$ admits an ℓ_∞-module structure if there is $C > 0$ such that for every $\{b_n\}_{n\in\mathbb{Z}} \in \mathcal{J}(\mathbf{X}, B)$ and $a \in \ell_\infty$, the following holds:

(1) $\{a \cdot b_n\}_{n\in\mathbb{Z}} \in \mathcal{J}(\mathbf{X}, B)$
(2) $\|\{a \cdot b_n\}_{n\in\mathbb{Z}}\|_{\mathcal{J}(\mathbf{X},B)} \le C\|a\|_\infty \|\{b_n\}_{n\in\mathbb{Z}}\|_{\mathcal{J}(\mathbf{X},B)}$

The natural example is $\mathbf{X} = \{\ell_{p_0}, \ell_{p_1}\}$ because one trivially has $\|a \cdot b_n\|_{B_j} \le \|a\|_\infty \|b_n\|_{B_j}, \quad j = 0, 1$; and thus, $\|\{a \cdot e^{jn} b_n\}_{n\in\mathbb{Z}}\|_{\ell_{p_j}(B_j)} \le \|a\|_\infty \|\{e^{jn} b_n\}_{n\in\mathbb{Z}}\|_{\ell_{p_j}(B_j)}$ for $j = 0, 1$. **End of the intermission**

Given $C \ge 1$ a C-extremal for a given $b \in B_{\mathbf{X},\theta}$ is a sequence $\{b_n\}_{n\in\mathbb{Z}}$ so that $\delta_\theta(\{b_n\}_{n\in\mathbb{Z}}) = b$ and $\|\{b_n\}_{n\in\mathbb{Z}}\| \le C\|b\|$. We write $S(b) = \{b_n\}_{n\in\mathbb{Z}}$ and say that S is C-*bounded selector* for the map δ_θ. We will work under the condition that the shift operator is an isometry on $\mathcal{X}_j(B_j)$ for $j = 0, 1$, see [15, Lemma 3.6.] (as it is the case of ℓ_p-spaces). This is enough to obtain a rather technical condition, $\mathbf{X}$ *admits differentiation*, [15, Lemma 3.11] which in practice yields that the associated *differential map* Ω can be obtained as the Σ-valued map

$$\Omega(b) = \delta'_\theta S(b) = \sum_{n\in\mathbb{Z}} n e^{\theta(n-1)} b_n. \tag{1}$$

Observe that Ω does not take (necessarily) values in $B_{\mathbf{X},\theta}$.

If $\mathbf{X}$ is a pseudolattice pair with ℓ_∞-module structure and $B = (B_0, B_1)$ is a pair of spaces with a joint 1-unconditional basis, the corresponding differential map Ω for $B_{\mathbf{X},\theta}$ is a *centralizer* in the sense of Kalton [18], which means that for all $\xi \in \ell_\infty$ and $x \in B_{\mathbf{X},\theta}$ and for some constant $C > 0$ one has $\Omega(\xi x) - \xi\Omega(x) \in B_{\mathbf{X},\theta}$ and $\|\Omega(\xi x) - \xi\Omega(x)\| \le C\|\xi\|_\infty \|x\|$. This follows easily from the fact that the pseudolattice $\{\ell_{p_0}, \ell_{p_1}\}$ has an ℓ_∞-module structure.

Let as before B be a compatible couple with ambient space Σ. The derived space $dB_{\mathbf{X},\theta}$ is the set of couples $(x, y) \in \Sigma \times B_{\mathbf{X},\theta}$ for which the following quasi-norm

$$\|x - \Omega(y)\|_{B_{\mathbf{X},\theta}} + \|y\|_{B_{\mathbf{X},\theta}}$$

makes sense and is finite, where Ω is the differential (1). This yields the exact sequence

$$0 \longrightarrow B_{\mathbf{X},\theta} \longrightarrow dB_{\mathbf{X},\theta} \longrightarrow B_{\mathbf{X},\theta} \longrightarrow 0$$

in which the isometric inclusion is $x \to (x, 0)$ and the "isometric" quotient map is $(\sigma, x) \to x$. If $B_{\mathbf{X},\theta}$ contains no (uniform) copies of ℓ_1^n for every $n \in \mathbb{N}$ then the quasi-norm of $dB_{\mathbf{X},\theta}$ is equivalent to a norm is equivalent to a norm by a result of Kalton [17] and $dB_{\mathbf{X},\theta}$ is thus a Banach space. In the case we will consider $B_{\mathbf{X},\theta} = \ell_2$ so the condition is satisfied.

