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Expositiones Mathematicae 41 (2023) 333–397

EXPOSITIONES MATHEMATICAE

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The Rochberg garden[☆]

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Received 8 November 2022; received in revised form 1 February 2023; accepted 3 April 2023

Abstract

In 1996, it was published the seminal work of Rochberg "Higher order estimates in complex interpolation theory" (Rochberg, 1996). Among many other things, the paper contains a new method to construct new Banach spaces having an intriguing behaviour: they are simultaneously interpolation spaces and twisted sums of increasing complexity. The fundamental idea of Rochberg is to consider for each $z \in \mathbb{S}$ the space formed by the arrays of the truncated sequence of the Taylor coefficients of the elements of the Calderón space. What was probably unforeseen is that the Rochberg constructions would lead to a deep theory connecting Interpolation theory, Homology, Operator Theory and the Geometry of Banach spaces. This work aims to synthetically present such connections, an up-to-date account of the theory and a list of significative open problems.

2010 Mathematics Subject Classification: 46M18; 46B10; 46B20; 46B70

Keywords: Rochberg spaces; Complex interpolation; Differentials; Twisted sums of Banach spaces

1. Introduction

In 1996, it was published the seminal work of Rochberg "Higher order estimates in complex interpolation theory" [97]. Among many other things, the paper contains a new method to construct new Banach spaces having an intriguing behaviour: they are simultaneously interpolation spaces and twisted sums of increasing complexity. From

☆ The research of the first author been supported in part MINCIN project PID2019-103961GB-C21. The research of both authors has been supported in part by Junta de Extremadura, Spain project IB20038.
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https://doi.org/10.1016/j.exmath.2023.04.002

this point of view we can think of the whole process of construction of Rochberg spaces unfolding at different levels: At level 0 (no derivatives involved) we have plain complex interpolation. More precisely, given a suitable pair of Banach spaces (X_0, X_1), if one considers the usual Calderón space $\mathcal{C}(X_0, X_1)$ of $X_0 + X_1$-valued analytic functions then the interpolation space $(X_0, X_1)_\theta = \{f(\theta) : f \in \mathcal{C}(X_0, X_1)\}$ is the space $\mathfrak{R}_1$ of values of the functions at θ. At level 1 (one derivative involved) one would think of the space $\mathfrak{R}_2$ of pairs of values $(f'(\theta), f(\theta))$ as $f \in \mathcal{C}(X_0, X_1)$. In general, at level n (n derivatives involved) one would consider the space

$$\mathfrak{R}_{n+1}(X_0, X_1)_\theta = \left\{\left(\frac{1}{n!} f^{(n)}(\theta), \dots, f'(\theta), f(\theta)\right) : f \in \mathcal{C}(X_0, X_1)\right\}$$

of truncated sequences of *Taylor coefficients* of elements of $\mathcal{C}(X_0, X_1)$ at θ. These spaces will be endowed with the quotient norm, referred to as *Rochberg spaces* and, unless it is required to specify (X_0, X_1) or θ, denoted $\mathfrak{R}_n$. The display and study of properties of Rochberg spaces is the main topic of this survey.

It turns out that many concepts, properties and results occurring in classical interpolation theory (level 0) have close analogues at higher levels. Other phenomena can only occur at higher levels. The following table depicts some of the analogies and correspondences:

Calderón space $\mathcal{C}(X_0, X_1)$	
Level 0	Level n
• Interpolation space $(X_0, X_1)_\theta$	• Rochberg space $\mathfrak{R}_n(X_0, X_1)_\theta$
• Evaluations $f(\theta)$	• Taylor Coefficients $\left(\frac{1}{(n-1)!} f^{(n-1)}(\theta), \dots, f(\theta)\right)$
• Interpolation property for operators	• Commutator Theorem
• Identity $(X_0, X_1)_\theta \to (X_0, X_1)_\theta$	• Differentials $\Omega_{k,l} : \mathfrak{R}_l \to \mathfrak{R}_k$ for $k + l = n$
• Reiteration Theorem	• Reiteration theorem for differential maps

The aim of this survey is to present an up-to-date account of the theory of Rochberg spaces. We will focus on two scenarios: Rochberg spaces generated by complex interpolation for pairs of spaces and Rochberg spaces generated by a family of abstract interpolators. We have strived to give proper credits and sources for all the results, while those unassigned are presented here for the first time. In some cases the results are in the literature but the proofs presented here are original. We will provide explicit proofs for the fundamental results, and the main ideas plus informal discussions for the rest.

We now explain the organization of the paper. For all unexplained terminology and concepts see the corresponding sections (or the end of this introduction). We assume that the reader has some acquaintance with Banach space theory, for which three authoritative references are [2,79,80], and the basics of complex interpolation as can be seen in [6,73].

The overall organization of the paper is: first, what occurs at level 0 (interpolation); then, what occurs at level 1 (twisted sums) and then what occurs at higher levels. In the three cases combining the complex interpolation and the abstract frame. Accordingly, in Section 2 we deal with the basics of complex interpolation for pairs: definitions, constructions, the reiteration theorem, Lozanovskii factorization and several examples including the Hilbert space, which is central to the theory in many regards. The 0 level is also considered from the abstract point of view through the concept of *interpolator*. In Section 3 we deal with level 1 and introduce the fundamental ideas of the theory of twisted sums, exact sequences of Banach spaces and the necessary homological techniques. We thus show how level 0 results can be generalized to level 1 using the previously introduced homological ideas. The central concept at this level 1 is that of differential associated to an interpolation scale. The mirroring of ideas between the two levels establishes, for instance, that Hilbert spaces correspond to twisted Hilbert spaces, that reiteration results from interpolation theory transform into reiteration results for differentials, etc. In Section 4 we formally introduce Rochberg spaces $\mathfrak{R}_n$ for the complex interpolation method as the space formed by truncated sequences of Taylor coefficients of functions in the corresponding Calderón space. Rochberg spaces can also be generated by certain higher order differentials $\Omega_{m,n}$. In turn, these differentials induce exact sequences that can be entwined in commutative diagrams of the form

$$\begin{array}{ccccc}
\mathfrak{R}_k & = & \mathfrak{R}_k & & \\
\downarrow & & \downarrow & & \\
\mathfrak{R}_n & \longrightarrow & \mathfrak{R}_{n+m} & \longrightarrow & \mathfrak{R}_m \\
\downarrow & & \downarrow & & \| \\
\mathfrak{R}_{n-k} & \longrightarrow & \mathfrak{R}_{n+m-k} & \longrightarrow & \mathfrak{R}_m
\end{array}$$

The discussion about the Rochberg spaces associated to a sequence of abstract interpolators points out specific properties of the interpolators associated to complex interpolation. Those specific properties crystallize in the notion of *compatibility* studied in Section 4.3. The rest of the Section contains a generalized form of the Commutator theorem and its consequences, an exposition of the inversion process and a few additional consideration about the effect on Rochberg spaces of reiteration and multiplication by scalars.

Sections 5 to 9 are devoted to present concrete examples. First, the case of weighted Hilbert spaces. This is the simplest situation since the associated differentials are all linear, and therefore all Rochberg spaces are isomorphic to ℓ_2. Then, Section 6 is devoted to the case of ℓ_p spaces, arguably the most important example. The associated Rochberg spaces are strange creatures: for instance, when fixed at $1/2$ the Rochberg space $\mathfrak{R}_n$ is a Banach space isomorphic to its dual, which has exactly n different types of basic sequences, has no complemented subspace with *G.L-lust.*, has a non-trivial symplectic structure and every operator $T : \mathfrak{R}_n \to X$ is strictly singular or invertible on some complemented subspace isomorphic to $\mathfrak{R}_n$. Section 7 treats the non-atomic L_p spaces, Section 8 deals with Orlicz spaces and Section 9 considers Tsirelson-like spaces, with particular interest in the 2-convexified Tsirelson space that produces new weak-Hilbert spaces [106].

The two final sections are devoted to (1) briefly discuss advanced topics closely related to Rochberg spaces: Stability, Homology and Nonlinear classification and (2) list open problems.

The following notation will be consistently used throughout the paper: $X = Y$ means that the two spaces X, Y coincide, $X \sim Y$ that they are isometric and $X \simeq Y$ that they are isomorphic. Given two real valued functions f and g, by $f \sim g$ means that there exist positive constants C and C' such that $C'g(x) \leq f(x) \leq Cg(x)$ for all x.

2. Complex interpolation for pairs of Banach spaces

In its essence, an interpolation method is a method to, given a pair (X_0, X_1) of Banach spaces and a parameter θ, produce an intermediate space $X_\theta = (X_0, X_1)_\theta$. The meaning of "intermediate" depends on the interests one has.

(1) From the point of view of convex analysis, the question of how is it possible to "continuously" transform a (convex) set C_0 into another C_1. Readers interested in this approach would largely benefit from the reading of [18], where a natural structure of normed space is placed on the space of quasinorms $\mathcal{Q}$ defined on a finite dimensional space $\mathbb{K}^n$. Then, consider, given the unit balls C_0, C_1 of two quasinorms, a continuous path $H : [0, 1] \longrightarrow \mathcal{Q}$ such that $H(0) = C_0$ and $H(1) = C_1$. If one sets $X_0 = (\mathbb{K}^n, C_0)$ and $X_1 = (\mathbb{K}^n, C_1)$ then each space $A_t = (\mathbb{K}^n, H(t))$ can be considered an intermediate space between X_0 and X_1.

(2) The classical operator theory point of view is rather interested in obtaining spaces X_θ with the following property: if one has another pair (B_0, B_1) and an operator T that is linear and continuous as an operator $A_i \to B_i$ for $i = 0, 1$ then it is also continuous as an operator $X_\theta \to B_\theta$. Spaces with this property are usually called interpolated spaces.

(3) The "differential equation" point of view considers interpolation methods as a kind of Banach valued forms of Dirichlet's problem for Laplace's equation: Given a function f that has values everywhere on the boundary of a region in $\mathbb{K}^n$, find a continuous function F twice continuously differentiable in the interior and continuous on the boundary, such that F is harmonic in the interior and $F = f$ on the boundary. Now, place at each point ω of the boundary of a domain $\mathbb{D}$ of the complex plane a normed space X_ω (say, a norm N_ω defined on a prefixed $\mathbb{K}^n$) instead of a scalar (one can think of this scalar as $\|1\|_\omega$) and determine a way to assign to each point $z \in \mathbb{D}$ a norm N_z with some additional continuity or regularity properties. Details about this approach can be followed at [52,96,102]. The papers [50,51] presented solutions for the so-called (sub-) interpolation families (in dimension one, the family of norms $\|\cdot\|_z$ is a (sub-) interpolation family when $\log \|1\|_z$ is (subharmonic) harmonic, so that what one is asking for is the solution for the Dirichlet problem for the equation $\Delta \log \|1\|_z = 0$); this lead to what is called nowadays the interpolation method for families. The case of pairs can be recovered placing, without giving too many details now, X_0 and X_1 on equally distributed arcs on the boundary of $\mathbb{D}$. The case of families is much more sophisticated and delicate than the case of pairs (X_0, X_1) since the relative distribution and positions of the spaces along the boundary have impact on the interpolated space one obtains.

(4) The differential geometry point of view attempts to think of the intermediate spaces $\{H(t) : t \in [0, 1]\}$ as conforming a geodesic, in a sense to be determined, between X_0 and X_1. See [102].

Moreover, the many methods (for pairs, for families, ...) appeared in the literature have led diverse unification attempts and from that to what could be called "abstract interpolation methods".

2.1. *The abstract setting*

As we have just said, there are many interpolation methods in the literature: in some of them the parameter θ is a complex number, in some θ is a pair, or three or four, real numbers, or a point in a prefixed Banach space, etc. At the abstract level, θ is just an element of the space D of parameters and most interpolation method adopt the following form: assign to each suitable pair (X_0, X_1) of Banach spaces a certain Banach space $\mathcal{H}$ of functions $D \to \Sigma$ from D to a given ambient Banach space Σ (see below for additional comments about the role of the ambient space) such that for each $\theta \in D$ the evaluation map $\delta_\theta : \mathcal{H} \to \Sigma$ is continuous. In this way, for each parameter θ one can form the space $X_\theta = \{f(\theta) : f \in \mathcal{H}\}$. Since δ_θ is a bounded operator, $X_\theta = \delta_\theta[\mathcal{H}]$ is a Banach space when endowed with the quotient topology [32] and $X_\theta = \mathcal{H}/\ker\delta_\theta$. Judicious choices of $\mathcal{H}$ will make the intermediate space X_θ to be an interpolation space in the sense of (2) above. Abstract methods make their entrance when one replaces δ_θ by a more general *interpolator* Φ; namely, a suitable operator $\Phi : \mathcal{H} \to \Sigma$ such that $X_\Phi = \Phi[\mathcal{H}]$ (endowed with the quotient norm) is an interpolation space. See the general setting for this abstract approach in [32] and the categorical approach in [85].

In this survey we will focus, half of the time at least, on Calderón complex interpolation method for pairs of Banach spaces [6,19] combined with Schechter's approach [99]. A certain bias towards abstract methods and some occasional appreciations of what other methods could yield, or even what could happen when families (instead of pairs) are considered, will be either desirable or unavoidable.

2.2. *The basics of complex interpolation*

Let us briefly describe the complex interpolation method for couples, following a combination of [6,73]. Let $\mathbb{S}$ denote the open strip $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ in the complex plane and let $\overline{\mathbb{S}}$ be its closure. A pair $(X_0, X_1, \imath, \jmath)$ of complex Banach spaces together with injective operators $\imath : X_0 \longrightarrow \Sigma$ and $\jmath : X_1 \longrightarrow \Sigma$ into some ambient Banach space Σ will be an interpolation pair. We will identify both X_0, X_1 with their continuous images in Σ without further mention. The Calderón space $\mathcal{C}(X_0, X_1)$ is the space of continuous bounded functions $f : \overline{\mathbb{S}} \longrightarrow \Sigma$ that are holomorphic on $\mathbb{S}$ and satisfy the following boundary condition: For $k = 0, 1$, $f(k + it) \in X_k$ for each $t \in \mathbb{R}$ and $\sup_t \|f(k+it)\|_{X_k} < \infty$. The Calderón space $\mathcal{C}(X_0, X_1)$ is complete under the norm $\|f\| = \sup\{\|f(k+it)\|_{X_k} : k = 0, 1; t \in \mathbb{R}\}$ The evaluation map $\delta_z : \mathcal{C}(X_0, X_1) \longrightarrow \Sigma$ is continuous for all $z \in \overline{\mathbb{S}}$. Given $\theta \in (0, 1)$, the *interpolation* space $X_\theta = (X_0, X_1)_\theta$ is defined as the Banach space

$$X_\theta = \{x \in \Sigma : x = f(\theta) \text{ for some } f \in \mathcal{C}(X_0, X_1)\} = \mathcal{C}(X_0, X_1)/\ker\delta_\theta$$

endowed with the quotient norm $\|x\|_{X_\theta} = \inf\big\{\|f\| : x = f(\theta),\ f \in \mathcal{C}(X_0, X_1)\big\}$.

