EXTRACTA MATHEMATICAE
Vol. **38**, Num. 2 (2023), 205 – 219

doi:10.17398/2605-5686.38.2.205
Available online September 25, 2023

Continua whose hyperspace of subcontinua is infinite dimensional and a cone

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Received December 30, 2022
Accepted July 20, 2023

Presented by A. Avilés

Abstract: We determine several classes of continua whose hyperspaces of subcontinua are infinite dimensional and homeomorphic to cones over (usually) other continuum. In particular, we obtain many Peano continua with such a property.

Key words: Cone, continuum, convex metric, Hilbert cube, Hilbert cube manifold, n-fold hyperspace, Peano continuum, smooth fan.

MSC (2020): 54B20, 54F15.

1. Introduction

In [19, Theorem 4.3] we determine exactly which fans have hyperspaces that are cones; namely, it is proven that for a fan F, the hyperspace of subcontinua of F is a cone if and only if F itself is a cone. In [15], it is shown that if a fan is a cone, then then each of its hyperspaces is homeomorphic to the cone over a compactum. We continue our study of classes of continua X such that the hyperspace of subcontinua of X is a cone. Our main results are Theorem 3.4, Theorem 3.8 and Theorem 3.10. We obtain many Peano continua with such a property which, in general, are different from the ones considered in [12].

Observe that the case when the n-fold hyperspaces ($n \geq 1$) of a continuum X is of finite dimension has essentially been done in [11, 12, 13, 14, 15, 16, 17, 18, 19]. Hence, we consider continua X whose hyperspace of subcontinua has infinite dimension.

This reseach started in the first half of 2001, while the first named author spent a sabbatical year at West Virginia University (2000-2001) with Professor Sam B. Nadler, Jr. After I left West Virginia, we worked on the paper via e-mail. After some time my computer broke and I lost the LaTeX file of

ISSN: 0213-8743 (print), 2605-5686 (online)

the manuscript and I only had a pdf file. While trying, for a very long time, to recover the source file, Professor Nadler's health deteriorated and, unfortunately, he passed away on 4 February, 2016. Once I recovered the LaTeX file, I decided to update it and to publish it as dedication to his memory and in gratitude to all the things he taught me over the years that I had the pleasure to work with him.

2. Notation and terminology

We denote the unit interval $[0, 1]$ by I. We denote the Hilbert cube by I^∞.

A *Hilbert cube manifold* is a separable metric space M such that each point of M has a closed neighborhood homeomorphic to I^∞.

A *continuum* is a nonempty compact connected metric space. A *Peano continuum* is a continuum which is locally connected.

The term *map* means a continuous function. We use the double headed arrow in $f: X \twoheadrightarrow Y$ to signify that the map f is surjective; i.e., $f(X) = Y$.

A closed subset A of a continuum X is a *Z-set* provided that for every $\varepsilon > 0$, there exists a map $f_\varepsilon: X \to X \setminus A$ such that $d(x, f(x)) < \varepsilon$ for all points x of X.

The *hyperspace of closed subsets of a continuum* X is the space 2^X of all nonempty closed subsets of X with the Hausdorff metric [22, Theorem (0.2)]; $\mathcal{H}$ always denotes the Hausdorff metric.

Notation 2.1. Let X be a continuum and let $k \geq 2$ be an integer. Then there exists a *union map*

$$u_k: \underbrace{2^X \times \cdots \times 2^X}_{k \text{ times}} \twoheadrightarrow 2^X$$

given by $u_k((A_1, \ldots, A_k)) = A_1 \cup \cdots \cup A_k$. It is easy to see that u_k is continuous for each $k \geq 2$ [22, Lemma (1.48)].

Given a positive integer n, the *n-fold hyperspace of a continuum* X is the space:

$$\mathcal{C}_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\},$$

a lot of information of this hyperspace may be found in [17]. Note that $\mathcal{C}_1(X)$ is the *hyperspace of subcontinua of* X.

Given a continuum X and a closed subset A of X, let

$$\mathcal{C}_1(A, X) = \{K \in \mathcal{C}_1(X) : A \subset K\}.$$

If $A = \{p\}$, we write $\mathcal{C}_1(p, X)$ instead of $\mathcal{C}_1(\{p\}, X)$.

Given a map $f\colon X \to Y$ between continua and a positive integer n, we define the *induced map* $\mathcal{C}_n(f)\colon \mathcal{C}_n(X) \to \mathcal{C}_n(Y)$ by $\mathcal{C}_n(f)(A) = f(A)$. It is known that $\mathcal{C}_n(f)$ is continuous [17, Corollary 8.2.3].

LEMMA 2.2. *Let X be a continuum and let A be a nonempty closed subset of X. If A is a Z-set in X, then $\mathcal{C}_1(A, X)$ is a Z-set in $\mathcal{C}_1(X)$.*

Proof. Let $\varepsilon > 0$. Since A is a Z-set in X, there exists a map $f_\varepsilon\colon X \to X\setminus A$ such that $d(x, f_\varepsilon(x)) < \varepsilon$. Hence, $\mathcal{C}_1(f_\varepsilon)\colon \mathcal{C}_1(X) \to \mathcal{C}_1(X)$ is a map. Also, since $f_\varepsilon(X) \cap A = \emptyset$, we have that $\mathcal{C}_1(f_\varepsilon)(\mathcal{C}_1(X)) \cap \mathcal{C}_1(A, X) = \emptyset$. Now, by [17, Lemma 8.2.13], we have that $\mathcal{H}(B, \mathcal{C}_1(f_\varepsilon)(B)) < \varepsilon$, for all B in $\mathcal{C}_1(X)$. Therefore, $\mathcal{C}_1(A, X)$ is a Z-set in $\mathcal{C}_1(X)$. ∎

Remark 2.3. It is well known that the manifold boundary, D, of an n-cell I^n is a Z-set of I^n. Hence, $\mathcal{C}_1(D, I^n)$ is a Z-set in $\mathcal{C}_1(I^n)$ (Lemma 2.2). Also, for each point $x \in D$, we have that $\mathcal{C}_1(x, I^n)$ is a Z-set of $\mathcal{C}_1(I^n)$. Another example is Sierpiński universal plane curve [17, pp. 230 − 231] $\mathbb{S}$. By [4, Example 3.10], one may prove that the boundary of the unit square, D (biggest square), is a Z-set of $\mathbb{S}$. Thus, $\mathcal{C}_1(D, \mathbb{S})$ is a Z-set of $\mathcal{C}_1(\mathbb{S})$. For this continuum, we also have that if $x \in D$, then $\mathcal{C}_1(x, \mathbb{S})$ is a Z-set of $\mathcal{C}_1(\mathbb{S})$. Observe that these two continua provide examples of continua B that satisfy the hypothesis of Theorem 3.4.