The CKMR method and the real interpolation method. Even the meaning of "the real interpolation method" is somewhat ambiguous since there are many *real* methods. This is not a problem since one usually uses the $K-$ and $J-$ methods, and these are equivalent [1, Chapter 3, 3.3]. In this paper however we will need to rely on the original "espaces de moyennes" real method of Lions and Peetre [20] out of which the $K-$ and $J-$ methods stem. This method involves a discrete and a continuous version and four parameters (ξ_0, ξ_1, p_0, p_1), that we will set at $p_0 = 1, p_1 = \infty, \xi_0 = 1, \xi_1 = -1$ for the pair (ℓ_1, ℓ_∞) and that when adequately fixed produce and equivalent method to the standard real interpolation [1, Chapter 3,3.12]. Regarding the equivalence with the CKMR method, the pseudolattice corresponding to the real method correspond to the choice $\mathcal{X}_j = \ell_{p_j}$ and admits differentiation since the shift operator is clearly an isometry on $\ell_{p_j}(B_j)$ for $j = 0, 1$, see [15, Lemma 3.6]. The equivalence with the CKRM method in practice means to multiply by the weight $e^{-n\theta}$. See the discussion in [15, Paragraphs 2 and 4, page 251] and keep this observation in mind for later. The advantage of using the CKRM method is the explicit existence of differentials with the manageably simple formula (1), something that is much harder to obtain from the descriptions in either [14] or [6].

If we denote by $\ell_2(\mathbb{R})$ the real infinite dimensional separable Hilbert space and by ℓ_2 the complex Hilbert space, one has:

- As it is well known [19,8] the complex method applied to the pair (ℓ_∞, ℓ_1) produces the interpolation space $(\ell_\infty, \ell_1)_{1/2} = \ell_2$ and the celebrated Kalton-Peck space Z_2 as derived space with associated differential
$$\mathsf{KP}(x) = 2x \log \frac{|x(n)|}{\|x\|}.$$
- The real interpolation method applied to the pair (ℓ_∞, ℓ_1) produces the interpolation space $(\ell_\infty, \ell_1)_{1/2,1/2} = \ell_2(\mathbb{R})$ with derived space Z_2^{real}, that will be referred to as *the real* Kalton-Peck space, and associated differential KP^{real} to be calculated in the next section.

2. The real Kalton-Peck space

What we will do is to approach the real method as a CKMR method.

Proposition 1. *(Lions-Peetre) The differential associated to $(\ell_{p_0}, \ell_{p_1})_{\theta,p} = \ell_p$ is*

$$\mathsf{KP}^{real}(a) = e^{-\theta} \sum_m -\left(\frac{p}{p_0} - \frac{p}{p_1}\right) \left[\log \frac{|a(m)|}{\|a\|}\right] a(m) e_m, \tag{2}$$

for $a \in \ell_p$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $0 < \theta < 1$ and $1 \leq p_0, p_1 < \infty$. Here $[\,\cdot\,]$ means "the entire part of".

Proof. That $\ell_p = (\ell_{p_0}, \ell_{p_1})_{\theta,p}$ is established in [20, Theorem (I.I), Chapitre VII]. With the same language and notation of the paper, observe that the starting point for the proof of this is that the space of moyennes is $S(p_0, \xi_0, \mathbb{R}; p_1, \xi_1, \mathbb{R}) = \mathbb{R}$ with the decomposition $a = (\dots, 0, a, 0, \dots)$, namely $a(m) = \sum_{-\infty}^{+\infty} w_n(m)$ with $w_0(m) = a(m)$ and $w_n(m) = 0$ otherwise.