Assume now that (X_0, X_1) and (Y_0, Y_1) are pairs of Banach spaces continuously injected in ambient spaces Σ and Σ', respectively. The fundamental property of interpolation spaces already mentioned at the beginning of this section is the following: if $T : \Sigma \to \Sigma'$ is a bounded operator such that $T|_{X_0} : X_0 \to Y_0$ and $T|_{X_1} : X_1 \to Y_1$ are bounded, then $T : X_\theta \to Y_\theta$ is bounded for any $\theta \in (0, 1)$. In [37, Section 3] it is explained a method so that, without loss of generality, one can assume that $\Sigma = \Sigma'$.

2.3. *Examples*

A few examples are of paramount importance for this survey, and all of them lie inside the category of Köthe spaces: Given a measure space (Σ, μ) we denote by L_0 the space of all μ-measurable functions. A Köthe space X is an L_∞-submodule of L_0 (in particular, a vector subspace X such that if $g \in L_0$ and $\|g\|_X < \infty$ then $g \in X$) containing the characteristic functions of measurable sets, and endowed with a norm such that if $f, g \in X$ and $|f| \le |g|$ then $\|f\|_X \le \|g\|_X$. To develop more sophisticated issues about twisted sums of Köthe spaces it will be needed, according to [71, p. 482], to additionally ask to X the existence of strictly positive functions $h, k \in L_0$ such that $\|hf\|_1 \le \|f\|_X \le \|kf\|_\infty$ for every $f \in L_0$. In particular, Banach spaces with a 1-unconditional basis with their associated ℓ_∞-module structure are Köthe spaces.

Let X, Y be two Köthe spaces on the same base space and let $0 < \theta < 1$. We can form the following quasi Banach spaces:

- $X^\theta = \{f \in L_0 : |f|^{1/\theta} \in X\}$ endowed with the norm $\|x\|_\theta = \||x|^{1/\theta}\|^\theta$.
- $XY = \{f \in L_0 : f = xy \text{ for some } x \in X, y \in Y\}$ endowed with the quasinorm $\|f\|_{XY} = \inf\{\|x\|_X\|y\|_Y : f = xy,\ x \in X, y \in Y\}$. According to [12, Lemma 2], if X, Y are Banach spaces then XY is (isomorphic to) a 1/2-Banach space.
- If X, Y are Banach spaces, the Banach space $X^{1-\theta}Y^\theta$ is therefore formed by the functions $f \in L_0$ such that $f = ab$ for some $a \in X^{1-\theta}, b \in Y^\theta$, so that $|f| = |x|^{1-\theta}|y|^\theta$ for $x = a^{1/1-\theta}$ and $y = b^{1/\theta}$. The space is thus endowed with the norm $\|f\| = \inf\{\|x\|_X^{1-\theta}\|y\|_Y^\theta : x \in X, y \in Y, |f| = |x|^{1-\theta}|y|^\theta\}$.

The pair (X_0, X_1) will be called *regular* in the terminology of Cwikel, if $X_0 \cap X_1$ is dense in both X_0 and X_1. Shestakov proved in [103] that $[X_0, X_1]_\theta = \overline{X_0 \cap X_1} \subset X_0^{1-\theta}X_1^\theta$ and Kalton [71, formula (3.2)] already observed that complex interpolation for regular couples and factorization are the same, thanks to Lozanovskii decomposition formula (see [73, Theorem 4.6]); precisely,

$$(X_0, X_1)_\theta = X_0^{1-\theta}X_1^\theta. \tag{1}$$

A fundamental example appears when one considers the regular pair (L_∞, L_1):

$$(L_\infty, L_1)_\theta = L_{\theta^{-1}}.$$

This is, in its complex interpolation version, the classical Riesz–Thorin theorem [6, Theorem 1.1.1], consequence of the Three-lines theorem [6, Lemma 1.1.2]. Still, this example can be seen as follows. Recall that the p-convexification of a Köthe function space X is defined by to be the space $X_{(p)}$ endowed with the norm $|||x||| = \||x|^p\|^{1/p}$. Conversely, when X is p-convex, the p-concavification of X is given by $|||x||| = \||x|^{1/p}\|^p$. Thus, if $0 < \theta < 1$, $1 < p < \infty$ and X is a Köthe space then

$$(L_\infty(\mu), X)_\theta = X_{(\theta^{-1})} \tag{2}$$

is the θ^{-1}-convexification of X. Conversely if X is p-convex and $X^{(p)}$ is the p-concavification of X then $X = (L_\infty(\mu), X^{(p)})_{1/p}$.

To present our second basic example, recall that a *weight* w is a positive function in $L_0(\mu)$. Given a Köthe function space X of μ-measurable functions, we denote by $X(w)$ the space of all measurable scalar functions f such that $wf \in X$, endowed with the norm $\|x\|_w = \|wx\|_X$. We get [6, Theorem 5.4.1] that if X is a Köthe space with the Radon–Nikodym property and w_0, w_1 are two weights then, for $0 < \theta < 1$, one has

$$(X(w_0), X(w_1))_\theta = X(w_0^{1-\theta} w_1^\theta) \tag{3}$$

Still intermediate between the special case of L_p-spaces and the general one of Köthe spaces is the case of Orlicz spaces (see [57,79,81,86] for details and the general theory). Recall that an *N-function* is a map $\varphi : [0, \infty) \to [0, \infty)$ which is strictly increasing, continuous, $\varphi(0) = 0$, $\varphi(t)/t \to 0$ as $t \to 0$, and $\varphi(t)/t \to \infty$ as $t \to \infty$. An N-function φ satisfies the Δ_2*-property* if there exists a number $C > 0$ such that $\varphi(2t) \le C\varphi(t)$ for all $t \ge 0$. When an N-function φ satisfies the Δ_2-property, the Orlicz space is $L_\varphi(\mu) = \{f \in L_0(\mu) : \varphi(|f|) \in L_1(\mu)\}$. endowed with the norm $\|f\| = \inf\{r > 0 : \int \varphi(|f|/r)d\mu \le 1\}$. It was proved in [63] (see also [20]) that if φ_0 and φ_1 are two N-functions satisfying the Δ_2-property, and let $0 < \theta < 1$, then the formula $\varphi^{-1} = \left(\varphi_0^{-1}\right)^{1-\theta}\left(\varphi_1^{-1}\right)^\theta$ defines an N-function φ satisfying the Δ_2-property, and

$$\left(L_{\varphi_0}(\mu), L_{\varphi_1}(\mu)\right)_\theta = L_\varphi(\mu). \tag{4}$$

Observe that the function $\varphi(t) = t^p$ for $1 < p < \infty$ defines an N-function satisfying the Δ_2-property and the corresponding Orlicz spaces are the usual L_p and ℓ_p spaces.

On the farthest shore of Banach spaces with unconditional basis we encounter the *family* of Schreier-like and Tsirelson-like Banach spaces. The Schreier space [101] is constructed as follows: call a finite subset $A \subset \mathbb{N}$ *admissible* if $|A| \le \min A$ and consider the family $\mathcal{A} \subset \{0, 1\}^{\mathbb{N}}$ of all finite admissible subsets of $\mathbb{N}$. Given $A \in \mathcal{A}$ and $x = (x_n)_n$ a sequence with finite support, denote by Ax the sequence formed by the elements of x having coordinates on A. The space $\mathcal{S}$ is the completion of the space of finitely supported sequence under the norm $\|x\|_{\mathcal{S}} = \sup_{A\in\mathcal{A}} \|Ax\|_1$. The canonical unit vectors form a shrinking basis for $\mathcal{S}$. The Tsirelson space $\mathcal{T}$ [109] was the first example of reflexive Banach space with no copies of c_0 or ℓ_p for $1 \le p < \infty$. A comprehensive treatment of Tsirelson space can be found in [27]. Given any finite subsets $E, F \subset \mathbb{N}$ denote by $E < F$ that $\max E < \min F$, and define as before $Ex = \sum_{n\in E} x_n$ for any $x = \sum_n x_k \in c_{00}$. The norm of Tsirelson space is defined inductively: one fixes $x \in c_{00}$

and consider the following sequence of norms $(\|\cdot\|_m)_{m=0}^{\infty}$

$$\begin{cases} \|x\|_0 = \|x\|_{c_0} \\ \\ \|x\|_{m+1} = \max\Big\{\|x\|_m, \frac{1}{2}\max\Big[\sum_{j=1}^k \|E_j x\|_m\Big]\Big\}, \quad \text{for } m \geq 0, \end{cases} \tag{5}$$

where the inner maximum is defined over all possible choices of finite subsets $E_1, \dots, E_k$ such that $k \leq E_1 < E_2 < \cdots < E_k$.

The norms thus defined are increasing in m, $\|x\|_{m+1} \geq \|x\|_m$, and bounded above by the ℓ_1 norm. Therefore the limit $\|x\|_{\mathcal{T}} := \lim_{m\to\infty} \|x\|_m$ exists for every $x \in c_{00}$ and defines a norm on the space of finitely supported sequences. The Tsirelson space $\mathcal{T}$ is the completion of c_{00} with the norm $\|\cdot\|_{\mathcal{T}}$. It is customary to say that, in fact, this definition is due to Figiel and Johnson [61] and describes the *dual space* of the original space obtained by Tsirelson. The canonical unit vectors form a 1-unconditional basis for $\mathcal{T}$. Moreover, the norm $\|\cdot\|_{\mathcal{T}}$ satisfies the following equation:

$$\|x\|_{\mathcal{T}} = \max\Big\{\|x\|_{c_0}, \frac{1}{2}\sup\Big[\sum_{j=1}^k \|E_j x\|_{\mathcal{T}}\Big]\Big\}, \tag{6}$$

where the supremum is taken over all choices $k \leq E_1 < \cdots < E_k$ with $k \in \mathbb{N}$. Thus, in contrast to all spaces considered up to this point, the norm of Tsirelson space is *implicitly defined* – it has been recently proved [25] that an explicit finitary definition is not possible –. The importance of Tsirelson space extends beyond these considerations. As Casazza and Shura state already in the preface of [27] about $\mathcal{T}$: "His example opened a Pandora's box of pathological variations, and has had a tremendous effect upon the study of Banach spaces". And such it is indeed the case: its fruitful modifications include, among others, the Schlumprecht space [100], which is the first arbitrarily distortable space that led to the discovery of Hereditarily Indecomposable spaces by Gowers and Maurey [62].

What is important for us is that due to reasons to be explained in the next section

$$(\mathcal{S}, \mathcal{S}^*)_{1/2} = \ell_2$$
$$(\mathcal{T}, \mathcal{T}^*)_{1/2} = \ell_2.$$

Tsirelson and Schreier spaces admit, of course, p-convexifications, and we will be especially interested in their 2-convexifications $\mathcal{T}_2$ and $\mathcal{S}_2$. Another interesting variation is the symmetrization of Tsirelson space. Recall that given a Banach space X with a basis $(e_n)_n$ there is a standard procedure to construct another Banach space having $(e_n)_n$ as a symmetric basis. Such space is defined, for any finitely supported vector $x = \sum_n a_n e_n$, as the completion of c_{00} under the norm

$$\|x\|_{S(X)} = \sup_{\sigma\in\sigma_\infty} \Big\| \sum_n |a_n| e_{\pi(n)} \Big\|_X$$

where σ_∞ denotes the set of all permutations of $\mathbb{N}$. The symmetrization of $\mathcal{T}_2$, denoted $\mathcal{T}_2^s$, was constructed by Casazza and Nielsen in [26] and is a reflexive Banach space with symmetric basis that contains neither ℓ_p nor c_0 [26,27]. One still has $(\mathcal{T}_2^s, (\mathcal{T}_2^s)^*)_{1/2} = \ell_2$.

2.4. Hilbert spaces

As we see, ℓ_2, the space of sequences $x = (x_n)$ of scalars such that $\|x\| = \left(\sum |x_n|^2\right)^{1/2}$ plays an important role in interpolation affairs. This is, up to isometries, the only infinite-dimensional separable Hilbert space. A Hilbert space is a complete normed space whose norm $\|\cdot\|$ comes induced by an inner product $(\cdot,\cdot)$ in the form $\|x\| = (x, x)^{1/2}$. The orthogonal projection (which should not be linear, but it is) provides a contractive projection onto every closed subspace. Infinite dimensional Hilbert spaces enjoy the following properties: They are reflexive, ℓ_2-saturated and isomorphic to its dual; they have type and cotype 2, all its subspaces are complemented, have a (generalized) unconditional basis and they only admit one type of basic sequence. Moreover, they are symplectic, in the following sense: A real Banach space X is said to be *symplectic* if there is a continuous alternating bilinear map $\omega : X \times X \to \mathbb{R}$ such that the induced map $L_\omega : X \to X^*$ given by $L_\omega(x)(y) = \omega(x, y)$ is an isomorphism onto. A symplectic Banach space is necessarily isomorphic to its dual and reflexive. The simplest symplectic structure is the obvious one on $Y \oplus_2 Y^*$ for Y a reflexive space; namely

$$\omega_Y[(z, z^*), (w, w^*)] = w^*(z) - z^*(w).$$

A symplectic Banach space (X, ω) is said *trivial* if there exist a reflexive Banach space Y and an isomorphism $T : X \to Y \oplus Y^*$ such that $\omega(x, y) = \omega_Y(Tx, Ty)$ for every $x, y \in X$. Thus, Hilbert spaces admit a symplectic but trivial structure.

Hilbert spaces are easily obtainable by complex interpolation and, in particular, ℓ_2 is easily obtainable by complex interpolation of Banach spaces with basis. The general idea is that complex interpolation between a Banach space and its dual gives, in many cases, a Hilbert space (see e.g., the comments at [95, around Theorem 3.1]). More precisely, Watbled shows [38,110] that if X is a Köthe function space on a complete σ-finite measure space S and we assume both that $X \cap X^*$ is dense in X and that $L_1(S)\cap L_\infty(S) \subset X\cap X^* \subset L_2(S) \subset X+X^* \subset L_1(S)+L_\infty(S)$ then $(X, X^*)_{1/2} = L_2(S)$. More precisely, given a Banach space X, we denote by $\overline{X}^*$ its *anti-dual*, namely, the dual space X^* endowed with the conjugate scalar multiplication $\alpha \cdot x = \overline{\alpha}x$. There are several results that yield a Hilbert space H as interpolated space of two Banach spaces. A typical situation occurs when $H \hookrightarrow X$ is continuously and densely embedded in X or, conversely, if $X \hookrightarrow H$ embeds continuously and densely in H (see [49,110]). In this case, $(X, \overline{X}^*)_{1/2} = H$, i.e, the interpolation of X and its anti-dual in $1/2$ produces the Hilbert space H. We state this explicitly (cf. [38, Prop. 6.1]):

Proposition 2.1. *Let X be a Banach space and H a Hilbert space. Suppose that either $X \hookrightarrow H$ or $H \hookrightarrow X$ define a densely continuous embedding. Then $(X, \overline{X}^*)_{1/2} = H$. In particular, if X is a Banach space with a monotone and shrinking basis then $(X, \overline{X}^*)_{1/2} = \ell_2$ with equality of norms.*

2.5. Reiteration for the complex method

We need to recall the classical reiteration theorem for the complex method, since it will play an important role in our exposition.