A *fan* is an arcwise connected hereditarily unicoherent continuum with only one ramification point; i.e., a point that is the only endpoint in common of at least three otherwise disjoint arcs. The unique ramification point of a fan F is called the *top of F*. The set of endpoints, in the classical sense, of a fan F is denoted by $E(F)$. Given two points x and y of a fan F, $\overline{xy}$ denotes the unique arc in F whose endpoints are x and y. For a fan F with top τ,

$$\mathcal{N}[\mathcal{C}_1(F)] = \{\overline{\tau x} \,:\, x \in F\};$$

$\mathcal{N}[\mathcal{C}_1(F)]$ is called *the natural part of* $\mathcal{C}_1(F)$. Also,

$$\mathcal{T}[\mathcal{C}_1(F)] = \bigcup_{e \in E(F)} \mathcal{C}_1(\overline{\tau e});$$

we call $\mathcal{T}[\mathcal{C}_1(F)]$ *the two-dimensional part of* $\mathcal{C}_1(F)$ (which is justified by [19, Lemma 3.1]).

A fan F is said to be *smooth* provided that whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in F converging to a point x of F, then $\{\overline{\tau x_n}\}_{n=1}^{\infty}$ converges to $\overline{\tau x}$, with respect to the Hausdorff metric. Thus, a fan is smooth if and only if the natural map $\varphi\colon F \twoheadrightarrow \mathcal{N}[\mathcal{C}_1(F)]$, given by $\varphi(x) = \overline{\tau x}$, is continuous and, in fact, a homeomorphism.

The *cone over a continuum* Y, denoted by $\mathrm{Cone}(Y)$, is the quotient space $(Y \times I)/(Y \times \{1\})$ obtained from the cartesian product $Y \times I$ by shrinking $Y \times \{1\}$ to a point v called the *vertex of the cone* [23, 3.15]. The *base of* $\mathrm{Cone}(Y)$ is $\{(y,0) : y \in Y\}$, which we denote by $\mathcal{B}(Y)$. A *coning arc* is the image under the quotient map of an arc in $Y \times I$ of the form $\{y\} \times I$.

Notation 2.4. Recall that if X is a Peano continuum with a convex metric ρ, then the function $K_\rho : [0,\infty) \times \mathcal{C}_1(X) \to \mathcal{C}_1(X)$ given by

$$K_\rho(t, A) = \{x \in X \,:\, \rho(x,a) \leq t \text{ for some } a \in A\}$$

is continuous [21, Corollary (3.4)]. We assume that if X is a Peano continuum, then the metric ρ on X is convex ([1] and [20]).

3. Continua whose one-fold hyperspace is a cone

Our first main result is Theorem 3.4. We begin with the following lemmas.

Lemma 3.1. *Let X, Y and Z be nondegenerate continua such that $X \cap Y \cap Z \neq \emptyset$. Assume $X \cap Y$ consists of at most two points and that $X \cap Z = Y \cap Z = \{p\}$. Let $\mathcal{G}_1 = \{H \cup E : H \in \mathcal{C}_1(p,X)$ and $E \in \mathcal{C}_1(p,Y)\}$ and let $\mathcal{G}_2 = \{H \cup E \cup G : H \in \mathcal{C}_1(p,X), E \in \mathcal{C}_1(p,Y)$ and $G \in \mathcal{C}_1(p,Z)\}$. Then $\mathcal{G}_1$ is homeomorphic to $\mathcal{C}_1(p,X) \times \mathcal{C}_1(p,Y)$ and $\mathcal{G}_2$ is homeomorphic to $\mathcal{C}_1(p,X) \times \mathcal{C}_1(p,Y) \times \mathcal{C}_1(p,Z)$.*

Proof. We show that $\mathcal{G}_2$ is homeomorphic to $\mathcal{C}_1(p,X) \times \mathcal{C}_1(p,Y) \times \mathcal{C}_1(p,Z)$. The proof of the fact that $\mathcal{G}_1$ is homeomorphic to $\mathcal{C}_1(p,X) \times \mathcal{C}_1(p,Y)$ is done in a similar way. Let $\varphi\colon \mathcal{C}_1(p,X) \times \mathcal{C}_1(p,Y) \times \mathcal{C}_1(p,Z) \twoheadrightarrow \mathcal{G}_2$ be given by $\varphi((H,E,G)) = H \cup E \cup G$ (Notation 2.1). Then φ is a homeomorphism. ∎

Lemma 3.2. *Let X be a nondegenerate Peano continuum, and let Y be any continuum. If $X \cap Y = \{q\}$, then $\mathcal{C}_1(q,Y)$ is a Z-set in $\mathcal{C}_1(X) \cup \mathcal{C}_1(q, X \cup Y)$.*

Proof. Let ρ be a convex metric for X ([1] and [20]). Let $\varepsilon > 0$ be given. Define $f_\varepsilon\colon \mathcal{C}_1(X) \cup \mathcal{C}_1(q, X \cup Y) \to \mathcal{C}_1(X) \cup \mathcal{C}_1(q, X \cup Y)$ as follows: For all

$A \in \mathcal{C}_1(X) \cup \mathcal{C}_1(q, X \cup Y)$, let

$$f_\varepsilon(A) = K_\rho\left(\frac{\varepsilon}{2}, A \cap X\right) \cup (A \cap Y)$$

(Notation 2.4). Then f_ε is continuous, $f_\varepsilon(\mathcal{C}_1(q,Y)) \cap \mathcal{C}_1(q,Y) = \emptyset$ and

$$\mathcal{H}(A, f_\varepsilon(A)) < \varepsilon,$$

for each $A \in \mathcal{C}_1(X) \cup \mathcal{C}_1(q, X \cup Y)$. Therfore, since $\mathcal{C}_1(q,Y)$ is a closed subset of $\mathcal{C}_1(X) \cup \mathcal{C}_1(q, X \cup Y)$, we have that $\mathcal{C}_1(q,Y)$ is a Z-set in $\mathcal{C}_1(X) \cup \mathcal{C}_1(q, X \cup Y)$. ∎

COROLLARY 3.3. *Let X be a nondegenerate Peano continuum, and let F be a smooth fan with top τ. If $X \cap F = \{\tau\}$, then $\mathcal{N}[\mathcal{C}_1(F)]$ is a Z-set in $\mathcal{C}_1(X) \cup \mathcal{C}_1(\tau, X \cup F)$.*

Proof. Since F is a smooth fan, $\mathcal{N}[\mathcal{C}_1(F)]$ is homeomorphic to F [8, p. 282]. Hence, $\mathcal{N}[\mathcal{C}_1(F)]$ is a closed subset of $\mathcal{C}_1(X) \cup \mathcal{C}_1(\tau, X \cup F)$. Thus, since $\mathcal{N}[\mathcal{C}_1(F)] \subset \mathcal{C}_1(\tau, F)$ and $\mathcal{C}_1(\tau, F)$ is a Z-set in $\mathcal{C}_1(X) \cup \mathcal{C}_1(\tau, X \cup F)$ (Lemma 3.2), we obtain that $\mathcal{N}[\mathcal{C}_1(F)]$ is a Z-set in $\mathcal{C}_1(X) \cup \mathcal{C}_1(\tau, X \cup F)$. ∎

THEOREM 3.4. *Let B be a nondegenerate Peano continuum without free arcs, and let F be a fan with top τ such that F is a cone. Let $X = B \cup F$. Suppose $B \cap F = \{\tau\}$ and $\mathcal{C}_1(\tau, B)$ is a Z-set in $\mathcal{C}_1(B)$. Then $\mathcal{C}_1(X)$ is a cone.*