Let us work with the CKMR method, fix $C > 1$ and obtain a C-extremal; namely, given $a = (a(m))_m$ we look for $\{v_n\}_{n\in\mathbb{Z}}$ such that

(a) $\delta_\theta(\{b_n\}_{n\in\mathbb{Z}}) = a$.
(b) $\|\{b_n\}_{n\in\mathbb{Z}}\| \leq C\|a\|_{\ell_p}$.

The proof of [20, Theorem (I.I), Chapitre VII] contains the idea to obtain the C-extremals for the moyennes method: set $\lambda = \frac{p}{p_0} - \frac{p}{p_1}$ so that

$$\begin{cases} p_0(1+\lambda\theta) = p, \\ p_1(1-\lambda(1-\theta)) = p, \end{cases}$$

$\|a\| = 1$. Define $\{v_n\}_{n\in\mathbb{Z}}$ by $v_n(m) = w_{n+[\lambda \log|a(m)|]}(m)$, which yields $v_n(m) = w_0(m) = a(m)$ when $n = -[\lambda \log|a(m)|]$ and 0 otherwise. The translation to the CKRM method is multiplication by the weight $e^{-n\theta}$ as mentioned above mentioned and thus we get:

$$b_n(m) = \begin{cases} e^{-n\theta} a(m), & n = -[\lambda \log|a(m)|] \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

It is clear that (a) holds. We check (b):

$$\begin{aligned} \|\{b_n\}_{n\in\mathbb{Z}}\|^{p_0}_{\ell_{p_0}(\ell_{p_0})} &= \sum_n \|b_n\|^{p_0}_{\ell_{p_0}} \\ &= \sum_{n=-[\lambda \log|a(m)|]} \left|e^{-n\theta} a(m)\right|^{p_0} \\ &\leq \sum_m e^{-\theta(-\lambda \log|a(m)|)p_0} |a(m)|^{p_0} \\ &= \sum_m |a(m)|^{p_0+\lambda\theta p_0} \\ &= \sum_m |a(m)|^p = 1. \end{aligned}$$

Analogously, $\|\{e^n b_n\}_{n\in\mathbb{Z}}\|_{\ell_{p_1}(\ell_{p_1})} \leq 1$ and thus this element is a C-extremal. Now, the corresponding differential will be, according to (1),

$$\begin{aligned} \mathsf{KP}^{real}(a) &= \sum_{n\in\mathbb{Z}} n e^{\theta(n-1)} b_n \\ &= \sum_{n\in\mathbb{Z}} n e^{\theta(n-1)} e^{-n\theta} a(m) e_m \\ &= e^{-\theta} \sum_{n\in\mathbb{Z}} a(m) e_m \\ &= e^{-\theta} \sum_{n\in\mathbb{Z}} -[\,\lambda \log|a(m)|\,] a(m) e_m \end{aligned}$$

$$= e^{-\theta} \sum_m - \left(\frac{p}{p_0} - \frac{p}{p_1} \right) [\, \log |a(m)| \,]\, a(m) e_m$$

for $\|a\| = 1$. Homogeneity yields (2). $\square$

We prove now the result in the title:

Theorem 1. *The space Z_2 is the complexification of Z_2^{real}.*

Proof. We show first that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ell_2 \arrow[r] & Z_2 \arrow[r] & \ell_2 \arrow[r] & 0 \\
0 \arrow[r] & \ell_2(\mathbb{R}) \arrow[r] \arrow[u, "e^\theta \imath"] & Z_2^{real} \arrow[r] \arrow[u, dashed] & \ell_2(\mathbb{R}) \arrow[r] \arrow[u, "\imath"'] & 0
\end{tikzcd}$$

in which the vertical arrow $\imath$ is plain inclusion of the real $\ell_2(\mathbb{R})$ into the complex ℓ_2. Indeed

$$\begin{aligned}
\left\| \left(\mathsf{KP}\imath - e^\theta \imath \mathsf{KP}^{real} \right) a \right\|_2 &= \left\| 2 \sum_m \log \frac{|a(m)|}{\|a\|} a(m) e_m - 2 e^\theta e^{-\theta} \sum_m \left[\log \frac{|a(m)|}{\|a\|} \right] a(m) e_m \right\|_2 \\
&= \left\| 2 \sum_m \left(\log \frac{|a(m)|}{\|a\|} - \left[\log \frac{|a(m)|}{\|a\|} \right] \right) a(m) e_m \right\|_2 \\
&\le 2 \left\| \sum_m \frac{1}{2} a(m) e_m \right\|_2 = \|a\|_2.
\end{aligned}$$

We have thus shown that $\mathsf{KP}\imath - e^\theta \imath \mathsf{KP}^{real} : \ell_2(\mathbb{R}) \longrightarrow \ell_2(\mathbb{R})$ is a bounded map, which yields the diagram above. The vertical middle dotted arrow exists by the nature of the diagram.