Proposition 2.2. *Let (X_0, X_1) be a pair of Banach spaces and consider, for $0 \leq \alpha \leq \beta \leq 1$, the interpolated spaces $(X_0, X_1)_\alpha$ and $(X_0, X_1)_\beta$. Then, for any $\alpha \leq \theta \leq \beta$, the interpolation identity*

$$\Big((X_0, X_1)_\alpha, (X_0, X_1)_\beta\Big)_\theta = (X_0, X_1)_{(1-\theta)\alpha+\theta\beta} \tag{7}$$

holds with equality of norms.

This was proved by Calderón in [19, 32.3] under supplementary hypothesis (see also [6, Th. 4.6.1]) and in full generality by Cwikel [54].

3. Exact sequences of Banach spaces and complex interpolation

For background on the theory of twisted sums and diagrams and their interaction with modern Banach space theory we refer to [8].

3.1. Exact sequences of Banach spaces

An exact sequence $0 \to A \to B \to C \to 0$ of Banach spaces and continuous operators is a diagram in which the kernel of each arrow coincides with the image of the preceding one. The simplest exact sequence is obtained taking $B = A \oplus_\infty C$ with embedding $y \to (y, 0)$ and quotient map $(y, x) \to x$. Two exact sequences $0 \to A \to B_1 \to C \to 0$ and $0 \to A \to B_2 \to C \to 0$ are said to be *equivalent* if there exists an operator $T : B_1 \to B_2$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \stackrel{\imath}{\longrightarrow} & B_1 & \stackrel{\rho}{\longrightarrow} & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & B_2 & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

is commutative. The classical algebraic 3-lemma (see [8]) shows that T must always be an isomorphism. An exact sequence is said to be *trivial*, or that it *splits*, if it is equivalent to $0 \to A \to A \oplus_\infty C \to C \to 0$. An exact sequence $0 \to A \to B \to C \to 0$ splits if and only if $\imath[A]$ is complemented in B.

Given an exact sequence $0 \to A \to B \to C \to 0$ and an operator $\tau : A \to A'$ one can form a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \stackrel{\imath}{\longrightarrow} & B & \stackrel{\rho}{\longrightarrow} & C & \longrightarrow & 0 \\ & & {\scriptstyle \tau}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A' & \underset{\overline{\imath}}{\longrightarrow} & \mathrm{PO} & \underset{\overline{\rho}}{\longrightarrow} & C & \longrightarrow & 0 \end{array}$$

in which $\mathrm{PO} = (A' \oplus_1 B)/\Delta$ with $\Delta = \{(\tau a, -\imath a) : a \in A\}$ is the *pushout* space, $\overline{\imath}(a') = (a', 0) + \Delta$ and $\overline{\rho}((a', b) + \Delta) = \rho b$. Since PO is a categorical object, any diagram having the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

is actually a pushout diagram, with the meaning that the lower sequence is equivalent to the pushout lower sequence in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A' & \longrightarrow & \mathrm{PO} & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Two equivalent sequences yield equivalent pushout sequences.

3.2. Twisted sums of Banach spaces

A *twisted sum of two Banach spaces* Y, X is a quasi-Banach space Z having a closed subspace isomorphic to Y and such that the quotient Z/Y is isomorphic to X. Therefore, a twisted sum Z of Y and X yields an exact sequence $0 \to Y \to Z \to X \to 0$ and, conversely, the open mapping theorem yields that the middle space Z in an exact sequence $0 \to Y \to Z \to X \to 0$ is a twisted sum of Y and X.

Kalton [68] developed first a theory of twisted sums of quasi Banach spaces through *quasi-linear* maps and later [70,71,75] for twisted sums for Köthe spaces through a special type of quasi-linear maps called *centralizers* (see below their definition). We briefly outline the fundamentals of the dictionary: A map $\Omega : X \to Y$ is called *quasi-linear* if it is homogeneous and there is a constant M such that $\|\Omega(u+v)-\Omega(u)-\Omega(v)\| \leq M\|u+v\|$ for all $u, v \in X$. A quasi-linear map $\Omega : X \to Y$ induces the exact sequence $0 \to Y \xrightarrow{\jmath} Y \oplus_\Omega X \xrightarrow{\pi} X \to 0$ in which $Y \oplus_\Omega X$ denotes the vector space $Y \times X$ endowed with the quasi-norm $\|(y, z)\|_\Omega = \|y - \Omega(z)\| + \|z\|$. The embedding is $\jmath(y) = (y, 0)$ while the quotient map is $\pi(y, z) = z$. Exact sequences $0 \to Y \to Z \to X \to 0$ of Banach spaces correspond to a special type of quasi-linear maps, called z-linear [14] or 1-linear [8], which are those satisfying $\|\Omega(\sum_{i=1}^n u_i) - \sum_{i=1}^n \Omega(u_i)\| \leq M \sum_{i=1}^n \|u_i\|$ for all finite sets $u_1, \ldots, u_n \in X$. Thus, when F is z-linear, the quasinorm above is equivalent to a norm [39, Chapter 1]. On the other hand, the process to obtain a z-linear map out from an exact sequence $0 \to Y \xrightarrow{\imath} Z \xrightarrow{\rho} X \to 0$ of Banach spaces is as follows: obtain a homogeneous bounded selection $b : X \to Z$ for π and then a linear selection $\ell : X \to Z$ for ρ. Then $\Omega = b - \ell$ is a z-linear map. The commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\imath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow[\jmath]{} & Y \oplus_\omega X & \xrightarrow[\pi]{} & X & \longrightarrow & 0 \end{array}$$

is obtained by setting $T : Z \to Y \oplus_\omega X$ as $T(x) = (x - \ell\rho x, \rho x)$.

Two quasi-linear maps $F, F' : X \to Y$ are said to be equivalent, something that will be denoted $F \equiv G$, if the difference $F - F'$ can be written as $B + L$, where $B : X \to Y$ is a homogeneous bounded map (not necessarily linear) and $L : X \to Y$ is a linear map (not necessarily bounded). Of course two quasi-linear maps are equivalent if and only if the associated exact sequences are equivalent.

Let X be a Köthe function space. A *centralizer on* X is a homogeneous map $\Omega : X \to L_0(\mu)$ for which there is a constant $C(\Omega)$ such that, given $f \in L_\infty(\mu)$ and $x \in X$, $\Omega(fx) - f\Omega(x) \in X$ and $\|\Omega(fx) - f\Omega(x)\|_X \le C(\Omega)\|f\|_\infty \|x\|_X$. A centralizer Ω on X generates an exact sequence

$$0 \longrightarrow X \xrightarrow{\jmath} X \oplus_\Omega X \xrightarrow{\pi} X \longrightarrow 0$$

where $X \oplus_\Omega X = \{(f,x) \in L_0 \times X : f - \Omega x \in X\}$ endowed with the quasi-norm $\|(f,x)\|_\Omega = \|f - \Omega x\|_X + \|x\|_X$, with inclusion $\jmath(y) = (y, 0)$ and quotient map $\pi(f,x) = x$. A centralizer is *trivial* if the exact sequence its induces is trivial. Namely, a centralizer $\Omega : X \to L_0(\mu)$ is trivial if and only if there exists a linear map $L : X \to L_0(\mu)$ such that $\Omega - L$ is a bounded map from X to X. Two centralizers Ω and Ω' defined on X are *boundedly equivalent* if $\Omega - \Omega'$ is bounded as map from X to X.

3.3. Differentials of interpolation processes

Interpolation methods do not provide by themselves twisted sums, in general, since there is not much that can be done with just one interpolator. One needs at least two (nevertheless, standard interpolation methods do not just provide one interpolator, but two or infinitely many). In [32] we developed an abstract theory to cover that case, and we briefly reproduce it here. Let (Ψ, Φ) be a couple of interpolators defined on a space $\mathcal{H}$ obtained from a pair of spaces (X_0, X_1). We consider them in that order: first Φ, then Ψ. So we first consider the interpolation space $\Phi[\mathcal{H}]$. Then, we can form the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\Phi & \longrightarrow & \mathcal{H} & \xrightarrow{\Phi} & \Phi[\mathcal{H}] & \longrightarrow & 0 \\ & & {\scriptstyle\Psi}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Psi[\ker\Phi] & \longrightarrow & \mathrm{PO} & \longrightarrow & \Phi[\mathcal{H}] & \longrightarrow & 0 \end{array} \tag{8}$$

in which we see that PO is a twisted sum of $\Psi[\ker\Phi]$ and $\Phi[\mathcal{H}]$. Precisely, if B_Φ denotes a homogeneous bounded selection for Φ then the lower sequence is generated by the quasilinear map $\Omega_{\Psi,\Phi} = \Psi B_\Phi$ so that $\mathrm{PO} = \Psi[\ker\Phi] \oplus_{\Omega_{\Psi,\Phi}} \Phi[\mathcal{H}]$.

Definition 3.1. The map $\Omega_{\Psi,\Phi}$ will be called the *differential* of the interpolation process generated by the two interpolators (Ψ, Φ), and the space $d\Omega_{\Psi,\Phi} = \Psi[\ker\Phi] \oplus_{\Omega_{\Psi,\Phi}} \Phi[\mathcal{H}]$ will be called the associated *derived* space.

The following problem is simultaneously deep and fuzzy:

Problem 1. Does an interpolation method provide derived twisted sum spaces?

3.4. Differentials of complex interpolation processes

If we focus on the complex interpolation method for pairs as described above (in which $\mathcal{H} = \mathcal{C}(X_0, X_1)$ is the associated Calderón space), the first interpolator is the evaluation map $\delta_\theta : \mathcal{C}(X_0, X_1) \longrightarrow \Sigma$ and $\delta_\theta[\mathcal{C}(X_0, X_1)] = X_\theta$. Schechter [99] observed that the evaluation of the nth derived function map $\delta_\theta^{(n)} : \mathcal{C}(X_0, X_1) \longrightarrow \Sigma$ provided new significative interpolators. The meaning of "significative" here can be condensed in the following cornerstone result (see [22]):

Proposition 3.2. $\delta'_\theta : \ker \delta_\theta \longrightarrow X_\theta$ *is bounded and onto for* $0 < \theta < 1$.

Proof. The crucial property we require from $\mathcal{C}(X_0, X_1)$ is that if $\varphi : \mathbb{S} \longrightarrow \mathbb{D}$ is a conformal equivalence vanishing at θ, then $\ker \delta_\theta = \varphi \cdot \mathcal{C}(X_0, X_1)$, in the sense that every $f \in \mathcal{C}(X_0, X_1)$ vanishing at θ has a factorization $f = \varphi\, g$, with $g \in \mathcal{C}(X_0, X_1)$ and $\|g\| = \|f\|$. If $f \in \ker \delta_\theta$ and we write $f = \varphi\, g$ then $f' = \varphi' g + \varphi g'$ and therefore $\delta'_\theta(f) = \varphi'(\theta)\delta_\theta(g)$, hence $\|\delta'_\theta : \ker \delta_\theta \longrightarrow X_\theta\| \leq |\varphi'(\theta)|$. That δ'_θ maps $\ker \delta_\theta$ onto X_θ is also clear: if $x \in X_\theta$, then $x = g(\theta)$ for some $f \in \mathcal{C}(X_0, X_1)$ and x is then the derivative of $\varphi(\theta)^{-1}\varphi\, f$ at θ. $\square$

Consequently, for each $\theta \in (0, 1)$ there is a pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \delta_\theta & \longrightarrow & \mathcal{C}(X_0, X_1) & \xrightarrow{\delta_\theta} & X_\theta & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta'_\theta} & & \downarrow & & \| & & \\ 0 & \longrightarrow & X_\theta & \longrightarrow & \mathrm{PO} & \longrightarrow & X_\theta & \longrightarrow & 0 \end{array} \tag{9}$$

and therefore whenever X is isomorphic to a space $X_\theta = (X_0, X_1)_\theta$ obtained by complex interpolation applied to a pair (X_0, X_1) there is always an exact sequence $0 \longrightarrow X \longrightarrow Z \longrightarrow X \longrightarrow 0$, more precisely

$$0 \longrightarrow X_\theta \longrightarrow X_\theta \oplus_{\delta'_\theta B_\theta} X_\theta \longrightarrow X_\theta \longrightarrow 0 \tag{10}$$

where B_θ is a homogeneous bounded selection for δ_θ. The exact sequence is or not trivial depending on whether the quasilinear map $\Omega_\theta = \delta'_\theta B_\theta$ can be approximated by a linear map.

3.5. Twisted Hilbert spaces

A twisted Hilbert space is a Banach space that is a twisted sum of two Hilbert spaces. Namely, the middle space Z in an exact sequence $0 \longrightarrow H \longrightarrow Z \longrightarrow H' \longrightarrow 0$ in which H and H' are Hilbert spaces. Things turn out spicy when it was shown that there exist twisted Hilbert spaces not isomorphic to Hilbert spaces. This was first proved by Enflo, Lindenstrauss and Pisier [58], with the following construction: they obtain quasi-linear map $g_n : \ell_2^n \to \ell_2^{2^n}$ increasingly (in n) far from linear maps. With these, they constructed the (finite dimensional) spaces $\ell_2^{2^n} \oplus_{g_n} \ell_2^n$ to then form the space $\ell_2\left(\ell_2^{2^n} \oplus_{g_n} \ell_2^n\right)$. Thus, $\mathsf{ELP} = \ell_2(\mathbb{N}, F_n)$ for certain finite dimensional spaces F_n. However, the construction of paramount importance to our purposes is that of the Kalton–Peck space Z_2 presented in [75]. And it is so because the space Z_2 was constructed obtaining a quasi-linear maps $\mathsf{KP} : \ell_2 \to \ell_2$ that cannot be approximated by linear maps. This map, called the Kalton–Peck map, is

$$\mathsf{KP}(x) = 2x \log \frac{|x|}{\|x\|}$$

The maverick argument that makes all that work is that, since $\mathsf{KP}(e_n) = 0$, the only linear map that can be close to KP is one having the form $L(e_n) = \lambda_n e_n$ for a certain bounded sequence (λ_n): The idea is averaging an arbitrary linear map L that is at finite distance from Ω to get a new linear map L' at the same distance from Ω as L and such that $L'(\varepsilon x) = \varepsilon L'(x)$ for every $\varepsilon \in \{-1, +1\}^{\mathbb{N}}$. It is straightforward that a linear map with that property must have the form $L'(x) = (\lambda_n x_n)$ for some sequence λ (see [8,22] or else [38] for details). Done that, the rest is simple: $\|\mathsf{KP}(\sum^N e_n) - L(\sum^N e_n)\| \leq M\sqrt{N}$ is mandatory for some $M < +\infty$: but $\|L(\sum^N e_n)\| = \|\sum \lambda_n e_n\| \leq C\sqrt{N}$ since (λ_n) is bounded while $\|\mathsf{KP}(\sum^N e_n)\| \sim N\sqrt{N}$ so no bound M is possible.