Proof. Since F is a cone, F is a smooth fan. Hence, $\mathcal{C}_1(F) = \mathcal{C}_1(\tau, F) \cup \mathcal{T}[\mathcal{C}_1(F)]$ and $\mathcal{C}_1(\tau, F) \cap \mathcal{T}[\mathcal{C}_1(F)] = \mathcal{N}[\mathcal{C}_1(F)]$ [8, Theorem 3.1]. Observe that

$$\begin{aligned}
\mathcal{C}_1(X) &= \mathcal{C}_1(B) \cup \mathcal{C}_1(\tau, X) \cup \mathcal{C}_1(F) \\
&= \mathcal{C}_1(B) \cup \mathcal{C}_1(\tau, X) \cup \mathcal{C}_1(\tau, F) \cup \mathcal{T}[\mathcal{C}_1(F)] \\
&= \mathcal{C}_1(B) \cup \mathcal{C}_1(\tau, X) \cup \mathcal{T}[\mathcal{C}_1(F)]
\end{aligned}$$

and

$$[\mathcal{C}_1(B) \cup \mathcal{C}_1(\tau, X)] \cap \mathcal{T}[\mathcal{C}_1(F)] = \mathcal{N}[\mathcal{C}_1(F)].$$

Since B is a Peano continuum without free arcs, $\mathcal{C}_1(B)$ is a Hilbert cube [6, Theorem 4.1]. Also, $\mathcal{C}_1(\tau, B)$ is a Hilbert cube [6, Theorem 5.2].

Since F is a smooth fan, $\mathcal{C}_1(\tau, F)$ is either an n-cell, for some positive integer n, or a Hilbert cube [8, Theorem 3.1 (3)].

Note that, by Lemma 3.1, $\mathcal{C}_1(\tau, X)$ is homeomorphic to $\mathcal{C}_1(\tau, B)\times\mathcal{C}_1(\tau, F)$. Since $\mathcal{C}_1(\tau, B)$ is a Hilbert cube and $\mathcal{C}_1(\tau, F)$ is either an n-cell or a Hilbert cube, $\mathcal{C}_1(\tau, B) \times \mathcal{C}_1(\tau, F)$ is a Hilbert cube. Therefore, $\mathcal{C}_1(\tau, X)$ is a Hilbert cube.

Since $\mathcal{C}_1(B)$, $\mathcal{C}_1(\tau, X)$ and $\mathcal{C}_1(\tau, B) = \mathcal{C}_1(B) \cap \mathcal{C}_1(\tau, X)$ are Hilbert cubes and since $\mathcal{C}_1(\tau, B)$ is a Z-set in $\mathcal{C}_1(B)$, by assumption, we have that $\mathcal{C}_1(B) \cup \mathcal{C}_1(\tau, X)$ is a Hilbert cube [9, Theorem 1].

Since F is fan that is a cone, F is the cone over $E(F)$ [19, Lemma 4.2]. Hence, $E(F)$ is a compact totally disconnected space. Therefore, $E(F)$ can be embedded in I, [10, Theorem V2, p. 56]

Let $g\colon E(F) \to I$ be an embedding of $E(F)$ in I. For each $e \in E(F)$, let L_e be the convex arc in I^∞ whose endpoints are $(0, 0, 0, \ldots)$ and $(1, g(e), 0, 0, \ldots)$. Let $L = \bigcup_{e\in E} L_e$. Then L is a Z-set in I^∞. Also, $\mathcal{N}[\mathcal{C}_1(F)]$ is a Z-set in $\mathcal{C}_1(B) \cup \mathcal{C}_1(\tau, X)$ (Lemma 3.2), and $\mathcal{N}[\mathcal{C}_1(F)]$ is homeomorphic to L (since F is a smooth fan). Hence, there exists a homeomorphism $\psi\colon \mathcal{C}_1(B)\cup\mathcal{C}_1(\tau, X) \twoheadrightarrow I^\infty$ such that $\psi(\mathcal{N}[\mathcal{C}_1(F)]) = L$ [3, Theorem 11.1].

For each $e \in E(F)$, let G_e be the convex arc in I^∞ whose endpoints are $(1, g(e), 0, 0, 0, \ldots)$ and $(1, g(e), 1, 0, 0, 0, \ldots)$.

For each $e \in E(F)$, let D_e be the convex hull in I^∞ of $L_e \cup G_e$. Let $\mathfrak{D} = \bigcup_{e\in E(F)} D_e$. It is easy to see that there exists a homeomorphism $\xi\colon \mathcal{T}[\mathcal{C}_1(F)] \twoheadrightarrow \mathfrak{D}$ such that $\xi(A) = \psi(A)$, for each $A \in \mathcal{N}[\mathcal{C}_1(F)]$.

Therefore, the map $\zeta\colon \mathcal{C}_1(X) \twoheadrightarrow I^\infty \cup \mathfrak{D}$ given by

$$\zeta(A) = \begin{cases} \psi(A), & \text{if } A \in \mathcal{C}_1(B) \cup \mathcal{C}_1(\tau, X), \\ \xi(A), & \text{if } A \in \mathcal{T}[\mathcal{C}_1(F)], \end{cases}$$

is a homeomorphism. Moreover, as is easy to see, $I^\infty \cup \mathfrak{D}$ is homeomorphic to the cone over $\left(\bigcup_{e\in E(F)} G_e\right) \cup \{(x_n)_{n=1}^\infty \in I^\infty : x_1 = 1\}$, with vertex $(0, 0, 0, \ldots)$. Therefore, $\mathcal{C}_1(X)$ is a cone. ∎

LEMMA 3.5. *Let Y and Z be continua and let A be an arc such that $Y \cap A = \{p, q\}$, where p and q are the endpoints of A, and $(Y \cup A) \cap Z = \{q\}$. If $\mathcal{G}_1 = \{K \cup R : K \in \mathcal{C}_1(q, Y)$ and $R \in \mathcal{C}_1(q, A)\}$, $\mathcal{G}_2 = \{K \cup R \cup G : K \in \mathcal{C}_1(q, Y), R \in \mathcal{C}_1(q, A)$ and $G \in \mathcal{C}_1(q, Z)\}$ and $\mathcal{E} = \{K \cup A : K \in \mathcal{C}_1(q, Y)\}$, then $\mathcal{E}$ is a Z-set in $\mathcal{G}_1$ and in $\mathcal{G}_2$.*

Proof. We prove that $\mathcal{E}$ is a Z-set in $\mathcal{G}_2$. The proof that $\mathcal{E}$ is a Z-set in $\mathcal{G}_1$ is similar.

Let $\varepsilon > 0$. Let $h\colon A \to A\setminus\{p\}$ be a map such that $h(q) = q$ and $d(z, h(z)) < \varepsilon$. Let $f_\varepsilon\colon \mathcal{G}_2 \to \mathcal{G}_2$ be given by $f_\varepsilon(K \cup R \cup G) = K \cup h(R) \cup G$. Then f_ε is continuous, $f_\varepsilon(\mathcal{E}) \cap \mathcal{E} = \emptyset$ and $\mathcal{H}(K \cup R \cup G, f_\varepsilon(K \cup R \cup G)) < \varepsilon$. Therefore, $\mathcal{E}$ is a Z-set in $\mathcal{G}_2$. ∎

Let X be a continuum and let C be a finite subset of X. We say that C is a *Z^*-set in X* provided that for each element c_0 of C and every $\varepsilon > 0$, there exists a map $f_\varepsilon\colon X \to X \setminus \{c_0\}$ such that $f_\varepsilon(c) = c$, for each $c \in C \setminus \{c_0\}$, and $d(x, f_\varepsilon(x)) < \varepsilon$, for all points x in X. Note that, by [3, Theorem 3.1 (3)], each Z^*-set in X is a Z-set in X. Also, observe that if F is a fan that is a cone and R is a finite subset of $\mathcal{B}(F)$, then R is a Z^*-set in F.