We now invoke the results in [7, Section 4] where it is shown that the complexification of the real twisted Hilbert space generated by the map $\mathsf{KP}^r(x) = x \log \frac{|x|}{\|x\|}$ on $\ell_2(\mathbb{R})$ is the Kalton-Peck Z_2 space (generated by KP) (and the same occurs after multiplying KP by any nonzero scalar). Since we have shown that $\mathsf{KP}_{|\ell_2(\mathbb{R}))} = \mathsf{KP}^r$ and KP^{real} are projectively equivalent, Z_2 is a complexification of Z_2^{real}. $\square$

We now translate three important properties from Z_2 to Z_2^{real} while adding a few original features to the proofs, something that could be interesting on its own. Recall that an exact sequence $0 \longrightarrow X \longrightarrow X_\Omega \longrightarrow X \longrightarrow 0$ with associated quasilinear map Ω (or the quasilinear map Ω) is called *singular* when the quotient map is a strictly singular operator. This occurs [12] if and only if the restriction of Ω to any infinite dimensional subspace is never the sum of a bounded plus a linear map [2].

Proposition 2.

(1) *Z_2^{real} is isomorphic to its dual.*
(2) *KP^{real} is singular.*
(3) *Z_2^{real} does not contain complemented copies of ℓ_2*

Proof. Assertion (1) is consequence of the Kalton-Peck inequality $|xy \log \frac{x}{y}| \le Cxy$ for positive $x, y > 0$ (see [19]) which means, as we will explain next, that the quasilinear map that defines the dual sequence $0 \longrightarrow \ell_2^* \longrightarrow Z_2^* \longrightarrow \ell_2^* \longrightarrow 0$ is $-\mathsf{KP}$. The same occurs to KP^r, hence to KP^{real}. The details can be found with different levels of depth in [19,3,11,13]. We follow [13]: Two quasilinear maps $\Omega : B \to A$ and $\Phi : A^* \to B^*$ are called bounded duals one of the other if there is $C > 0$ such that for every $b \in B, a^* \in A^*$ one has

$$|\langle \Omega b, a^* \rangle + \langle b, \Phi a^* \rangle| \le C\|b\|\|a^*\|.$$

If Ω generates the exact sequence $0 \to A \xrightarrow{\imath} A \oplus_\Omega B \xrightarrow{\pi} B \to 0$ with inclusion $\imath(a) = (a, 0)$ and quotient map $\pi(a, b) = b$; and Φ is a bounded dual of Ω then Φ generates the dual sequence $0 \to B^* \xrightarrow{\pi^*} (A \oplus_\Omega B)^* \xrightarrow{\imath^*} A^* \to 0$, with the meaning that there is an operator $D : B^* \oplus_\Phi A^* \longrightarrow (A \oplus_\Omega B)^*$ given by

$$\langle D(b^*, a^*), (a, b)\rangle = \langle b^*, b\rangle + \langle a^*, a\rangle$$

making the diagram (here $\imath'(b^*) = (b^*, 0)$ and $\pi'(b^*, a^*) = a^*$)

$$\begin{tikzcd}
0 \arrow[r] & B^* \arrow[r, "\imath'"] \arrow[d, equal] & B^* \oplus_\Phi A^* \arrow[r, "\pi'"] \arrow[d, "D"] & A^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B^* \arrow[r, "\pi^*"'] & (A \oplus_\Omega B)^* \arrow[r, "\imath^*"'] & A^* \arrow[r] & 0
\end{tikzcd}$$

commutative. Now, the Kalton-Peck inequality above shows that $-\mathsf{KP}$ is a bounded dual of KP.