Why this map is interesting? Because is the differential map that corresponds to the pair (ℓ_∞, ℓ_1). Indeed, a selector for the interpolator $\delta_{1/2}$ acting on the pair (ℓ_∞, ℓ_1) can be easily given as follows: assume x is positive and $\|x\|_2 = 1$ and define $B(x)(z) = x^{2z} \in \mathcal{C}(\ell_\infty, \ell_1)$. Therefore, an associated differential is $B(x)'(1/2) = 2x \log x$ which, after suitable homogenization, yields $\Omega(x) = \mathsf{KP}(x)$.

Observe that an exact sequence $0 \longrightarrow \ell_2 \longrightarrow Z \longrightarrow \ell_2 \longrightarrow 0$ splits if and only if Z is a Hilbert space. Equivalently, $\ell_2 \oplus_F \ell_2$ is (isomorphic to) a Hilbert space if and only if F is at finite distance from a linear map. Thus, a way for obtaining nontrivial sequences as above is to show that Z is not a Hilbert space. This makes important to determine which properties twisted Hilbert spaces enjoy. Are twisted Hilbert spaces similar to Hilbert spaces in some sense or can they be very different? Both. Obviously, all 3-space properties enjoyed by Hilbert spaces, such as superreflexivity, near Hilbert etc (see [39]) are shared by all twisted Hilbert spaces. On the other hand, there are obvious properties that no twisted Hilbert space can have: those implying that copies of ℓ_2 are complemented, such as Maurey extension property. In between, one encounters

- Properties of Hilbert spaces that some nontrivial twisted Hilbert enjoy but that we do not know whether every twisted Hilbert space enjoys, such as: (1) to be isomorphic to its dual, (2) to be isomorphic to its square; (3) to be isomorphic to its hyperplanes; (4) to admit complex structures [34]; (5) to be a weak Hilbert space; (6) to be ergodic [60].
- Properties that we do not know if can be possessed by a nontrivial non-Hilbert twisted Hilbert space, such as to have GL-l.u.st. [72] (see below) or to be prime (i.e. isomorphic to its infinite dimensional complemented subspaces).
- Properties P for which it is known that a twisted Hilbert space with P must be Hilbert, such as to have unconditional basis (proved by Kalton [72]); to have type 2 or to have cotype 2 in either case the canonical copy of ℓ_2 is complemented by Maurey extension theorem.

For instance, the Kalton–Peck space Z_2 has properties (1, 2, 4) [34,75] but not (5) [39, pp. 95]; whether Z_2 has property (3) is a long-standing problem (see Section 6.6). The Enflo–Lindenstrauss–Pisier space has property (3), not (5) and we do not know about the rest. The twisted Hilbert constructed by Suárez in [106] has property (5) and, thus, has (6) by a result of Anisca [4]. We shall review this example in Section 9. Kalton and Swanson [76] solved the problem raised by Weinstein [111] by showing Z_2 is a symplectic space with a nontrivial structure and we shall return later to this topic. We shall explain later that all twisted Hilbert spaces generated with a centralizer are isomorphic to their duals. More precisely: if X is a Köthe space and $(X, X^*)_{1/2} = H$ is a Hilbert space with associated differential $\Omega_{1/2}$ then the twisted Hilbert space $H \oplus_{\Omega_{1/2}} H$ is isomorphic to its dual.

Problem 2. Is every twisted Hilbert space isomorphic to its dual?

The space ELP is the only twisted Hilbert space in sight for which we do not know if it can be generated with a centralizer, so it makes sense to ask:

Problem 3. Is ELP isomorphic to its dual?

We refer the reader to [24] for an authoritative account of approximation properties in Banach spaces. We will need the Bounded Approximation Property (BAP), the Finite Dimensional Decomposition (FDD) (there exist a sequence of closed subspaces $(X_n)_n$ of X such that any $x \in X$ can be represented uniquely as $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ for all $n \in \mathbb{N}$) and the basis; as well as their unconditional variations: Unconditional Finite Dimensional Decomposition (UFDD) if, for every $x \in X$, the expansion $x = \sum_{n=1}^{\infty} x_n$ in the FDD converges unconditionally and the notion of unconditional basis. Note that a (unconditional) basis for X is just an (unconditional) FDD where $\dim(X_n) = 1$ for all $n \in \mathbb{N}$.

All twisted Hilbert spaces generated with a centralizer have an unconditional 2-FDD formed by the subspaces $F_n = [(e_n, 0), (0, e_n)]$ (in [106] is already observed that the vectors $(0, e_n)_n$ form an unconditional basis for its closed span): pick a finite sequence of elements $f_n = (u_n, v_n) \in F_n$ and pick $\varepsilon_n \in \{-1, +1\}$. One has:

$$
\begin{aligned}
\left\|\sum \varepsilon_n\lambda_n f_n\right\| &= \left\|\left(\sum \varepsilon_n\lambda_n u_n, \sum \varepsilon_n\lambda_n v_n\right)\right\| \\
&= \left\|\sum \varepsilon_n\lambda_n u_n e_n - \Omega\left(\sum \varepsilon_n\lambda_n v_n e_n\right)\right\| + \left\|\sum \varepsilon_n\lambda_n v_n e_n\right\| \\
&= \Big\| \sum \varepsilon_n\lambda_n u_n e_n - (\varepsilon_n)\cdot\Omega(\sum \lambda_n v_n e_n) \\
&\qquad + (\varepsilon_n)\cdot\Omega(\sum \lambda_n v_n e_n) - \Omega\left((\varepsilon_n)\cdot(\sum \lambda_n v_n e_n)\right) \Big\| \\
&\qquad + \left\|\sum \lambda_n v_n e_n\right\| \\
&\le \left\|(\varepsilon_n)\cdot\sum \lambda_n u_n e_n - (\varepsilon_n)\cdot\Omega(\sum \lambda_n v_n e_n)\right\| \\
&\qquad + \left\|(\varepsilon_n)\cdot\Omega(\sum \lambda_n v_n e_n) - \Omega\left((\varepsilon_n)\cdot(\sum \lambda_n v_n e_n)\right)\right\| \\
&\qquad + \left\|\sum \lambda_n v_n e_n\right\| \\
&\le \left\|\sum \lambda_n u_n e_n - \Omega(\sum \lambda_n v_n e_n)\right\| + C(\Omega)\left\|\sum \lambda_n v_n e_n\right\| \\
&\qquad + \left\|\sum \lambda_n v_n e_n\right\| \\
&\le (C(\Omega)+1)\left\|\sum \lambda_n f_n\right\|.
\end{aligned}
$$

Consequently, twisted Hilbert spaces generated with a centralizer have a basis (in particular, Z_2 has a basis [75]). The space ELP has BAP since $\mathsf{ELP} = \ell_2(\mathbb{N}, F_n)$. It makes once again sense to ask:

Problem 4. Does every twisted Hilbert space have the BAP?

3.6. Derived spaces and differentials

We are ready to resume the construction of the derived space (cf. [22, Proposition 7.2] and [38, Proposition 3.2]).

Proposition 3.3.

$$\begin{aligned} d_{\Omega_\theta} X_\theta &= X_\theta \oplus_{\Omega_\theta} X_\theta \\ &= \{(w, x) \subset \Sigma \times X_\theta : w - \Omega_\theta x \in X_\theta\} \\ &\simeq \{(f'(\theta), f(\theta)) : f \in \mathcal{C}(X_0, X_1)\}. \end{aligned}$$

Proof. Keep in mind that $\Omega_\theta = \delta'_\theta B_\theta$. Given $f \in \mathcal{C}(X_0, X_1)$, since $f - B_\theta(f(\theta)) \in \ker \delta_\theta$, by Proposition 3.2 we have

$$f'(\theta) - \Omega_\theta(f(\theta)) = \delta'_\theta(f - B_\theta(f(\theta))) \in X_\theta,$$

hence $\big(f'(\theta), f(\theta)\big) \in X_\theta \oplus_{\Omega_\theta} X_\theta$. Conversely, if $(w, x) \in X_\theta \oplus_{\delta'_\theta B_\theta} X_\theta$ then $B_\theta x \in \mathcal{C}(X_0, X_1)$. Since $w - \Omega_\theta x \in X_\theta$, there exists $g \in \ker \delta_\theta$ such that $w - \Omega_\theta x = g'(\theta)$. Thus, picking $f = B_\theta x + g$ we obtain $f(\theta) = x$ and $f'(\theta) = w$. To prove the equivalence of quasinorms, pick $(w, x) \in X_\theta \oplus_{\Omega_\theta} X_\theta$ and $f \in \mathcal{C}(X_0, X_1)$ with $\|f\| \leq 2\mathrm{dist}(f, \ker \delta_\theta \cap \ker \delta'_\theta)$ such that $w = f'(\theta)$ and $x = f(\theta)$. Then $\|x\|_{X_\theta} = \mathrm{dist}(f, \ker \delta_\theta)$ and

$$\|w - \Omega_\theta x\|_{X_\theta} = \|\delta'_\theta(f - B_\theta x)\|_{X_\theta}.$$

Since $f - B_\theta x \in \ker \delta_\theta$, we get

$$\begin{aligned} \|(w, x)\|_d &\leq \|\delta'_{\theta\,|\ker \delta_\theta}\|(1 + \|B_\theta\|)\|f\| + \|f\| \\ &\leq 2(\|\delta'_{\theta\,|\ker \delta_\theta}\|(1 + \|B_\theta\|) + 1)\mathrm{dist}(f, \ker \delta_\theta \cap \delta'_\theta), \end{aligned}$$

and there exists a constant C so that $\mathrm{dist}(f, \ker \delta_\theta \cap \delta'_\theta) \leq C\|(w, x)\|$ by the open-mapping theorem. □

And since ideas are more powerful that realizations, any Banach space X such that $(X, X^*)_{1/2}$ is a Hilbert space generates a differential $\Omega_{1/2}$ and then a twisted Hilbert space. The following problem is a little less ambitious than Problem 2:

Problem 5. Let X be a Banach space such that $(X, X^*)_{1/2} = H$ is a Hilbert space with associated differential Ω. Is $H \oplus_\Omega H$ isomorphic to its dual?

Let us transport these ideas to the abstract setting.

3.7. The abstract setting

The abstract setting is sometimes clearer than any specific case. Let (Ψ, Φ) be a pair of interpolators on a space $\mathcal{H}$, let $\langle \Psi, \Phi \rangle : \mathcal{H} \to \Sigma \times \Sigma$ be the map defined by $\langle \Psi, \Phi \rangle f = (\Psi f, \Phi f)$, and let us write $X_\Phi = \Phi[\mathcal{H}]$ and $X_{\Psi,\Phi} = \langle \Psi, \Phi \rangle[\mathcal{H}] =$

$\{(\Psi(f), \Phi(f)) : f \in \mathcal{H}\}$, both endowed with the respective quotient norms. One thus has the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \ker\Psi\cap\ker\Phi & = & \ker\langle\Psi,\Phi\rangle & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \ker\Phi & \longrightarrow & \mathcal{H} & \xrightarrow{\Phi} & X_\Phi & \longrightarrow & 0\\
 & & {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\langle\Psi,\Phi\rangle} & & \| & & \\
0 & \longrightarrow & \Psi[\ker\Phi] & \xrightarrow[\imath]{} & X_{\Psi,\Phi} & \xrightarrow[\pi]{} & X_\Phi & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{11}$$

As we know, the differential associated to the pair (Ψ, Φ) is the quasilinear map $\Omega_{\Psi,\Phi} = \Psi B_\Phi$, where B_Φ is a homogeneous bounded selection for Φ. We have the following generalization of Proposition 3.3 (cf. [32]):

$$\begin{aligned} d\Omega_{\Psi,\Phi} &= \{(w, x) \in \Sigma \times X_\Phi : w - \Omega_{\Psi,\Phi}x \in \Psi[\ker\Phi]\} \\ &\simeq \{(\Psi(f), \Phi(f)) : f \in \mathcal{H}\} \\ &= X_{\Psi,\Phi} \end{aligned}$$

(with equivalence of quasinorms).

3.8. *Reiteration results for differential maps*

The reiteration formula (7) for the complex method has the following counterpart for differential maps:

$$\Omega_\theta = (\beta - \alpha)\Omega_{(1-\theta)\alpha+\theta\beta}.$$

In [38, Prop. 3.7] there is a proof via factorization for Köthe spaces. We cover now the general case.

Proposition 3.4. *Let (X_0, X_1) be a pair of Banach spaces and $\alpha < \theta < \beta$. Denote by Ω_t the differential associated to $(X_0, X_1)_t$ and $\widehat{\Omega}_t$ the differential associated to $\Big((X_0, X_1)_\alpha, (X_0, X_1)_\beta\Big)_t$. If $(X_0, X_1)_{(1-\theta)\alpha+\beta\theta} = \Big((X_0, X_1)_\alpha, (X_0, X_1)_\beta\Big)_\theta$ then*

$$\widehat{\Omega}_\theta = (\beta - \alpha)\Omega_{(1-\theta)\alpha+\theta\beta}.$$

Proof. Denote by $X_\alpha = (X_0, X_1)_\alpha$ and $X_\beta = (X_0, X_1)_\beta$ and note that given $f \in \mathcal{C}(X_0, X_1)$, the operator $C : \mathcal{C}(X_0, X_1) \to \mathcal{C}(X_\alpha, X_\beta)$ given by

$$C(f)(z) = f\big((1 - z)\alpha + z\beta\big)$$

is bounded with $\|C(f)\| \leq \|f\|$ (see [19, 32.3]). Consider now a homogeneous bounded selection $B : X_{(1-\theta)\alpha+\beta\theta} \to \mathcal{C}(X_0, X_1)$ for the evaluation operator $\delta_{(1-\theta)\alpha+\beta\theta} : \mathcal{C}(X_0, X_1) \to X_{(1-\theta)\alpha+\beta\alpha}$. Taking into account the reiteration identity $X_{(1-\theta)\alpha+\beta\theta} = (X_\alpha, X_\beta)_\theta$, we conclude that the map $CB : (X_\alpha, X_\beta)_\theta \to \mathcal{C}(X_\alpha, X_\beta)$, defined for each $x \in X_{(1-\theta)\alpha+\beta\theta}$ by

$$CB(x)(z) = B(x)\big((1-z)\alpha + \beta z\big),$$

is a homogeneous bounded selection for the evaluation map $\widehat{\delta_\theta} : \mathcal{C}(X_\alpha, X_\beta) \to (X_\alpha, X_\beta)_\theta$. Therefore, the differential map $\widehat{\Omega}_\theta$ is defined, for each $x \in (X_\alpha, X_\beta)_\theta$, by:

$$\begin{aligned}\widehat{\Omega}_\theta(x) &= \widehat{\delta}'_\theta CB(x)(z) = \widehat{\delta}'_\theta B(x)\big((1-z)\alpha + \beta z\big)\\ &= B'(x)\big((1-z)\alpha + \beta z\big)\,(-\alpha+\beta)|_{z=\theta}\\ &= (\beta-\alpha)\delta'_{((1-\theta)\alpha+\beta\theta)}B(x)(z)\\ &= (\beta-\alpha)\Omega_{(1-\theta)\alpha+\beta\theta}. \quad \square\end{aligned}$$