LEMMA 3.6. *Let B be a nondegenerate Peano continuum without free arcs. If $\{p, q, r\}$ is a Z^*-set in B, then $\mathcal{C}_1(p, B) \cup \mathcal{C}_1(q, B)$, $\mathcal{C}_1(p, B) \cup \mathcal{C}_1(q, B) \cup \mathcal{C}_1(r, B)$, and $\mathcal{C}_1(p, B) \cup \mathcal{C}_1(\{q, r\}, B)$ are Hilbert cubes.*

Proof. To show that $\mathcal{C}_1(p, B) \cup \mathcal{C}_1(q, B)$ is a Hilbert cube, note that $\mathcal{C}_1(p, B)$ and $\mathcal{C}_1(q, B)$ are Hilbert cubes [6, Theorem 5.2]. Also, observe that $\mathcal{C}_1(p, B) \cap \mathcal{C}_1(q, B) = \mathcal{C}_1(\{p, q\}, B)$, and $\mathcal{C}_1(\{p, q\}, B)$ is a Hilbert cube [5, Theorem 5.2]. Since $\{p, q\}$ is a Z^*-set in B, for each $\varepsilon > 0$ there exists a map $f_\varepsilon\colon B \to B \setminus \{q\}$ such that $f_\varepsilon(p) = p$ and $d(x, f_\varepsilon(x)) < \varepsilon$. Hence, the induced map $\mathcal{C}_1(f_\varepsilon)\colon \mathcal{C}_1(B) \to \mathcal{C}_1(B)$ satisfies $\mathcal{H}(A, \mathcal{C}_1(f_\varepsilon)(A)) < \varepsilon$, for each $A \in \mathcal{C}_1(B)$ [17, Lemma 8.2.13], and $\mathcal{C}_1(f_\varepsilon)(\mathcal{C}_1(p, B)) \subset \mathcal{C}_1(p, B) \setminus \mathcal{C}_1(\{p, q\}, B)$. Thus, $\mathcal{C}_1(\{p, q\}, B)$ is a Z-set in $\mathcal{C}_1(p, B)$. Hence, $\mathcal{C}_1(p, B) \cup \mathcal{C}_1(q, B)$ is a Hilbert cube [9, Theorem 1].

Now, we prove that $\mathcal{C}_1(p, B) \cup \mathcal{C}_1(q, B) \cup \mathcal{C}_1(r, B)$ is a Hilbert cube. By the previous paragraph, $\mathcal{C}_1(p, B) \cup \mathcal{C}_1(q, B)$ is a Hilbert cube. Also, $\mathcal{C}_1(r, B)$ is a Hilbert cube [6, Theorem 5.2]. Observe that

$$\begin{aligned}[]
[\mathcal{C}_1(p, B) \cup \mathcal{C}_1(q, B)] \cap \mathcal{C}_1(r, B) &= [\mathcal{C}_1(p, B) \cap \mathcal{C}_1(r, B)] \cup [\mathcal{C}_1(q, B) \cap \mathcal{C}_1(r, B)] \\
&= \mathcal{C}_1(\{p, r\}, B) \cup \mathcal{C}_1(\{q, r\}, B).
\end{aligned}$$

A similar argument to the one given in the previous paragraph shows that both $\mathcal{C}_1(\{p, r\}, B)$ and $\mathcal{C}_1(\{q, r\}, B)$ are Z-sets in $\mathcal{C}_1(r, B)$. Hence, $\mathcal{C}_1(p, B) \cup \mathcal{C}_1(q, B)$ is a Z-set in $\mathcal{C}_1(r, B)$ [3, Theorem 3.1 (3)]. Thus, $\mathcal{C}_1(p, B) \cup \mathcal{C}_1(q, B) \cup \mathcal{C}_1(r, B)$ is a Hilbert cube [9, Theorem 1].

To see that $\mathcal{C}_1(p, B) \cup \mathcal{C}_1(\{q, r\}, B)$ is a Hilbert cube, note that $\mathcal{C}_1(p, B)$ and $\mathcal{C}_1(\{q, r\}, B)$ are Hilbert cubes ([6, Theorem 5.2] and [5, Theorem 5.2], resp.). Also, observe that $\mathcal{C}_1(p, B) \cap \mathcal{C}_1(\{q, r\}, B) = \mathcal{C}_1(\{p, q, r\}, B)$, and this set is a Hilbert cube [5, Theorem 5.2]. A similar argument to the one given in the first paragraph shows that $\mathcal{C}_1(\{p,q,r\},B)$ is a Z-set in $\mathcal{C}_1(p,B)$. Thus, $\mathcal{C}_1(p,B)\cup\mathcal{C}_1(\{q,r\},B)$ is a Hilbert cube [9, Theorem 1]. ∎

Lemma 3.7. *Let B be a nondegenerate Peano continuum without free arcs and let A be an arc such that $B\cap A=\{p,q\}$, where p and q are the endpoints of A, and $\{p,q\}$ is a Z^*-set in B. Let F be a fan with top q such that F is a cone and $(B\cup A)\cap F=\{q\}$. If $X_1=B\cup A$ and $X_2=B\cup A\cup F$, then $\mathcal{D}_j=\mathcal{C}_1(B)\cup\mathcal{C}_1(p,X_j)\cup\mathcal{C}_1(q,X_j)$ is a Hilbert cube, $j\in\{1,2\}$.*

Proof. We show that $\mathcal{D}_2$ is a Hilbert cube; the proof for $\mathcal{D}_1$ is similar. Note that $\mathcal{D}_2$ is a contractible compact space. We show that $\mathcal{D}_2$ is a Hilbert cube manifold.

Let A_p and A_q be two subarcs of A such that $p\in A_p$, $q\in A_q$ and $A_p\cap A_q=\emptyset$. First, let us consither the following sets:

$$\mathcal{C}_2(A)^q_p=\{E\cup R\in\mathcal{C}_2(A)\,:\,p\in E\text{ and }q\in R\},$$
$$\mathcal{G}_1=\{K\cup E\,:\,K\in\mathcal{C}_1(p,B)\text{ and }E\in\mathcal{C}_1(p,A)\},$$
$$\mathcal{G}_{1p}=\{K\cup E\,:\,K\in\mathcal{C}_1(p,B)\text{ and }E\in\mathcal{C}_1(p,A_p)\},$$
$$\mathcal{G}_2=\{H\cup R\cup G\,:\,H\in\mathcal{C}_1(q,B),\ R\in\mathcal{C}_1(q,A)\text{ and }G\in\mathcal{C}_1(q,F)\},$$
$$\mathcal{G}_{2q}=\{H\cup R\cup G\,:\,H\in\mathcal{C}_1(q,B),\ R\in\mathcal{C}_1(q,A_q)\text{ and }G\in\mathcal{C}_1(q,F)\},$$
$$\mathcal{G}_3=\{K\cup(E\cup R)\cup G\,:\,K\in\mathcal{C}_1(\{p,q\},B)\,,E\cup R\in\mathcal{C}_2(A)^q_p\text{ and }G\in\mathcal{C}_1(q,F)\},$$
$$\mathcal{G}_4=\{L\cup G\,:\,L\in\mathcal{C}_1(A,B\cup A)\text{ and }G\in\mathcal{C}_1(q,F)\}.$$