To prove (2) it is enough to show that there no subspace $W \subset \ell_2$ so that the restriction $\mathsf{KP}^{real}_{|W}$ is bounded. And this occurs, by the transfer principle [12,5,2], if and only if there is no sequence (u_n) of consecutive blocks so that the restriction of KP^{real} to the subspace $W = [(u_n)_n]$ spanned by those blocks is bounded. We work with KP, which is slightly simpler, with an argument which is immediately valid for KP^r, hence for KP^{real}.

Normalize the elements u_n in c_0 so that $\|u_n\|_\infty = 1$. Let $|x|$ denote the size of a finite element x, i.e. the cardinal of its support, and set $u_n = \sum_{j\in F_n} \lambda_{n,j} e_j$ with $|F_n| = |u_n|$. The element $\sum^N u_n$ is such that $\|\sum^N u_n\|_\infty = 1$, $\|\sum^N u_n\|_1 = \sum^N \|u_n\|_1$ and $\|\sum^N u_n\|_2 = \left(\sum^N \|u_n\|_2^2\right)^{1/2}$. The holomorphic function

$$f(z) = \left(\sum_{n=1}^{n=N} \sum_{j\in F_n} |\lambda_{n,j}|^{2z}\right)^{(1/2)(2z-1)} \sum^N u_n$$

is obviously a selector for $\sum^N u_n$ since $f(1/2) = \sum^N u_n$. We estimate its norm. Since

$$f(1) = \left(\sum_{n=1}^{n=N} \sum_{j\in F_n} |\lambda_{n,j}|^2\right)^{1/2} \sum^N u_n = \left(\sum^N \|u_n\|_2^2\right)^{1/2} \sum^N u_n$$

and thus $\|f(1)\|_\infty = \left(\sum^N \|u_n\|_2^2\right)^{1/2} = \left\|\sum^N u_n\right\|_2$, the same occurs on points $1 + it$. On the other hand, since

$$f(0) = \left(\sum_{n=1}^{n=N} \sum_{j\in F_n} 1\right)^{-1/2} \sum^N u_n = \left(\sum_{n=1}^{n=N} |u_n|\right)^{-1/2} \sum^N u_n$$

and thus

$$
\begin{aligned}
\|f(0)\|_1 &= \frac{\left\|\sum^N u_n\right\|_1}{\left(\sum^N |u_n|\right)^{1/2}} = \frac{\sum^N \|u_n\|_1}{\left(\sum^N |u_n|\right)^{1/2}} \\
&= \frac{\sum^N \sum_{j\in F_n} |\lambda_{n,j}|}{\left(\sum^N |u_n|\right)^{1/2}} \leq \frac{\sum^N |u_n|^{1/2}\|u_n\|_2}{\left(\sum^N |u_n|\right)^{1/2}} \\
&\leq \frac{\left(\sum^N |u_n|\right)^{1/2}\left(\sum^N \|u_n\|_2^2\right)^{1/2}}{\left(\sum^N |u_n|\right)^{1/2}} \\
&= \left(\sum^N \|u_n\|_2^2\right)^{1/2} = \left\|\sum^N u_n\right\|_2
\end{aligned}
$$

the same occurs on points e^{it}. This yields that $\Omega\left(\sum^N u_n\right) = f'(1/2)$ is an acceptable differential. Now, if

$$
g(z) = \left(\sum_{n=1}^{n=N}\sum_{j\in F_n} |\lambda_{n,j}|^{2z}\right)^{(1/2)(2z-1)} = \left(\sum_{n,j} |\lambda_{n,j}|^{2z}\right)^{z-1/2)}
$$

then $\log g(z) = (z-\frac{1}{2})\log\left(\sum_{n,j}|\lambda_{n,j}|^{2z}\right)$ and thus

$$
\frac{g'(z)}{g(z)} = (z-\frac{1}{2})\frac{\sum_{n,j} 2|\lambda_{n,j}|^{2z}\log|\lambda_{n,j}}{\sum_{n,j}|\lambda_{n,j}|^{2z}} + \log\left(\sum_{n,j}|\lambda_{n,j}|^{2z}\right)
$$

which yields

$$
\frac{g'(1/2)}{g(z)} = \log\left(\sum_{n,j}|\lambda_{n,j}|\right) = \log\sum_n \|u_n\|_1
$$

Therefore

$$
f'(1/2) = \log\sum_n \|u_n\|_1 \left(\sum_n u_n\right).
$$

This implies that $\Omega_{|W}$ is unbounded. The same occurs to KP^r and KP^{real}. Consequently, KP^{real} is singular. To prove (3) we make a detour.