3.9. *Differentials for p-convexifications*

This especially interesting situation was treated from the interpolation side in 1.3. We obtain now the associated differential. Since the p-convexified space $X_{(p)}$ of a Köthe space X can be identified with the interpolated space $(\ell_\infty, X)_{1/p}$, we have that any positive and normalized $x \in X_{(p)}$ admits a factorization of the form $x = a_0(x)^{1-1/p}a_1(x)^{1/p}$, where $a_0(x) = 1_{supp}(x) \in \ell_\infty$ is the indicator function on the support of x and $a_1(x) = x^p \in X$ (see [38, Prop. 3.6] for details). In particular, $\|x\|_{1/p} = \|a_0(x)\|_{\ell_\infty}^{1-1/p}\|a_1(x)\|_X^{1/p} = \|x^p\|_X^{1/p} = \|x\|_{X_{(p)}}$. Thus, for normalized and positive $x \in X_{(p)}$ the map $B(x)(z) = a_0(x)^{1-z}a_1(x)^z$ is an homogeneous bounded selection for the evaluation map on $1/p$, hence

$$\Omega_{(p)}(x) = \delta'_{1/p}B(x)(z) = x\log\frac{a_1(x)}{a_0(x)} = x\log x^p = px\log x.$$

For general $x \in X_{(p)}$ we may define the selection $\hat{B}(x)(z) = sign(x)\|x\|_{(p)}B(\frac{|x|}{\|x\|_{(p)}})(z)$, which yields the associated differential

$$\Omega_{(p)}(x) = px\log\Big(\frac{|x|}{\|x\|_{X_{(p)}}}\Big).$$

3.10. *The butterfly lemma*

The formula for the differential associated to a p-convexification is a particular case of a phenomenon called in [44] "the butterfly lemma" that we explain now. We assume we are dealing with a Köthe space X such that $(X, \overline{X}^*)_{1/2} = \ell_2$. We have, by reiteration, the following interpolation identities

$$(X_{(p)}, X^*_{(p)})_{1/2} = \Big((\ell_\infty, X)_{1/p}, (\ell_1, X^*)_{1/p}\Big)_{1/2} = \Big((X, X^*)_{1/2}, (\ell_\infty, \ell_1)_{1/2}\Big)_{1/p} = \ell_2$$

Let us consider the interaction between the associated differentials to the three scales involved:

- Let Ω_X be the differential associated to the scale $(X, \overline{X}^*)$ at $1/2$.
- Let $\Omega_{(p)}$ be the differential associated to the scale $(X_{(p)}, X^*_{(p)})$ at $1/2$.
- And, as always, let KP be the differential associated to the scale (ℓ_∞, ℓ_1) at $1/2$.

The Butterfly Lemma [44, Prop. 8] states that then

$$\Omega_{(p)} = (1 - \frac{1}{p})\Omega_X + \frac{1}{p}\mathsf{KP}. \tag{12}$$

The Butterfly Lemma can be regarded as a reiteration formula for differential maps of interpolation scales (see [44, Lemma 3]).

4. The Rochberg spaces

The paper [19] of Calderón established in its point 1 the foundations for abstract interpolation. The foundations were expanded by Schechter in [99] since he considers as interpolators distributions with compact support; in particular the evaluation of the nth-derivative. Still, compatibility conditions between several interpolators and their connection with exact sequences and twisted sums did not begin to emerge until [55]. In that paper are posed questions about the existence of the associated differential, of the derived space and its representations. For instance, the problem of whether Proposition 3.3 (or its generalized abstract form) was true appears formulated in [55, Problem VIII.6]. The work of Kalton, mostly [70,71], pushed hard in the right direction: the mysterious Ω-operator of [55] is the associated differential to the interpolation problem and a quasilinear map that generates a twisted sum that is the derived space. The paper [22] articulated these ideas in the notion of compatible interpolators, obtained Proposition 3.3 in [22, Proposition 7.2] and showed that the standard interpolation methods (K- and J- real method, complex method) fitted into this scheme. The setting reached cruising speed with the work of Rochberg [97], who introduced the successive differentials and their associated derived spaces, from now on called Rochberg spaces.

4.1. Rochberg spaces for the complex method

The fundamental idea of Rochberg [97] is to consider for each $z \in \mathbb{S}$ the space formed by the arrays of the truncated sequence of the Taylor coefficients of the elements of $\mathcal{C}(X_0, X_1)$, namely

$$\mathfrak{R}_n = \mathfrak{R}_n(X_0, X_1)_z = \left\{ \left(\frac{f^{(n-1)}(z)}{(n-1)!}, \ldots, f^{(1)}(z), f(z) \right) : f \in \mathcal{C}(X_0, X_1) \right\}$$

endowed with the natural quotient norm. We will omit the pair (X_0, X_1) and the point z from now on unless there is absolute necessity. In this form, the space $\mathfrak{R}_1$ of arrays of length one is the space of the values of the functions of $\mathcal{C}(X_0, X_1)$ at z, namely, the interpolated space X_z and the space $\mathfrak{R}_2$ of arrays of length two at z constitutes the derived space d_{Ω_z}.

Fix z. An idea, implicit in [97] and explicit in [17], about the spaces $\mathfrak{R}_n$ is that they can be arranged into exact sequences in a very natural way. Indeed, if for $1 \leq n, k < m$ we denote by $\imath_{n,m} : \Sigma^n \to \Sigma^m$ the inclusion on the left given by $\imath_{n,m}(x_n, \ldots, x_1) = (x_n, \ldots, x_1, 0 \ldots, 0)$ and by $\pi_{m,k} : \Sigma^m \to \Sigma^k$ the projection on the right given by $\pi_{m,k}(x_m, \ldots, x_k, \ldots, x_1) = (x_k, \ldots, x_1)$, then $\pi_{m,k}$ restricts to an isometric quotient map of $\mathfrak{R}_m$ onto $\mathfrak{R}_k$ and $\imath_{n,m}$ is an isomorphic embedding of $\mathfrak{R}_n$ into $\mathfrak{R}_m$ [17, Proposition 2(a)], which is even an isometric space when the base domain is the complex unit circle $\mathbb{D}$ instead of the unit strip [97, Proposition 3.1]. Let us rescue here Rochberg's original proof.

Rochberg's proof. For the reasons we have just explained, we have to set as base domain the complex unit circle $\mathbb{D}$. Let $\mathbb{T} = \partial\mathbb{D}$ be the unit sphere. Consider the arcs $A_0 = \{e^{i\theta} : 0 < \theta < \pi\}$ and $A_1 = \{e^{i\theta} : \pi < \theta < 2\pi\}$ and set X_k on A_k, $k = 0, 1$, so that $\|\cdot\|_\theta = \|\cdot\|_k$ when $\theta \in A_k$. Fix Σ as ambient space and then form the Calderón space $\mathcal{D}(X_0, X_1)$ of holomorphic functions $f : \mathbb{D} \longrightarrow \Sigma$ having radial boundary values and such that

$$\|f\| = \sup\{\|f(e^{i\theta})\|_\theta : e^{i\theta} \in \mathbb{T}\} < \infty$$

Along this proof we will consider the associated Rochberg spaces

$$\mathfrak{R}_n = \left\{ \left(\frac{f^{(n-1)}(0)}{(n-1)!}, \ldots, f^{(1)}(0), f(0) \right) : f \in \mathcal{D}(X_0, X_1) \right\}$$

Now, it is enough to show that if $(x, 0, \ldots, 0) \in \mathfrak{R}_{n+1}$ then $x \in \mathfrak{R}_1$ with $\|x\|_{\mathfrak{R}_1} = \|(x, 0, \ldots, 0)\|_{\mathfrak{R}_{n+1}}$. To this end, pick $f \in \mathcal{D}(X_0, X_1)$ such that $f^{(k)}(0) = 0$ for $k < n$ and $f^{(n)}(0) = n!x$. This f has a zero of order $n-1$ at 0 and therefore $f = z^{n-1}g$ for some $g \in \mathcal{D}(X_0, X_1)$ with $\|g\|_{\mathcal{D}(X_0,X_1)} = \|f\|_{\mathcal{D}(X_0,X_1)}$. We are thus done since

$$x = \frac{1}{n!} f^{(n)}(0) = \frac{n!}{n!} g(0) = g(0). \quad \square$$

The proof at any other $z \in \mathbb{D}$ only requires ad-hoc modifications. However, if one attempts to perform this same proof on an arbitrary point z of either $\mathbb{S}$ (as in [17, lemma 3]) or an unbounded domain $\mathbb{U}$ of the complex plane then multiplication by z^n is not allowed and one has to use a conformal mapping $\varphi : \mathbb{U} \to \mathbb{D}$ such that $\varphi(z) = 0$. The proof goes then smoothly except that a coefficient $\varphi'(z)^n$ appears. This is why, in general, embeddings $\mathfrak{R}_k \longrightarrow \mathfrak{R}_n$ for $k < n$ are not isometric. In [9, Section 6.4] the reader can found the general change of variable formulae from a domain $\mathbb{U}$ to $\mathbb{D}$, something that is, in its essence, a combination of Chain and Leibnitz rule.

All together, see [17, Theorem 4], for each $n, k \in \mathbb{N}$ there is an exact sequence of Banach spaces and operators

$$0 \longrightarrow \mathfrak{R}_n \xrightarrow{\imath_{n,n+k}} \mathfrak{R}_{n+k} \xrightarrow{\pi_{n+k,k}} \mathfrak{R}_k \longrightarrow 0 \tag{13}$$

This sequence can be described via its associated differential. If Δ_k denotes the interpolator $\Delta_k = \frac{1}{k!}\delta_k$ and $B_{(k-1,\dots,0)}$ is a homogeneous bounded selector for the interpolator $\langle \Delta_{k-1}, \dots, \Delta_0 \rangle$ then the differential associated to the sequence (13) is, as usual, the map $\mathfrak{R}_k \longrightarrow \Sigma^n$ defined as

$$\Omega_{k,n} = \langle \Delta_{n+k-1}, \dots, \Delta_k \rangle B_{(k-1,\dots,0)}.$$

We can reformulate this construction: let us set the map $\tau_{(n,0]}(f) = \left(\frac{f^{(n-1)}}{(n-1)!}, \dots, f\right)$. Fix $\varepsilon \in (0,1)$ and, for each $x = (x_{k-1}, \dots, x_0)$ in $\mathfrak{R}_k$, select f_x in the Calderón space such that $x = \tau_{(k,0]} f_x(z)$ with $\|f_x\| \le (1+\varepsilon)\|x\|$ – something we will call a $(1+\varepsilon)$-extremal for x, or just an extremal if no confusion arises –. Let us do that in such a way that f_x depends homogeneously on x. Thus,

$$\Omega_{k,n}(x) = \tau_{(n+k,k]} f_x(z).$$

It is clear that this map depends on the choice of f_x, but different choices of f_x only produce bounded perturbations of the same map. The map $\Omega_{k,n}$ defined in this way is a quasilinear $\mathfrak{R}_k \curvearrowright \mathfrak{R}_n$ and defines the twisted sum space

$$\mathfrak{R}_n \oplus_{\Omega_{k,n}} \mathfrak{R}_k = \left\{(y, x) \in \Sigma^{n+k} : y - \Omega_{k,n}(x) \in \mathfrak{R}_n, x \in \mathfrak{R}_k\right\},$$

endowed with the quasinorm

$$\|(y, x)\|_{\Omega_{k,n}} = \left\|y - \Omega_{k,n}(x)\right\|_{\mathfrak{R}_n} + \|x\|_{\mathfrak{R}_k}. \tag{14}$$

We arrived thus far to show:

Proposition 4.1. $\mathfrak{R}_n \oplus_{\Omega_{k,n}} \mathfrak{R}_k = \mathfrak{R}_{n+k}$ *with equivalent quasinorms.*

Proof. Assume one is working at point $z \in \mathbb{S}$. Fix a conformal mapping $\varphi : \mathbb{S} \longrightarrow \mathbb{D}$ such that $\varphi(z) = 0$. Pick $(y, x) = (y_{n-1}, \dots, y_0, x_{k-1}, \dots, x_0) \in \mathfrak{R}_{n+k}$. Since $x \in \mathfrak{R}_k$ and $(\Omega_{k,n}(x), x) \in \mathfrak{R}_{n+k}$ we get that $(y, x) - (\Omega_{k,n}(x), x) = (y - x, \Omega_{k,n}(x), 0) \in \mathfrak{R}_{n+k}$ and therefore $y - \Omega_{k,n}(x) \in \mathfrak{R}_n$. Moreover

$$\begin{aligned}\|y - \Omega_{k,n}(x)\|_{\mathfrak{R}_n} &\le \|\imath_{n,n+k}\|(\|(y, x)\|_{\mathfrak{R}_{n+k}} - \|(\Omega_{k,n}(x), x)\|_{\mathfrak{R}_{n+k}}) \\ &\le (\|\imath_{n,n+k}\| + 1)\|(y, x)\|_{\mathfrak{R}_{n+k}}.\end{aligned}$$

Hence $\|(y,x)\|_{\Omega_{k,n}} \leq (\|\imath_{n,n+k}\| + 2)\|(y,x)\|_{\mathfrak{R}_{n+k}}$. As for the other containment, pick $(y,x) \in \mathfrak{R}_n \oplus_{\Omega_{k,n}} \mathfrak{R}_k$. Let $f \in \mathcal{C}(X_0, X_1)$ be a $(1+\varepsilon)$-extremal for x so that we may assume $\Omega_{k,n}(x) = \tau_{(n+k,k]} f_x(z)$ and let $g \in \mathcal{C}(X_0, X_1)$ be a $(1+\varepsilon)$-extremal for $y - \Omega_{k,n}(x) \in \mathfrak{R}_n$. We have that if $S_-^k : \mathcal{C} \longrightarrow \bigcap_{j=0}^{k-1} \ker \Delta_j$ is the shift map from Proposition 4.5 and $(y,x) \in \mathfrak{R}_{n+k}$ then (y,x) is the list of Taylor coefficients of $f + S_-^k(g)$ and for some $M > 0$ we have

$$\|(y,x)\|_{\mathfrak{R}_{n+k}} \leq \|f + S_-^k(g)\|_{\mathcal{C}(X_0,X_1)} \leq (1+\varepsilon)\left(M\|y - \Omega_{k,n}(x)\|_{\mathfrak{R}_n} + \|x\|_{\mathfrak{R}_k}\right). \quad \square$$

The Rochberg sequences (13) can be, in turn entwined in commutative diagrams [17, Theorem 4] (we omit the initial and final $0's$ from now on)