R. Schori has shown that $\mathcal{C}_2(A)^q_p$ is a two-cell (for a proof, see [17, Theorem 6.9.12]). Now, we show that

$$\mathcal{G}_1,\ \mathcal{G}_{1p},\ \mathcal{G}_2,\ \mathcal{G}_{2q},\ \mathcal{G}_3\text{ and }\mathcal{G}_4\text{ are Hilbert cubes.}\qquad(\star)$$

By Lemma 3.1, $\mathcal{G}_1$ ($\mathcal{G}_{1p}$, resp.) is homeomorphic to $\mathcal{C}_1(p,B)\times\mathcal{C}_1(p,A)$ ($\mathcal{C}_1(p,B)\times\mathcal{C}_1(p,A_p)$, resp.) and $\mathcal{G}_2$ ($\mathcal{G}_{2q}$, resp.) is homeomorphic to $\mathcal{C}_1(q,B)\times\mathcal{C}_1(q,A)\times\mathcal{C}_1(q,F)$ ($\mathcal{C}_1(q,B)\times\mathcal{C}_1(q,A_q)\times\mathcal{C}_1(q,F)$, resp.). Since $\mathcal{C}_1(p,B)$ ($\mathcal{C}_1(q,B)$, resp.) is a Hilbert cube [6, Theorem 5.2], $\mathcal{C}_1(p,A)$ ($\mathcal{C}_1(p,A_p),\mathcal{C}_1(q,A)$, $\mathcal{C}_1(q,A_q)$, resp.) is an arc [7, Example 1, p. 267] (and $\mathcal{C}_1(q,F)$ is an either a Hilbert cube or an n-cell, for some positive integer n [8, Theorem 3.1 (3)]), we have that $\mathcal{G}_1$ ($\mathcal{G}_{1p}$, $\mathcal{G}_2$, $\mathcal{G}_{2q}$, resp.) is a Hilbert cube.

Using the union map (Notation 2.1), we see that $\mathcal{G}_3$ is homeomorphic to $\mathcal{C}_1(\{p,q\},B)\times\mathcal{C}_2(A)^q_p\times\mathcal{C}_1(q,F)$; also, $\mathcal{C}_1(\{p,q\},B)$ is a Hilbert cube

[5, Theorem 5.2], $\mathcal{C}_2(A)_p^q$ is a two-cell [17, Theorem 6.9.12], and $\mathcal{C}_1(q,F)$ is either a Hilbert cube or an n-cell, for some positive integer n [8, Theorem 3.1 (3)]. Thus, $\mathcal{G}_3$ is a Hilbert cube.

Using the union map, we see that $\mathcal{G}_4$ is homeomorphic to $\mathcal{C}_1(A, B\cup A)\times\mathcal{C}_1(q,F)$; also, since $B\setminus A$ contains no free arc, $\mathcal{C}_1(A,B\cup A)$ is a Hilbert cube [6, Theorem 5.2] and $\mathcal{C}_1(q,F)$ is either a Hilbert cube or an n-cell, for some positive integer n [8, Theorem 3.1 (3)]. Thus, $\mathcal{G}_4$ is a Hilbert cube.

This finishes the proof of $(\star)$.

Let $D\in\mathcal{D}_2$. We show that D has closed neighborhood $\mathcal{U}$ in $\mathcal{D}_2$ such that $\mathcal{U}$ is a Hilbert cube. We divide the proof of this fact into five main cases.

CASE (1): *$D\subset B\setminus\{p,q\}$.*

In this case, $\mathcal{U}=\mathcal{C}_1(B)$ is a neighborhood of D in $\mathcal{D}_2$ such that $\mathcal{U}$ is a Hilbert cube [6, Theorem 4.1].

CASE (2): *$D\in\mathcal{C}_1(p,X_2)$ and $q\notin D$.*

Observe that $D\cap B\in\mathcal{C}_1(p,B)$. Also, note that $\mathcal{C}_1(B)\cap\mathcal{G}_1=\mathcal{C}_1(p,B)$. Since $\{p\}$ is a Z-set in B, by Lemma 2.2, $\mathcal{C}_1(p,B)$ is a Z-set in $\mathcal{C}_1(B)$. Observe that $\mathcal{C}_1(B)$ is a Hilbert cube [6, Theorem 4.1], $\mathcal{G}_2$ is a Hilbert cube (by $(\star)$), $\mathcal{C}_1(B)\cap\mathcal{G}_1=\mathcal{C}_1(p,B)$ is a Hilbert cube [6, Theorem 5.2], and $\mathcal{C}_1(B)\cap\mathcal{G}_1$ is a Z-set in $\mathcal{C}_1(B)$. Hence, $\mathcal{U}=\mathcal{C}_1(B)\cup\mathcal{G}_1$ is a neighborhood of $D\in\mathcal{D}_2$ such that $\mathcal{U}$ is a Hilbert cube [9, Theorem 1].

CASE (3): *$D\in\mathcal{C}_1(q,X_2)$ and $p\notin D$.*

Note that $D\cap B\in\mathcal{C}_1(q,B)$. Also, observe that $\mathcal{C}_1(B)\cap\mathcal{G}_2=\mathcal{C}_1(q,B)$. Since $\{q\}$ is a Z-set in B, by Lemma 2.2, $\mathcal{C}_1(q,B)$ is a Z-set in $\mathcal{C}_1(B)$. Note that $\mathcal{C}_1(B)$ is a Hilbert cube [6, Theorem 4.1], $\mathcal{G}_1$ is a Hilbert cube (by $(\star)$), $\mathcal{C}_1(B)\cap\mathcal{G}_2=\mathcal{C}_1(q,B)$ is a Hilbert cube [6, Theorem 5.2], and $\mathcal{C}_1(B)\cap\mathcal{G}_2$ is a Z-set in $\mathcal{C}_1(B)$. Hence, $\mathcal{U}=\mathcal{C}_1(B)\cup\mathcal{G}_2$ is a neighborhood of D in $\mathcal{D}_2$ such that $\mathcal{U}$ is a Hilbert cube [9, Theorem 1].

CASE (4): *$D\in\mathcal{C}_1(\{p,q\},X_2)$ and $D\cap B$ is connected.*

Observe that $D\cap B\in\mathcal{C}_1(\{p,q\},B)$. We consider four subcases.