Lemma 1. *If* $0 \longrightarrow \ell_2 \xrightarrow{\jmath} Z \xrightarrow{\rho} \ell_2 \longrightarrow 0$ *is an exact sequence with* ρ *strictly singular and* $\jmath$ *strictly cosingular then* Z *does not contain complemented copies of* ℓ_2

Proof. Assume then that Z contains a complemented copy B of ℓ_2 and let $P: Z \to B$ be a projection. Either $\ker\rho_{|B}$ is finite or infinite dimensional. If it is finite dimensional, ρ is an isomorphism on some infinite dimensional subspace of Z, which is impossible. If it is infinite dimensional, let $P': B \to \ker\rho_{|B}$ be a continuous linear projection, that exists because B is Hilbert. Thus $P'P: Z \to \ker\rho_{|B}$ is a projection, as well as $P'P_{|\ker\rho}$, which is once more impossible since $\jmath$ is cosingular $\square$

Therefore, if $0 \longrightarrow \ell_2 \xrightarrow{\jmath} Z \xrightarrow{\rho} \ell_2 \longrightarrow 0$ is a singular exact sequence for which there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ell_2 \arrow[r, "\jmath"] \arrow[d, "\alpha"'] & Z \arrow[r, "\rho"] & \ell_2 \arrow[r] & 0 \\
0 \arrow[r] & \ell_2 \arrow[r, "\rho^*"'] & Z^* \arrow[r, "\jmath^*"'] & \ell_2 \arrow[r] \arrow[u, "\gamma"'] & 0
\end{tikzcd}$$

in which α, γ are isomorphisms then Z cannot contain complemented copies of ℓ_2 because an operator T such that T^* is strictly singular must be strictly cosingular. Thus, the sequence $0 \longrightarrow \ell_2(\mathbb{R}) \longrightarrow Z_2^{real} \longrightarrow \ell_2(\mathbb{R}) \longrightarrow 0$ is singular and cosingular, so the previous lemma applies. $\square$

3. Concluding remarks

Having singular differential is a rather demanding condition. For instance:

Proposition 3. *Let (X, X^*) be a real or complex interpolation pair of Banach spaces with a common unconditional basis for which there exists a continuous inclusion $X^* \to X$ and such that $(X^*, X)_{1/2} = \ell_2$ with differential Ω. If Ω is singular then X^* does not contain ℓ_2.*

Proof. IN what follows, $\sim$ means "proportional". Let (u_n) be blocks in X^* so that $[(u_n)] \simeq \ell_2$. Pick $u \in [(u_n)]$ and observe that

$$\begin{aligned}
\|u\|_X &= \sup\{< y, u >: \|y\| \leq 1;\ y \in X^*\} \\
&\geq \sup\{< y, u >: \|y\| \leq 1:\ y \in [(u_n)]\} \\
&= \|u\|_{[(u_n)]^*} \sim \|u\|_2.
\end{aligned}$$

Since $\|u\|_X \leq \|u\|_2$ it turns out that $\|u\|_{X^*} \sim \|u\|_2$. Thus, the norms of X and X^* are equivalent on $[(u_n)]$, and this obliges $\Omega_{|[(u_n)]}$ to be bounded. $\square$

Optimistic readers could now easily believe the following conjecture:

Conjecture. *Let (X, X^*) be an interpolation pair of Banach spaces with a common unconditional basis for which there exists a continuous inclusion $X^* \to X$ and such that $(X^*, X)_{1/2} = \ell_2$ with differential Ω. If Ω is singular then X and X^* are incomparable.*

We do not have a proof for that. Optimistic readers should be warned: an example in [3] provides a space X incomparable with X^*, so that none of them contains ℓ_2 but such that the complex differential at $1/2$ is an isomorphism on a complemented copy of ℓ_2. Unfortunately, neither X nor X^* have unconditional basis and $(X, X^*)_{1/2}$ is not ℓ_2.