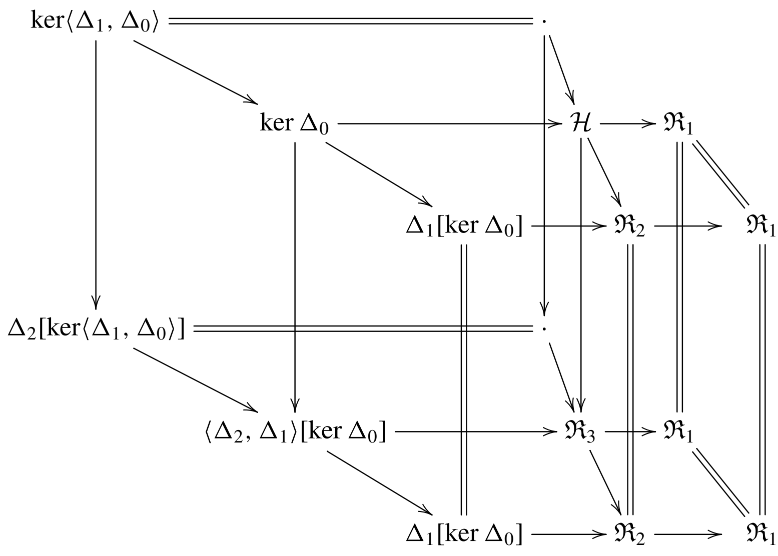
$$\begin{array}{ccccc} \mathfrak{R}_k & = & \mathfrak{R}_k & & \\ \downarrow & & \downarrow & & \\ \mathfrak{R}_n & \longrightarrow & \mathfrak{R}_{n+m} & \longrightarrow & \mathfrak{R}_m \\ \downarrow & & \downarrow & & \| \\ \mathfrak{R}_{n-k} & \longrightarrow & \mathfrak{R}_{n+m-k} & \longrightarrow & \mathfrak{R}_m \end{array} \tag{15}$$

as one can readily check. The attentive reader will not leave unnoticed that the sequences of Rochberg spaces appear in different positions. For instance, in the simplest case

$$\begin{array}{ccccc} \mathfrak{R}_1 & = & \mathfrak{R}_1 & & \\ \downarrow & & \downarrow & & \\ \mathfrak{R}_2 & \longrightarrow & \mathfrak{R}_3 & \longrightarrow & \mathfrak{R}_1 \\ {\scriptstyle \pi_{2,1}}\downarrow & & \downarrow & & \| \\ \mathfrak{R}_1 & \underset{\imath_{1,2}}{\longrightarrow} & \mathfrak{R}_2 & \longrightarrow & \mathfrak{R}_1 \end{array} \tag{16}$$

the sequence $0 \longrightarrow \mathfrak{R}_1 \longrightarrow \mathfrak{R}_2 \longrightarrow \mathfrak{R}_1 \longrightarrow 0$ appears both as the lowest row and as the left column. To know how exactly their associated differentials are related, observe that the lower row appearance has associated differential $\pi_{2,1}\Omega_{1,2}$, while the left column appearance has $\Omega_{2,1}\imath_{1,2}$. The identity $\pi_{2,1}\Omega_{1,2} = \Omega_{1,1}$ follows from the definition. But knowing $\Omega_{2,1}\imath_{1,2}$ requires to know $\Omega_{2,1}$, which, in turn, requires to know a homogeneous bounded selection $B_{1,0}$ for $\langle \Delta_1, \Delta_0 \rangle$. There is however another way around:

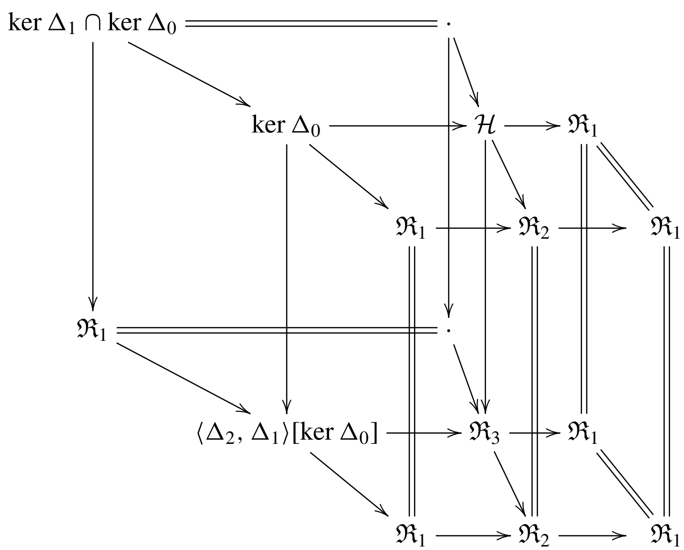
drawing. Observe the commutative 3D diagram:



Since

- $\Delta_1[\ker \Delta_0] = \mathfrak{R}_1$
- $\Delta_2[\ker\langle \Delta_1, \Delta_0\rangle] = \Delta_2[\ker \Delta_1 \cap \ker \Delta_0] = \mathfrak{R}_1$

it becomes



Thus, the left column we wanted to identify is the pushout sequence in the diagram

$$\begin{array}{ccccc} \ker \Delta_1 \cap \ker \Delta_0 & \longrightarrow & \ker \Delta_0 & \xrightarrow{\Delta_1} & \mathfrak{R}_1 \\ {\scriptstyle \Delta_2}\downarrow & & \downarrow & & \| \\ \mathfrak{R}_1 & \longrightarrow & \langle \Delta_2, \Delta_1 \rangle[\ker \Delta_0] & \longrightarrow & \mathfrak{R}_1 \end{array}$$

and therefore we can calculate its associated quasilinear map finding a homogeneous bounded selection V_1 for $\ker \Delta_0 \xrightarrow{\Delta_1} \mathfrak{R}_1$ and setting $\Delta_2 V_1$. This selection, by the magic of Proposition 3.2 (later called "compatibility") is considerably easier than finding a homogeneous bounded selection for $\mathcal{H} \xrightarrow{\Delta_1} \Delta_1[\mathcal{H}]$. Indeed, simply realize that $V_1(x) = \varphi B_0(x)$ works, and therefore $V_1(x)' = \varphi B_0' + \varphi' B_0$ and then $V_1(x)'' = \varphi B_0'' + \varphi' B_0' + \varphi' B_0' + \varphi'' B_0$. When evaluated at θ one gets

$$\Omega(x) = 2\varphi'(\theta)\Omega_{1,1}(x) + \varphi''(\theta)x.$$

Namely, Ω is a multiple of $\Omega_{1,1}$ plus a linear map and thus Ω is equivalent to $\Omega_{1,1}$. See the examples in Sections 5–7 for more details. We cannot resist the temptation of concluding this section pointing out that the following question is uncharted:

Problem 6. Obtain a generalized Butterfly lemma for higher order differentials.

4.2. Rochberg space for Abstract sequences of interpolators

Diagrams (15) and (11) are, from a homological bird's eye view, identical. The process to clarify this issue has been long and winding, starting in [17], passing through [9,31–33], up to arrive to [85]. In its abstract formulation, let $\mathcal{H}$ be the space on which one has defined a sequence $(\Phi_n)_n$ of interpolators, namely, operators $\Phi_n : \mathcal{H} \longrightarrow \Sigma$. The associated Rochberg space $\mathfrak{R}_n$ is

$$\mathfrak{R}_n = \langle \Phi_n, \dots, \Phi_1 \rangle[\mathcal{H}] = \{(x_j) \in \Sigma^n : \exists f \in \mathcal{H} : \Phi_j(f) = x_j \quad 1 \le j \le n\}$$

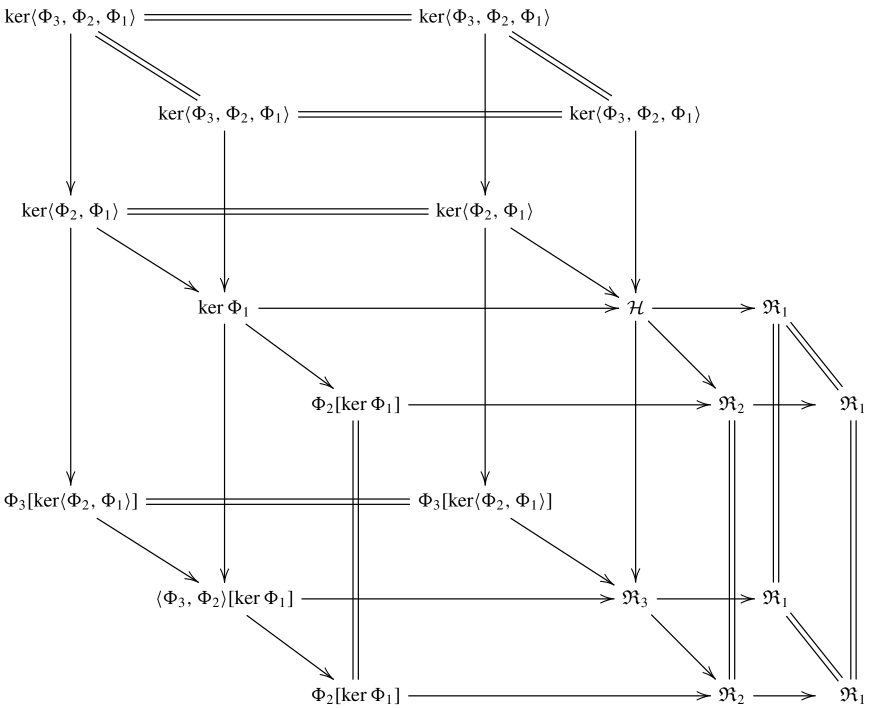
endowed with its natural quotient norm. Alternatively, given $n, k \in \mathbb{N}$ and a finite number of interpolators $\Phi_i : \mathcal{H} \to \Sigma$ $(i = 1, \dots, n+k)$, consider the new interpolators $\Psi = \langle \Phi_{k+n}, \dots \Phi_{k+1} \rangle : \mathcal{H} \to \Sigma^n$ and $\Phi = \langle \Phi_k, \dots \Phi_1 \rangle : \mathcal{H} \to \Sigma^k$ to set

$$\mathfrak{R}_{n+k} = \langle \Psi, \Phi \rangle[\mathcal{H}]$$

The elegant simplicity of the theory developed in this way has a shortcoming, though: the associated exact sequences are more awkward to describe, let us see why. Diagram (11) is now

$$\begin{array}{ccccc} \ker \Phi_2 \cap \ker \Phi_1 & = & \ker\langle \Phi_2, \Phi_1 \rangle & & \\ \downarrow & & \downarrow & & \\ \ker \Phi_1 & \longrightarrow & \mathcal{H} & \xrightarrow{\Phi_1} & \mathfrak{R}_1 \\ {\scriptstyle \Phi_2}\downarrow & & \downarrow{\scriptstyle \langle \Phi_2, \Phi_1 \rangle} & & \| \\ \Phi_2[\ker \Phi_1] & \longrightarrow & \mathfrak{R}_2 & \longrightarrow & \mathfrak{R}_1 \end{array} \qquad (17)$$

and therefore the first sequence is $0 \longrightarrow \Phi_2[\ker\Phi_1] \longrightarrow \mathfrak{R}_2 \longrightarrow \mathfrak{R}_1 \longrightarrow 0$ instead of $0 \longrightarrow \mathfrak{R}_1 \longrightarrow \mathfrak{R}_2 \longrightarrow \mathfrak{R}_1 \longrightarrow 0$. If one picks now three interpolators Φ_3, Φ_2, Φ_1 and forms the cubic diagram

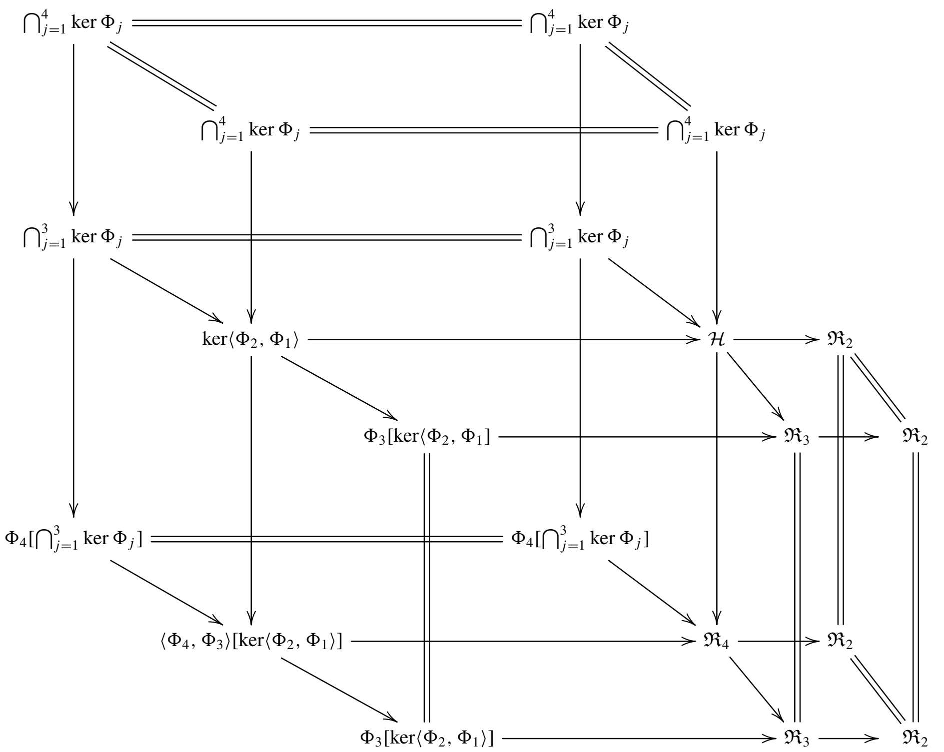


then we observe that the bottom diagram for $m = k = 1$ and $n = 2$ is

$$\begin{array}{ccccc} \Phi_3[\ker\langle\Phi_2,\Phi_1\rangle] & = & \Phi_3[\ker\langle\Phi_2,\Phi_1\rangle] & & \\ \downarrow & & \downarrow & & \\ \langle\Phi_3,\Phi_2\rangle[\ker\Phi_1] & \longrightarrow & \mathfrak{R}_3 & \longrightarrow & \mathfrak{R}_1 \\ \downarrow & & \downarrow & & \| \\ \Phi_2[\ker\Phi_1] & \longrightarrow & \mathfrak{R}_2 & \longrightarrow & \mathfrak{R}_1 \end{array} \tag{18}$$

instead of (15). The lesson to be learnt from this is the fact already underlined in [17] that the mystery in the entwined exact sequences lies in the embedding, not in the quotient map. In particular, we overlooked the fact, suggested by the first sequence $0 \longrightarrow \Phi_2[\ker\Phi_1] \longrightarrow \mathfrak{R}_2 \longrightarrow \mathfrak{R}_1 \longrightarrow 0$, that $\mathfrak{R}_m$ could not be a subspace of $\mathfrak{R}_n$ when $m < n$, as it occurs when the interpolators are compatible. The comparison between diagrams (15) and (18) makes this more evident, as the beautiful symmetry present in (15) – say, $\mathfrak{R}_3$ is simultaneously a twisted sum of $\mathfrak{R}_1$ and $\mathfrak{R}_2$ and of $\mathfrak{R}_2$ and $\mathfrak{R}_1$ – disappears in (18). If we reproduce the cubic diagram working with the interpolators