SUBCASE (4.i): *$\{p\}$ is not a component of $D\cap A$ and $\{q\}$ is not a component of $D\cap F$.*

Note that $\mathcal{G}_3\cap\mathcal{G}_4=\{K\cup A\cup G : K\in\mathcal{C}_1(\{p,q\},B)$ and $G\in\mathcal{C}_1(q,F)\}$. Using the union map (Notation 2.1) $\mathcal{G}_3\cap\mathcal{G}_4$ is homeomorphic to $\mathcal{C}_1(\{p,q\},B)\times\mathcal{C}_1(q,F)$; moreover, $\mathcal{C}_1(\{p,q\},B)$ is a Hilbert cube [5, Theorem 5.2] and $\mathcal{C}_1(q,F)$ is either an n-cell (for some positive integer n) or a Hilbert cube [8, Theorem 3.1 (3)]. Thus, $\mathcal{G}_3\cap\mathcal{G}_4$ is a Hilbert cube. It is easy to see that $\mathcal{G}_3\cap\mathcal{G}_4$ is a

Z-set in $\mathcal{G}_3$. Hence, $\mathcal{U} = \mathcal{G}_3 \cup \mathcal{G}_4$ is a neighborhood of D in $\mathcal{D}_2$ such that $\mathcal{U}$ is a Hilbert cube [9, Theorem 1].

SUBCASE (4.ii): *$\{p\}$ is a component of $D \cap A$ and $\{q\}$ is not a component of $D \cap F$.*

Observe that

$$\mathcal{G}_2 \cap \mathcal{G}_3 = \big\{K \cup R \cup G \ : \ K \in \mathcal{C}_1(\{p,q\}, B),\ R \in \mathcal{C}_1(q, A),\ G \in \mathcal{C}_1(q, F)\big\}.$$

Using the union map (Notation 2.1), $\mathcal{C}_1(\{p,q\}, B) \times \mathcal{C}_1(q, A) \times \mathcal{C}_1(q, F)$ is homeomorphic to $\mathcal{G}_2 \cap \mathcal{G}_3$; moreover, $\mathcal{C}_1(\{p,q\}, B)$ is a Hilbert cube [5, Theorem 5.2], $\mathcal{C}_1(q, A)$ is an arc [7, Example 1, p. 267] and $\mathcal{C}_1(q, F)$ is either a Hilbert cube or an n-cell, for some positive integer n [8, Theorem 3.1 (3)]. Thus, $\mathcal{G}_2 \cap \mathcal{G}_3$ is a Hilbert cube. It is easy to see that $\mathcal{G}_2 \cap \mathcal{G}_3$ is a Z-set in $\mathcal{G}_3$. Hence, $\mathcal{U} = \mathcal{G}_2 \cup \mathcal{G}_3$ is a neighborhood of D in $\mathcal{D}_2$ such that $\mathcal{U}$ is a Hilbert cube [9, Theorem 1].

SUBCASE (4.iii): *$\{p\}$ is a component of $D \cap A$, $\{q\}$ is a component of $D \cap F$ and $\{q\}$ is not a component of $D \cap A$.*

In this case, $\mathcal{U} = \mathcal{G}_2 \cup \mathcal{G}_3$ is a neighborhood of D in $\mathcal{D}_2$ such that $\mathcal{U}$ is a Hilbert cube (see Subcase (4.ii):

SUBCASE (4.iv): *$D \cap (A \cup F) = \{p, q\}$.*

Observe that $\mathcal{G}_{2q} \cup \mathcal{G}_3$ is a Hilbert cube (by a similar argument to the one given in Subcase (4.ii) for $\mathcal{G}_2 \cup \mathcal{G}_3$). Also note that

$$\begin{aligned}
&\mathcal{G}_{1p} \cap [\mathcal{G}_{2q} \cup \mathcal{G}_3] = [\mathcal{G}_{1p} \cap \mathcal{G}_{2q}] \cup [\mathcal{G}_{1p} \cap \mathcal{G}_3] \\
&\quad = \{K \ : \ K \in \mathcal{C}_1(\{p,q\}, B)\} \cup \{K \cup E \ : \ K \in \mathcal{C}_1(\{p,q\}, B),\, E \in \mathcal{C}_1(p, A_p)\} \\
&\quad = \{K \cup E \ : \ K \in \mathcal{C}_1(\{p,q\}, B),\, E \in \mathcal{C}_1(p, A_p)\}.
\end{aligned}$$

Thus, by the union map (Notation 2.1), we have that $\mathcal{G}_{1p} \cap [\mathcal{G}_{2q} \cup \mathcal{G}_3]$ is homeomorphic to $\mathcal{C}_1(\{p,q\}, B) \times \mathcal{C}_1(p, A_p)$; also, since $\mathcal{C}_1(\{p,q\}, B)$ is a Hilbert cube [5, Theorem 5.2] and $\mathcal{C}_1(p, A_p)$ is an arc [7, Example 1, p. 267], $\mathcal{C}_1(\{p,q\}, B) \times \mathcal{C}_1(p, A_p)$ is a Hilbert cube. Since $\{p, q\}$ is a Z^*-set in B, we have that $\mathcal{G}_{1p} \cap [\mathcal{G}_{2q} \cup \mathcal{G}_3]$ is a Z-set in $\mathcal{G}_{1p}$. Hence, $\mathcal{G}_{1p} \cup \mathcal{G}_{2q} \cup \mathcal{G}_3$ is a Hilbert cube [9, Theorem 1].

Now, note that

$$\begin{aligned}
\mathcal{C}_1(B) \cap [\mathcal{G}_{1p} \cup \mathcal{G}_{2q} \cup \mathcal{G}_3] &= [\mathcal{C}_1(B) \cap \mathcal{G}_{1p}] \cup [\mathcal{C}_1(B) \cap \mathcal{G}_{2q}] \cup [\mathcal{C}_1(B) \cap \mathcal{G}_3] \\
&= \mathcal{C}_1(p, B) \cup \mathcal{C}_1(q, B) \cup \mathcal{C}_1(\{p,q\}, B) \\
&= \mathcal{C}_1(p, B) \cup \mathcal{C}_1(q, B).
\end{aligned}$$

By Lemma 3.6, $\mathcal{C}_1(p,B)\cup\mathcal{C}_1(q,B)$ is a Hilbert cube. Since $\{p,q\}$ is a Z^*-set in B, we have that $\mathcal{C}_1(B)\cap[\mathcal{G}_{1p}\cup\mathcal{G}_{2q}\cup\mathcal{G}_3]$ is a Z-set in $\mathcal{C}_1(B)$ (Lemma 2.2). Therefore, $\mathcal{U}=\mathcal{C}_1(B)\cup\mathcal{G}_{1p}\cup\mathcal{G}_{2q}\cup\mathcal{G}_3$ is a neighborhood of D in $\mathcal{D}_2$ such that $\mathcal{U}$ is a Hilbert cube [9, Theorem 1].

CASE (5): *$D\in\mathcal{C}_1(\{p,q\},X_2)$ and $D\cap B$ is not connected.*

Observe that in this case $A\subset D$. We consider five subcases.

SUBCASE (5.i): *$D\cap B$ has two nondegenerate components.*

Here, $\mathcal{U}=\mathcal{G}_4$ is a neighborhood of D in $\mathcal{D}_2$ such that $\mathcal{U}$ is a Hilbert cube (by $(\star)$).