References

[1] J. Bergh, J. Löfström, Interpolation Spaces. An Introduction, Grundlehren der Mathematischen Wissenschaften, vol. 223, Springer-Verlag, Berlin-New York, 1976.

[2] F. Cabello, J.M.F. Castillo, Homological Methods in Banach Space Theory, Cambridge Studies in Advanced Math., vol. 203, 2022, Online ISBN. 9781108778312.

[3] F. Cabello, J.M.F. Castillo, W.H.G. Correa, Higher order derivatives of analytic families of Banach spaces, arXiv:1906.06677v2.
[4] F. Cabello Sánchez, J.M.F. Castillo, N.J. Kalton, Complex interpolation and twisted twisted Hilbert spaces, Pac. J. Math. 276 (2015) 287–307.
[5] F. Cabello, J.M.F. Castillo, J. Suárez, On strictly singular exact sequences, Nonlinear Anal. 75 (2012) 2313–2321.
[6] M.J. Carro, J. Cerdà, F. Soria, Commutators and interpolation methods, Ark. Mat. 33 (1995) 199–216.
[7] J.M.F. Castillo, V. Ferenczi, Y. Moreno, W. Cuellar, Complex structures on twisted Hilbert spaces, Isr. J. Math. 222 (2017) 787–814.
[8] J.M.F. Castillo, V. Ferenczi, M. González, Singular exact sequences generated by complex interpolation, Trans. Am. Math. Soc. 369 (2017) 4671–4708.
[9] J.M.F. Castillo, W.H.G. Correa, V. Ferenczi, M. González, On the stability of the differential process generated by complex interpolation, J. Inst. Math. Jussieu 21 (2022) 303–334.
[10] J.M.F. Castillo, W.H.G. Correa, V. Ferenczi, M. González, Differential processes generated by two interpolators, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 114 (2020) 183, https://doi.org/10.1007/s13398-020-00920-5.
[11] J.M.F. Castillo, H.W.G. Correa, V. Ferenczi, M. González, Interpolator symmetries and new Kalton-Peck spaces, arXiv:2111.10640.
[12] J.M.F. Castillo, Y. Moreno, Strictly singular quasi-linear maps, Nonlinear Anal. 49 (2002) 897–904.
[13] J.M.F. Castillo, Y. Moreno, Twisted dualities in Banach space theory, in: Banach Spaces and Their Applications in Analysis, in: Walter de Gruyter Proceedings in Maths., vol. 59-76, 2007.
[14] M. Cwikel, B. Jawerth, M. Milman, R. Rochberg, Differential estimates and commutators in interpolation theory, in: E.R. Berkson, N.T. Peck, J. Uhl (Eds.), Analysis at Urbana II, in: London Math. Soc. Lecture Note Series, vol. 138, Cambridge Univ. Press, Cambridge, 1989, pp. 170–220.
[15] M. Cwikel, N. Kalton Nigel, M. Milman, R. Rochberg, A unified theory of commutator estimates for a class of interpolation methods, Adv. Math. 169 (2) (2002) 241–312.
[16] E. Hilton, K. Stammbach, A Course in Homological Algebra, GTM, vol. 4, Springer-Verlag, 1971.
[17] N.J. Kalton, The three-space problem for locally bounded F-spaces, Compos. Math. 37 (1978) 243–276.
[18] N.J. Kalton, Differentials of complex interpolation processes for Köthe function spaces, Trans. Am. Math. Soc. 333 (2) (1992) 479–529.
[19] N.J. Kalton, N.T. Peck, Twisted sums of sequence spaces and the three-space problem, Trans. Am. Math. Soc. 255 (1979) 1–30.
[20] J.L. Lions, J. Peetre, Sur une classe d'espaces d'interpolation, Publ. Math. IHÉS 19 (1964) 5–68.
[21] R. Rochberg, Higher order estimates in complex interpolation theory, Pac. J. Math. 174 (1) (1996) 247–267.