$\langle \Phi_4, \Phi_3 \rangle$ and $\langle \Phi_2, \Phi_1 \rangle$ then we get



whose bottom face is

$$\begin{array}{ccccc} \Phi_4[\bigcap_{j=1}^{3} \ker \Phi_j] & = & \Phi_4[\bigcap_{j=1}^{3} \ker \Phi_j] & & \\ \downarrow & & \downarrow & & \\ \langle \Phi_4, \Phi_3 \rangle[\ker\langle \Phi_2, \Phi_1 \rangle] & \longrightarrow & \mathfrak{R}_4 & \longrightarrow & \mathfrak{R}_2 \\ \downarrow & & \downarrow & & \| \\ \Phi_3[\ker\langle \Phi_2, \Phi_1 \rangle] & \longrightarrow & \mathfrak{R}_3 & \longrightarrow & \mathfrak{R}_2 \end{array} \tag{19}$$

The diagrams for $n = m + k$ and $m = u + v$ are

$$\begin{array}{ccccc} \langle \Phi_{m+k}, \ldots, \Phi_{1+v+k} \rangle \left[\bigcap_{j=1}^{v+k} \ker \Phi_j\right] & = & \langle \Phi_{m+k}, \ldots, \Phi_{1+v+k} \rangle \left[\bigcap_{j=1}^{v+k} \ker \Phi_j\right] & & \\ \downarrow & & \downarrow & & \\ \langle \Phi_{m+k}, \ldots, \Phi_{1+k} \rangle \left[\bigcap_{j=1}^{k} \ker \Phi_j\right] & \longrightarrow & \mathfrak{R}_{m+k} & \longrightarrow & \mathfrak{R}_k \\ \downarrow & & \downarrow & & \| \\ \langle \Phi_{v+k}, \ldots, \Phi_{1+k} \rangle \left[\bigcap_{j=1}^{k} \ker \Phi_j\right] & \longrightarrow & \mathfrak{R}_{v+k} & \longrightarrow & \mathfrak{R}_k \end{array} \tag{20}$$

which, as we see, are much more intricate than (15) and, probably, hide a good number of unexpected symmetries (see [31] for an appetizer). The lesson, however, to be kept in mind that explains our interest in this abstract approach is that all the exact sequences

$$0 \longrightarrow \langle \Phi_{m+k}, \ldots, \Phi_{1+k} \rangle \left[\bigcap_{j=1}^{k} \ker \Phi_j \right] \longrightarrow \mathfrak{R}_{m+k} \longrightarrow \mathfrak{R}_k \longrightarrow 0$$

are actually the lower sequence in diagram (11)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \ker \Psi \cap \ker \Phi & = & \ker\langle \Psi, \Phi \rangle & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \ker \Phi & \longrightarrow & \mathcal{H} & \xrightarrow{\Phi} & X_\Phi & \longrightarrow & 0 \\
 & & {\scriptstyle \Psi}\downarrow & & \downarrow{\scriptstyle \langle \Psi, \Phi \rangle} & & \| & & \\
0 & \longrightarrow & \Psi[\ker \Phi] & \longrightarrow & X_{\Psi,\Phi} & \longrightarrow & X_\Phi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \qquad (21)$$

and therefore all differentials $\Omega_{k,n}$ are of the form $\Omega_{\Psi,\Phi}$.

4.3. Compatibility conditions

What do we need to restore this last diagram to a manageable diagram (15)? Just ask the sequence (Φ_n) of interpolators to be *compatible*. Compatibility conditions were introduced by Carro, Cerda and Soria in [22], who strived to put in sound mainland some of the ideas of [55]: A pair of interpolators (Ψ, Φ) defined on the same space is called *almost compatible* when $\Psi[\ker \Phi] \subset X_\Phi$, and it is called *compatible* when $\Psi[\ker \Phi] = X_\Phi$. After that, the identity $X_{\Psi,\Phi} = d\Omega_{\Psi,\Phi}$ appears in [22, Prop. 7.2] for compatible pairs of interpolators, answering the question 6 of [55]. Our diagrammatic approach yields that if we want the sequence $0 \longrightarrow \Phi_2[\ker \Phi_1] \longrightarrow \mathfrak{R}_2 \longrightarrow \mathfrak{R}_1 \longrightarrow 0$ to be $0 \longrightarrow \mathfrak{R}_1 \longrightarrow \mathfrak{R}_2 \longrightarrow \mathfrak{R}_1 \longrightarrow 0$, the first compatibility condition to impose is $\Phi_2[\ker \Phi_1] = X_{\Phi_1} = \Phi_1[\mathcal{H}]$. To do the same when more interpolators are involved, [23] they say that a family of interpolators $(\Phi_n, \ldots, \Phi_1)$ is compatible if for each $j \geq 1$ and each $k \geq 0$ each subset $(\Phi_{k+j}, \ldots, \Phi_{k+1}, \Phi_k)$ one has $\Phi_{k+j}[\ker \Phi_k \cap \ldots \ker \Phi_{k+j-1}] = \Phi_k[\mathcal{H}]$. However, as remarked in [85] this is not enough to turn all diagrams (20) into (15). To amend this, Moreno and Simões introduce in [85] the following notion:

Definition 4.2. The family of interpolators $(\Phi_n, \ldots, \Phi_1)$ is called *compatible* if:

(1) For each $n \geq 1$ and each $k \geq 0$ such that $n + k \leq N$ one has

$$\langle \Phi_{k+n}, \ldots, \Phi_{k+1} \rangle : \ker\langle \Phi_k, \ldots, \Phi_1 \rangle \longrightarrow \langle \Phi_n, \ldots, \Phi_1 \rangle[\mathcal{H}]$$

is linear continuous and surjective (i.e, the pair (Ψ, Φ) of multi-interpolators $\Psi = \langle \Phi_{k+n}, \ldots, \Phi_{k+1} \rangle$ and $\Phi = \langle \Phi_k, \ldots, \Phi_1 \rangle$ is compatible in the sense of [22]).

(2) Given $n, k, m \in \mathbb{N}$ such that $n + k + m \leq N$, if B_n is a homogeneous bounded selector for

$$\langle \Phi_n, \ldots, \Phi_1 \rangle : \mathcal{H} \longrightarrow \langle \Phi_n, \ldots, \Phi_1 \rangle[\mathcal{H}]$$

and $V_{n,k}$ is a homogeneous bounded selector for

$$\begin{aligned} \langle \Phi_{k+n}, \ldots, \Phi_{k+1} \rangle : \bigcap_{j=1}^{k} \ker \Phi_j \longrightarrow \langle \Phi_{k+n}, \ldots, \Phi_{k+1} \rangle \left[\bigcap_{j=1}^{k} \ker \Phi_j \right] \\ = \langle \Phi_n, \ldots, \Phi_1 \rangle[\mathcal{H}] \end{aligned}$$

then

$$\langle \Phi_{k+n+m}, \ldots, \Phi_{k+n+1} \rangle V_{n,k} \equiv \langle \Phi_{n+m}, \ldots, \Phi_{n+1} \rangle B_n$$

and show in [85, Proposition 6.1] that the family of Schechter interpolators is compatible. Observe, however, that compatibility conditions always refer to an ordered sequence. For instance, (Ψ, Φ) can be compatible but (Φ, Ψ) not. In fact, one has [32, Lemma 6.2] that if the two pairs (Ψ, Φ) and (Φ, Ψ) are compatible then $X_\Phi = X_\Psi$ and both $\Omega_{\Psi,\Phi}$ and $\Omega_{\Phi,\Psi}$ are bounded. But this Pandora's box also contains a gem at its bottom since it is impossible not to ask, as it was done in [32] about the symmetric pairs (Ψ, Φ) and (Φ, Ψ). A part of the beautiful theory that emerged is explained in the next section. Before that, let us dig an extra mile into the compatibility condition.

Definition 4.3. A family of interpolators $(\Phi_n, \ldots, \Phi_1)$ is called *strongly compatible* if:

(1) For any $n, k \in \mathbb{N}$, there exist a bounded linear operator $S_-^k : \mathcal{H} \to \bigcap_{j=1}^{k} \ker \Phi_j$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{S_-^k} & \bigcap_{j=1}^{k} \ker \Phi_j \\ & \searrow^{\langle \Phi_n, \ldots, \Phi_1 \rangle} & \downarrow {\scriptstyle \langle \Phi_{n+k}, \ldots, \Phi_{k+1} \rangle} \\ & & \langle \Phi_n, \ldots, \Phi_1 \rangle(\mathcal{H}) \end{array}$$

In particular, given $f \in \mathcal{H}$ we have that

$$\langle \Phi_{n+k}, \ldots, \Phi_{k+1} \rangle (S_-^k(f)) = \langle \Phi_n, \ldots, \Phi_1 \rangle (f).$$

(2) For any $n, k \in \mathbb{N}$, there exist a bounded linear operator $S_+^k : \bigcap_{j=1}^{k} \ker \Phi_j \to \mathcal{H}$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{H} & \xleftarrow{S_+^k} & \bigcap_{j=1}^{k} \ker \Phi_j \\ & \searrow^{\langle \Phi_n, \ldots, \Phi_1 \rangle} & \downarrow {\scriptstyle \langle \Phi_{n+k}, \ldots, \Phi_{k+1} \rangle} \\ & & \langle \Phi_n, \ldots, \Phi_1 \rangle(\mathcal{H}) \end{array}$$

In particular, given $f \in$ we have that

$$\langle \Phi_{n+k}, \ldots, \Phi_{k+1} \rangle (f) = \langle \Phi_n, \ldots, \Phi_1 \rangle (S_+^k(f)).$$

The maps S_+^k and S_-^k will be referred to as the shift maps. Note that in the abstract setting, these shift maps are not necessarily the inverse of the other, but in most natural cases they will be.

Proposition 4.4. *Every strongly compatible family is compatible.*

Proof. Let $n \geq 1$ and $k \geq 0$ be fixed and $f \in \bigcap_{j=1}^k \ker \Phi_j$. Since the family is strongly compatible, there exist $g \in \mathcal{H}$ such that $g = S_+^k(f)$ and

$$(\Phi_{n+k}, \dots, \Phi_{k+1})(f) = \langle \Phi_n, \dots, \Phi_1 \rangle (g) \in \langle \Phi_n, \dots, \Phi_1 \rangle (\mathcal{H}).$$

Boundedness follows by the inequality $\|g\| = \|S_+^k f\| \leq C\|f\|$. To show that such map is surjective we use the other shift: given $x = (x_n, \dots, x_1) \in \langle \Phi_n, \dots, \Phi_1 \rangle (\mathcal{H})$ there exist $f \in \mathcal{H}$ such that $\langle \Phi_n, \dots, \Phi_1 \rangle (f) = x$. Then $S_-^k(f) \in \bigcap_{j=1}^k \ker \Phi_j$ and

$$\langle \Phi_{n+k}. \dots, \Phi_{k+1} \rangle (S_-^k(f)) = \langle \Phi_n, \dots, \Phi_1 \rangle (f) = x.$$

To prove the second condition just take $B_n : \langle \Phi_n, \dots, \Phi_1 \rangle [\mathcal{H}] \to \mathcal{H}$ and $V_{n,k} : \langle \Phi_{n+k}, \dots, \Phi_{k+1} \rangle [\mathcal{H}] \to \bigcap_{j=1}^k \ker \Phi_j$ bounded homogeneous selections of the maps $\langle \Phi_n, \dots, \Phi_1 \rangle$ and $\langle \Phi_{n+k}, \dots, \Phi_{k+1} \rangle |_{\bigcap_{j=1}^k \ker \Phi_j}$, respectively. Then note that $S_-^k(B_n)$ is an homogeneous bounded selection of

$$\langle \Phi_{n+k}, \dots, \Phi_{k+1} \rangle : \bigcap_{j=1}^{k} \ker \Phi_j \to \langle \Phi_n, \dots, \Phi_1 \rangle [\mathcal{H}] \tag{22}$$

since given $x \in \langle \Phi_n, \dots, \Phi_1 \rangle [\mathcal{H}]$ one has

$$\langle \Phi_{n+k}. \dots, \Phi_{k+1} \rangle (S_-^k(B_n(x))) = \langle \Phi_n, \dots, \Phi_1 \rangle (B_n(x)) = x.$$

Moreover, considering the family $(\Phi_{m+n}, \dots, \Phi_1)$ we have

$$\langle \Phi_{n+m+k}, \dots, \Phi_{k+1} \rangle (S_-^k(B_n(x))) = \langle \Phi_{m+n}, \dots, \Phi_1 \rangle (B_n(x))$$

which means that $\Phi_{l+k}(S_-^k(B_n(x))) = \Phi_l(B_n(x))$ for all $n+1 \leq l \leq n+m$, and thus

$$\langle \Phi_{k+n+m}, \dots, \Phi_{k+n+1}) S_-^k(B_n(x)) \rangle = \langle \Phi_{n+m}, \dots, \Phi_{n+1} \rangle B_n(x).$$

Since any two bounded homogeneous selections of the map (22) define boundedly equivalent quasilinear maps, the result follows:

$$\begin{aligned}\big\|(\Phi_{k+n+m},\dots,\Phi_{k+n+1})V_{n,k}(x)-(\Phi_{k+n+m},\dots,\Phi_{k+n+1})(S_-^k(B_n(x)))\big\|_m\\ =\|(\Phi_{k+n+m},\dots,\Phi_{k+n+1})(V_{n,k}-S_-^k(B_n))(x)\|_m\\ \le\|(\Phi_{k+n+m},\dots,\Phi_{k+n+1})|_{\cap_{j=0}^k\ker\Phi_j}\|(\|V_{n,k}\|+\|B_n\|)\|x\|_n.\quad\square\end{aligned}$$

Now, the proof of [85] actually yields:

Proposition 4.5. *The Schechter interpolators $(\Delta_n)_n$ are strongly compatible.*

Indeed, the key to prove the result lies in the following lemma of [17]:

Lemma 4.6. *Let $\varphi:\mathbb{S}\to\mathbb{D}$ a conformal mapping vanishing at θ. Then given m and $0\le k\le m$ there exist a polynomial P_k of degree at most m such that $\Delta_i(P_k(\varphi))=\delta_{i,k}$ for every $0\le i\le m$.*

Set now the following linear bounded operators:

- $S_-^k:\mathcal{C}\longrightarrow\bigcap_{j=0}^{k-1}\ker\Delta_j$ given by $S_-^k(f)=P_k(\varphi)f$.
- $S_+^k:\bigcap_{j=0}^{k-1}\ker\Delta_j\to\mathcal{C}$ given by $S_+^k(f)=S_+^k(\varphi^k g)=\sum_{j=0}^{n-1}\Delta_{k+j}(\varphi^k)P_j(\varphi)g$.

and check following [85] that they have the desired properties. The fact that the family of Schechter interpolators is strongly compatible together with the fact that the generalized CKMR method [56] implies that most methods (with the remarkable exception of Orbits method) are based on the Schechter sequence of interpolators suggests both a specific technical not overwhelmingly interesting question and a woozy but quite important one:

Problem 7. Is every compatible family of interpolators strongly compatible?