SUBCASE (5.ii): *$\{p\}$ is the only degenerate component of $D\cap B$.*

Note that $\mathcal{G}_2\cap\mathcal{G}_4=\{A\cup H\cup G : H\in\mathcal{C}_1(q,B)$ and $G\in\mathcal{C}_1(q,F)\}$. Using the union map (Notation 2.1), we obtain that $\mathcal{G}_2\cap\mathcal{G}_4$ is homeomorphic to $\mathcal{C}_1(q,B)\times\mathcal{C}_1(q,F)$; moreover, $\mathcal{C}_1(q,B)$ is a Hilbert cube [6, Theorem 5.2] and $\mathcal{C}_1(q,F)$ is either a Hilbert cube or an n-cell, for some positive integer n [8, Theorem 3.1 (3)]. Thus, $\mathcal{G}_2\cap\mathcal{G}_4$ is a Hilbert cube. It is easy to see that $\mathcal{G}_2\cap\mathcal{G}_4$ is a Z-set in $\mathcal{G}_2$. Hence $\mathcal{U}=\mathcal{G}_2\cup\mathcal{G}_4$ is a neighborhood of D in $\mathcal{D}_2$ such that $\mathcal{U}$ is a Hilbert cube [9, Theorem 1].

SUBCASE (5.iii): *$\{q\}$ is the only degenerate component of $D\cap B$.*

Observe that $\mathcal{G}_1\cap\mathcal{G}_4=\{K\cup A : K\in\mathcal{C}_1(p,B)\}$. Thus, $\mathcal{G}_1\cap\mathcal{G}_4$ is homeomorphic to $\mathcal{C}_1(p,B)$. Hence, $\mathcal{G}_1\cap\mathcal{G}_4$ is a Hilbert cube [6, Theorem 5.2]. By Lemma 3.5, $\mathcal{G}_1\cap\mathcal{G}_4$ is a Z-set in $\mathcal{G}_1$. Thus, since $\mathcal{G}_1$ and $\mathcal{G}_4$ are Hilbert cubes (by $(\star)$), $\mathcal{G}_1\cap\mathcal{G}_4$ is a Hilbert cube and $\mathcal{G}_1\cap\mathcal{G}_4$ is a Z-set in $\mathcal{G}_1$, we have that $\mathcal{U}=\mathcal{G}_1\cup\mathcal{G}_4$ is a neighborhood of D in $\mathcal{D}_2$ such that $\mathcal{U}$ is a Hilbert cube [9, Theorem 1].

SUBCASE (5.iv): *$D=A\cup G$, where $G\in\mathcal{C}_1(q,F)\setminus\{\{q\}\}$.*

In this case, $\mathcal{U}=\mathcal{G}_2\cup\mathcal{G}_4$ is a neighborhood of D in $\mathcal{D}_2$ such that $\mathcal{U}$ is a Hilbert cube (see Subcase (5.ii)).

SUBCASE (5.v): *$D=A$.*

We show that $\mathcal{G}_1\cup\mathcal{G}_4\cup\mathcal{G}_2$ is a Hilbert cube. Observe that $\mathcal{G}_1\cup\mathcal{G}_4$ is a Hilbert cube (see Subcase (5.iii)).

Now, observe that

$$\begin{aligned}[\mathcal{G}_1\cup\mathcal{G}_4]\cap\mathcal{G}_2&=[\mathcal{G}_1\cap\mathcal{G}_2]\cup[\mathcal{G}_4\cap\mathcal{G}_2]\\&=\{K\cup A : K\in\mathcal{C}_1(\{p,q\},B)\}\cup\{A\cup H\cup G : H\in\mathcal{C}_1(q,B),\ G\in\mathcal{C}_1(q,F)\}\\&=\{A\cup H\cup G : H\in\mathcal{C}_1(q,B),\ G\in\mathcal{C}_1(q,F)\}.\end{aligned}$$

Thus, $[\mathcal{G}_1 \cup \mathcal{G}_4] \cap \mathcal{G}_2$ is homeomorphic to $\mathcal{C}_1(q, B) \times \mathcal{C}_1(q, F)$, by the union map (Notation 2.1). Hence, $[\mathcal{G}_1 \cup \mathcal{G}_4] \cap \mathcal{G}_2$ is a Hilbert cube (see [6, Theorem 5.2] and [8, Theorem 3.1 (3)], resp.). By Lemma 3.5, $[\mathcal{G}_1 \cup \mathcal{G}_4] \cap \mathcal{G}_2$ is a Z-set in $\mathcal{G}_2$. Thus, since $\mathcal{G}_1 \cup \mathcal{G}_4$ is a Hilbert cube, $\mathcal{G}_2$ is a Hilbert cube (by ($\star$)), $[\mathcal{G}_1 \cup \mathcal{G}_4] \cap \mathcal{G}_2$ is a Hilbert cube, and $[\mathcal{G}_1 \cup \mathcal{G}_4] \cap \mathcal{G}_2$ is a Z-set in $\mathcal{G}_2$, we have that $\mathcal{G}_1 \cup \mathcal{G}_4 \cup \mathcal{G}_2$ is a Hilbert cube [9, Theorem 1].

Therefore, $\mathcal{U} = \mathcal{G}_1 \cup \mathcal{G}_4 \cup \mathcal{G}_2$ is a neighborhood of D in $\mathcal{D}_2$ such that $\mathcal{U}$ is a Hilbert cube. Hence, $\mathcal{D}_2$ is a compact contractible Hilbert cube manifold. Therefore, $\mathcal{D}_2$ is a Hilbert cube [2, Corollary 5]. ∎

Theorem 3.8. *Let B be a nondegenerate Peano continuum without free arcs. Let A be an arc such that $B \cap A = \{p, q\}$, where p and q are the endpoints of A, and $\{p, q\}$ is a Z^*-set in B. Let F be a fan with top q such that F is a cone. Suppose that $(B \cup A) \cap F = \{q\}$. If $X_1 = B \cup A$ and $X_2 = B \cup A \cup F$, then $\mathcal{C}_1(X_j)$ is a cone, $j \in \{1, 2\}$.*

Proof. We show that $\mathcal{C}_1(X_2)$ is a cone; the proof for $\mathcal{C}_1(X_1)$ is similar. Observe that

$$\begin{aligned} \mathcal{C}_1(X_2) &= \mathcal{C}_1(B) \cup \mathcal{C}_1(p, X_2) \cup \mathcal{C}_1(q, X_2) \cup \mathcal{C}_1(A) \cup \mathcal{C}_1(F) \\ &= \mathcal{C}_1(B) \cup \mathcal{C}_1(p, X_2) \cup \mathcal{C}_1(q, X_2) \cup \mathcal{C}_1(A) \cup \mathcal{C}_1(q, F) \cup \mathcal{T}[\mathcal{C}_1(F)] \\ &= \mathcal{C}_1(B) \cup \mathcal{C}_1(p, X_2) \cup \mathcal{C}_1(q, X_2) \cup \mathcal{C}_1(A) \cup \mathcal{T}[\mathcal{C}_1(F)], \end{aligned}$$

and

$$\begin{aligned} &[\mathcal{C}_1(B) \cup \mathcal{C}_1(p, X_2) \cup \mathcal{C}_1(q, X_2)] \cap [\mathcal{C}_1(A) \cup \mathcal{T}[\mathcal{C}_1(F)]] \\ &\qquad\qquad = \mathcal{C}_1(p, A) \cup \mathcal{C}_1(q, A) \cup \mathcal{N}[\mathcal{C}_1(F)]. \end{aligned}$$

Let $\mathcal{D}_2 = \mathcal{C}_1(B) \cup \mathcal{C}_1(p, X_2) \cup \mathcal{C}_1(q, X_2)$. By Lemma 3.7, $\mathcal{D}_2$ is a Hilbert cube. We show that

$$\mathcal{C}_1(p, A) \cup \mathcal{C}_1(q, A) \cup \mathcal{N}[\mathcal{C}_1(F)] \text{ is a } Z\text{-set in } \mathcal{D}_2. \qquad (\star\star)$$

Let $\varepsilon > 0$ and let $f_\varepsilon \colon \mathcal{D}_2 \to \mathcal{D}_2$ be given by $f_\varepsilon(D) = K_\rho(\frac{\varepsilon}{2}, D \cap (B \cup A)) \cup (D \cap F)$ (Notation 2.4). Then f_ε is continuous,

$$f_\varepsilon(\mathcal{C}_1(p, A) \cup \mathcal{C}_1(q, A) \cup \mathcal{N}[\mathcal{C}_1(F)]) \cap [\mathcal{C}_1(p, A) \cup \mathcal{C}_1(q, A) \cup \mathcal{N}[\mathcal{C}_1(F)]] = \emptyset,$$

and $\mathcal{H}(D, f_\varepsilon(D)) < \varepsilon$, for each $D \in \mathcal{D}_2$. Therefore, $\mathcal{C}_1(p, A) \cup \mathcal{C}_1(q, A) \cup \mathcal{N}[\mathcal{C}_1(F)]$ is a Z-set in $\mathcal{D}_2$. This proves ($\star\star$).

Since F is a fan that is a cone, F is the cone over $E(F)$ [19, Lemma 4.2]. Hence, $E(F)$ is a compact totally disconnected space. Let $E' = E(F) \cup \{p\}$. Then E' is a compact totally disconnected space. Therefore, E' can be embedded in I [10, Theorem V2, p. 56].

Let $g\colon E' \to I$ be an embedding of E' in I. For each $e \in E'$, let L_e be the convex arc in I^∞ whose end points are $(0,0,0,\ldots)$ and $(1, g(e), 0, 0, 0, \ldots)$. Let $L = \bigcup_{e\in E'} L_e$. Then L is a Z-set in I^∞. Recall that $\mathcal{D}_2$ is a Hilbert cube (Lemma 3.7). Also, $\mathcal{C}_1(p, A) \cup \mathcal{C}_1(q, A) \cup \mathcal{N}[\mathcal{C}_1(F)]$ is a Z-set in $\mathcal{D}_2$ (by $(\star\star)$), $\mathcal{C}_1(p, A) \cup \mathcal{C}_1(q, A)$ is homeomorphic to L_p, and $\mathcal{N}[\mathcal{C}_1(F)]$ is homeomorphic to $(L \setminus L_p) \cup \{(0,0,0,\ldots)\}$ (since F is a smooth fan). Hence, since $[\mathcal{C}_1(p, A) \cup \mathcal{C}_1(q, A)] \cap \mathcal{N}[\mathcal{C}_1(F)] = \{\{q\}\}$, there exists a homeomorphism $\psi\colon \mathcal{D}_2 \twoheadrightarrow I^\infty$ such that $\psi(\{q\}) = (0,0,0,\ldots)$, $\psi(\mathcal{C}_1(p, A) \cup \mathcal{C}_1(q, A)) = L_p$ and $\psi(\mathcal{N}[\mathcal{C}_1(F)]) = (L \setminus L_p) \cup \{(0,0,0,\ldots)\}$ [3, Theorem 11.1].

For each $e \in E'$, let G_e be the convex arc in I^∞ whose end points are $(1, g(e), 0, 0, 0, \ldots)$ and $(1, g(e), 1, 0, 0, 0, \ldots)$.

For every $e \in E'$, let D_e be the convex hull in I^∞ of $L_e \cup G_e$. Let $\mathfrak{D} = \bigcup_{e\in E'} D_e$. It is easy to see that there exists a homeomorphism $\xi\colon \mathcal{C}_1(A) \cup \mathcal{T}[\mathcal{C}_1(F)] \twoheadrightarrow \mathfrak{D}$ such that $\xi(K) = \psi(K)$, for each $K \in \mathcal{C}_1(p, A) \cup \mathcal{C}_1(q, A) \cup \mathcal{N}[\mathcal{C}_1(F)]$.

Therefore, the map $\zeta\colon \mathcal{C}_1(X) \twoheadrightarrow I^\infty \cup \mathfrak{D}$ given by

$$\zeta(K) = \begin{cases} \psi(K), & \text{if } K \in \mathcal{D}_2, \\ \xi(K), & \text{if } K \in \mathcal{C}_1(A) \cup \mathcal{T}[\mathcal{C}_1(F)], \end{cases}$$

is a homeomorphism. Also, as is easy to see, $I^\infty \cup \mathfrak{D}$ is homeomorphic to the cone over $\left(\bigcup_{e\in E'} G_e\right) \cup \{(x_n)_{n=1}^\infty \in I^\infty : x_1 = 1\}$, with vertex $(0,0,0,\ldots)$. Therefore, $\mathcal{C}_1(X_2)$ is a cone. ∎

The proof of the following lemma is similar to the one given for Lemma 3.7, we omit it because it involves many more sets and cases.

Lemma 3.9. *Let B be a nondegenerate Peano continuum without free arcs. Let A_1 and A_2 be two arcs such that $B \cap A_1 = \{p, q\}$, $B \cap A_2 = \{q, r\}$, $A_1 \cap A_2 = \{q\}$, where p and q are the endpoints of A_1 and q and r are the endpoints of A_2. Suppose that $\{p, q, r\}$ is a Z^*-set in B. Let F be a fan with top $\{q\}$ such that F is a cone and $(B \cup A_1 \cup A_2) \cap F = \{q\}$. If $X = B \cup A_1 \cup A_2 \cup F$, then $\mathcal{D} = \mathcal{C}_1(B) \cup \mathcal{C}_1(p, X) \cup \mathcal{C}_1(q, X) \cup \mathcal{C}_1(r, X)$ is a Hilbert cube.*

A similar proof to the one given for Theorem 3.8 shows the following theorem. Note that we need to use Lemma 3.9 instead of Lemma 3.7.

Theorem 3.10. *Let B be a nondegenerate Peano continuum without free arcs. Let A_1 and A_2 be two arcs such that $B \cap A_1 = \{p, q\}$, $B \cap A_2 = \{q, r\}$, $A_1 \cap A_2 = \{q\}$, where p and q are the endpoints of A_1 and q and r are the endpoints of A_2. Suppose that $\{p, q, r\}$ is a Z^*-set in B. Let F be a fan with top $\{q\}$ such that F is a cone and $(B \cup A_1 \cup A_2) \cap F = \{q\}$. If $X = B \cup A_1 \cup A_2 \cup F$, then $\mathcal{C}_1(X)$ is a cone.*

Question 3.11. Can the hypothesis "Z^*-set" be changed to "Z-set" in Lemma 3.7, Theorem 3.8, Lemma 3.9 and Theorem 3.10?

Acknowledgements

The authors thank the referee for the valuable suggestions made that improved the paper.

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