Problem 8. Do there appear non-strongly compatible families of interpolators in nature?

Under compatibility, the Rochberg space associated to $(\Phi_n,\dots,\Phi_1)$ remains unaltered no matter which representation one chooses:

$$\begin{aligned}\mathfrak{R}_n&=\left\{(x_n,\dots,x_1):\begin{cases}x_1\in\Phi_1[\mathcal{H}]\\(x_n,\dots,x_2)-\Omega_{\langle\Phi_n,\dots,\Phi_2\rangle,\langle\Phi_1\rangle}(x_1)\in\mathfrak{R}_{n-1},\end{cases}\right\}\\&=\left\{((x_n,\dots,x_3),(x_2,x_1)):\begin{cases}(x_2,x_1)\in\mathfrak{R}_2,\\(x_n,\dots,x_3)-\Omega_{\langle\Phi_n,\dots,\Phi_3\rangle,\langle\Phi_2,\Phi_1\rangle}(x_2,x_1)\in\mathfrak{R}_{n-2},\end{cases}\right\}\\&\dots\\&=\{\big((x_n,\dots,x_{j+1}),(x_j,\dots,x_1)\big):\\&\quad\begin{cases}(x_j,\dots,x_0)\in\mathfrak{R}_j,\\(x_n,\dots,x_{j+1})-\Omega_{\langle\Phi_n,\dots,\Phi_{j+1}\rangle,\langle\Phi_j,\dots,\Phi_1\rangle}(x_j,\dots,x_1)\in\mathfrak{R}_{n-j},\end{cases}\}\end{aligned}$$

Moreover, as it is remarked in [85], compatibility conditions pay off making the bunch of spaces appearing in the diagrams generated by a sequence $(\Phi_n,\dots,\Phi_1)$ of

n interpolators reduce to just the n Rochberg spaces $\mathfrak{R}_j$ for $1 \le j \le n$ spaces. We will see in Sections 5 and 6 that these spaces can be all isomorphic, as it occurs with the scale of weighted ℓ_2-spaces, or all non-isomorphic, as it is the case of the scale of ℓ_p-spaces.

4.4. Domains, ranges and inverses

The seminal question is simple: if a compatible pair (Ψ, Φ) generates the exact sequence

$$0 \longrightarrow \mathfrak{R}_1 \longrightarrow \mathfrak{R}_2 \longrightarrow \mathfrak{R}_1 \longrightarrow 0$$

what occurs with (Φ, Ψ), who is no longer compatible? And we keep fresh in our fronthead the example of the pair $(\delta'_{1/2}, \delta_{1/2})$ applied to the pair (ℓ_∞, ℓ_1) that yields the sequence

$$0 \longrightarrow \ell_2 \longrightarrow Z_2 \longrightarrow \ell_2 \longrightarrow 0$$

The beautiful theory of symmetries in Rochberg diagrams surfaces crawling through the papers [11,22,55,75] to see daylight in [32]. In the first of those papers Kalton and Peck observe that in the particular case of the differential KP associated to pair $(\delta'_{1/2}, \delta_{1/2})$ applied to the pair (ℓ_∞, ℓ_1) the space $\{x \in \ell_2 : \mathsf{KP}(x) \in \ell_2\}$ endowed with the quasi-norm $\|x\| = \|\mathsf{KP}x\|_2 + \|x\|_2$ is isomorphic to the Orlicz space ℓ_{f_1} generated by the function $f_1(t) = t^2 \log^2 t$. In [55] the notions of Domain and Range for the so-called Ω-operator generated by some interpolation method (a precursor of our differential $\Omega_{\Psi,\Phi}$) defined on the space X appear introduced as $\mathrm{Dom}\Omega = \{x \in X : \Omega(x) \in X\}$ endowed with the quasi-norm $\|x\|_{\mathrm{Dom}} = \|\Omega x\|_X + \|x\|_X$. The range of Ω is defined as $\mathrm{Ran}\Omega = \{\Omega(x) \in \Sigma \ : x \in X\}$ endowed with the quasi-norm $\|w\|_{\mathrm{Ran}} = \inf\{\|x\|_X \ : w = \Omega(x)\}$. This definition, reproduced in [22], is however a wrong one since, as [22] remarks, $\mathrm{Ran}\Omega$ is not necessarily a vector space. The right definition is given in [11]: $\mathrm{Ran}\Omega = \{w \in \Sigma : \exists x \in X \ : w - \Omega(x) \in X\}$ endowed with $\|w\|_{\mathrm{Ran}} = \inf\{\|w - \Omega(x)\|_X + \|x\|_X : x \in X,\ w - \Omega(x) \in X\}$. All this yields $\mathrm{Dom}\mathsf{KP} = \ell_{f_1}$ is a subspace of Z_2 and, as identified in [11], $Z_2/\ell_f \simeq \ell^*_{f_1} = \mathrm{Ran}\mathsf{KP}$, which yields another representation of Z_2 as a twisted sum

$$0 \longrightarrow \ell_{f_1} \longrightarrow Z_2 \longrightarrow \ell^*_{f_1} \longrightarrow 0$$

This sequence is generated by the non-compatible pair of interpolators $(\delta_{1/2}, \delta'_{1/2})$ because, as it show in [32], that is what always happens:

Definition 1. Let (Ψ, Φ) be a pair of interpolators on $\mathcal{H}$. The *domain* and *range* of $\Omega_{\Psi,\Phi}$ are defined as follows:

$$\mathrm{Dom}\Omega_{\Psi,\Phi} = \{x \in X_\Phi : \Omega_{\Psi,\Phi}(x) \in \Psi[\ker \Phi]\}$$

endowed with the quasi-norm $\|x\|_{\mathrm{Dom}} = \|\Omega_{\Psi,\Phi}x\|_{\Psi|_{\ker\Phi}} + \|x\|_\Phi$ and

$$\mathrm{Ran}\Omega_{\Psi,\Phi} = \{w \in \Sigma : \exists x \in X_\Phi,\ w - \Omega_{\Psi,\Phi}(x) \in \Psi[\ker \Phi]\}$$

endowed with $\|w\|_{\mathrm{Ran}} = \inf\{\|w - \Omega_{\Psi,\Phi}(x)\|_{\Psi|_{\ker\Phi}} + \|x\|_\Phi : x \in X_\Phi,\ w - \Omega_{\Psi,\Phi}(x) \in \Psi(\ker\Phi)\}$.

Thus, everything is clear now: the maps $\jmath x = (0, x)$ and $Q(w, y) = w$ define an exact sequence

$$0 \longrightarrow \mathrm{Dom}\Omega_{\Psi,\Phi} \xrightarrow{\ \jmath\ } \mathfrak{R}_2 \xrightarrow{\ Q\ } \mathrm{Ran}\Omega_{\Psi,\Phi} \longrightarrow 0.$$

and one therefore has two perfectly symmetric representations of the space $\mathfrak{R}_2$

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \uparrow & & & & \\ & & & & X_\Psi = \mathrm{Ran}\Omega_{\Psi,\Phi} & & & & \\ & & & & \uparrow & & & & \\ 0 & \longrightarrow & \Psi[\ker\Phi] = \mathrm{Dom}\Omega_{\Phi,\Psi} & \longrightarrow & \mathfrak{R}_2 & \longrightarrow & X_\Phi = \mathrm{Ran}\Omega_{\Phi,\Psi} & \longrightarrow & 0 \\ & & & & \uparrow & & & & \\ & & & & \Phi[\ker\Psi] = \mathrm{Dom}\Omega_{\Psi,\Phi} & & & & \\ & & & & \uparrow & & & & \\ & & & & 0 & & & & \end{array} \tag{23}$$

These ideas can in turn be generalized as it has been done in [40]: let $\Omega : X \longrightarrow \Sigma$ be a quasilinear map $X \curvearrowright Y$ generating an exact sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ and let us call Ω^{-1} the quasilinear map generating the "inverse" sequence

$$0 \longrightarrow \mathrm{Dom}\Omega \longrightarrow Z \longrightarrow \mathrm{Ran}\Omega \longrightarrow 0\ .$$

Beyond the fact already proved in [32, Proposition 3.5] that $\Omega^{-1}_{\Psi,\Phi} = \Omega_{\Phi,\Psi}$ the symmetries do not end here since duality is also involved. In [40] it is proved that $(\Omega^*)^{-1} = (\Omega^{-1})^*$, under rather acceptable conditions. A forerunner for this formula is in [55, Corollary 3.2.1].

4.5. Commutator theorems

Let A, B be two maps (not necessarily linear or continuous). Their commutator is defined to be the map $[A, B] = AB - BA$ provided that all terms make sense. Commutators are important regarding actions of groups on exact sequences since they are derivations on the group; they are also important regarding interpolation affairs as tools to produce operators on Rochberg spaces. This is how: Let (X_0, X_1) be an interpolation pair with ambient space Σ. An operator $\tau : \Sigma \longrightarrow \Sigma$ is said to act on the interpolation scale generated by the pair (X_0, X_1), or to be an operator on the scale, if it acts continuously as an operator $X_0 \to X_0$ and $X_1 \to X_1$. The following generalized commutator theorem has been obtained in [43] and has a technical and tedious proof:

Theorem 4.7 (*Generalized Commutator Theorem*)**.** *Let τ be an operator acting on the scale generated by the pair (X_0, X_1) and fix $\theta \in (0, 1)$ and $n \in \mathbb{N}$. Then*

(1) *The diagonal operator*

$$\tau_n = \begin{pmatrix} \tau & & & 0 \\ & \tau & & \\ & & \ddots & \\ 0 & & & \tau \end{pmatrix}$$

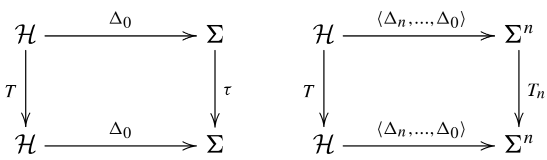
acting as $\tau_n(x_1, \dots, x_n) = (\tau x_1, \dots, \tau x_n)$ *is bounded on the Rochberg space* $\mathfrak{R}_n$.

(2) *For any* $k, m \in \mathbb{N}$ *such that* $k + m = n$*, there exist* $C > 0$ *so that*

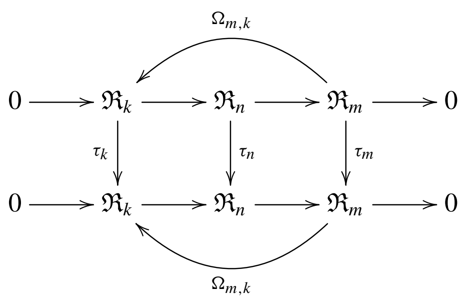
$$\left\| \tau_k \Omega_{m,k}(x) - \Omega_{m,k}(\tau_m x) \right\|_{\mathfrak{R}_k} \le C \|x\|_{\mathfrak{R}_m}.$$

Observe that the case $n = 1$ is just the interpolation property for operators mentioned at the end of Section 2.2: any operator $\tau : \Sigma \to \Sigma$ such that $\tau[X_0] \subset X_0$ and $\tau[X_1] \subset X_1$ boundedly, satisfies that $\tau : \mathfrak{R}_1 \to \mathfrak{R}_1$ is bounded, i.e., the operator $\tau : X_\theta \to X_\theta$ is bounded; and the case $n = 2$ is the classical Commutator Theorem of Rochberg and Weiss [98].

However, the key to prove Theorem 4.7 is the simple observation that if τ is an operator on the scale (X_0, X_1) then it induces an operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ defined by $T(f)(z) = \tau(f(z))$ for $f \in \mathcal{C}(X_0, X_1)$ and these operators makes the diagrams



commute. From the pictorial point of view, the Commutator Theorem asserts the commutativity of the diagrams:



From the hard analysis point of view, Commutator theorems are interesting since they provide the hard-to-obtain estimate (2) above. Just to give an example, in the case of the scale of ℓ_p spaces in which $\Omega_{m,k} = \mathsf{KP}_{m,k}$ have been already computed [9,31] one has

$$\left\| \tau_k \mathsf{KP}_{m,k}(x) - \mathsf{KP}_{m,k}(\tau_m x) \right\|_{\mathfrak{R}_k} \le C \|x\|_{\mathfrak{R}_m}.$$

Other unexpected consequences of the Commutator Theorem can be presented too:

Proposition 4.8.

- $\mathrm{Dom}\Omega_{1,n-1}$ *is an interpolation space for the pair* (X_0, X_1).
- $\mathrm{Ran}\Omega_{n-1,1}$ *is an interpolation space for the pair* (X_0, X_1)

Proof. If $x \in \mathrm{Dom}\Omega_{1,n-1}$ then $\Omega_{1,n-1}(x) \in \mathfrak{R}_{n-1}$ by definition, and we must show that $Tx \in \mathrm{Dom}\Omega_{1,n-1}$, namely, that $\Omega_{1,n-1}(Tx) \in \mathfrak{R}_{n-1}$. By the Commutator Theorem 4.7 we have

$$\left\| T_{n-1}\Omega_{1,n-1}(x) - \Omega_{1,n-1}(Tx) \right\|_{\mathfrak{R}_{n-1}} \leq C\|x\|_{\mathfrak{R}_1}$$

and thus we deduce that

$$\Omega_{1,n-1}(Tx) = \Omega_{1,n-1}(Tx) - T_{n-1}\Omega_{1,n-1}(x) + T_{n-1}\Omega_{1,n-1}(x) \in \mathfrak{R}_{n-1}.$$

The proof of the statement for range spaces is similar: Given $x \in \mathrm{Ran}\Omega_{n-1,1}$ there exist $y \in \mathfrak{R}_{n-1}$ so that $x - \Omega_{n-1,1}y \in \mathfrak{R}_1$. Then $T(x) \in \mathfrak{R}_1$ since, by the Commutator Theorem

$$T(x) - \Omega_{n-1,1}(T_{n-1}y) = T(x) - \Omega_{n-1,1}(T_{n-1}y) + T(\Omega_{n-1,1}y) - T(\Omega_{n-1,1}y) \in \mathfrak{R}_1. \quad \square$$

Since Domain and Range spaces behave as interpolation spaces with respect to the original pair (X_0, X_1) it makes sense to ask

Problem 9. Do Domains and Ranges form interpolation scales on their own?

This is, we admit, a slightly vague question. We would probably mean to find out whether $(\mathrm{Dom}\mathsf{KP}_p, \mathrm{Dom}\mathsf{KP}_{p^*})_{1/2} = \mathrm{Dom}\mathsf{KP}_2$ or rather $(\mathrm{Dom}\mathsf{KP}_p, \mathrm{Dom}\mathsf{KP}_{p^*})_{1/2} = \ell_2$. In the light of [9], probably both things are true.

4.6. Harmless and lethal variations of Rochberg spaces

The abstract approach makes impossible to avoid the following question: Does it make any difference to work with the sequence of interpolators (δ_n) instead of (Δ_n)? Namely, what occurs if instead of using the Taylor coefficients of a function one simply uses the successive values of the derivatives? This phenomenon is invisible when only one interpolator is considered, since the interpolation space $\Phi[\mathcal{H}]$ and $\lambda\Phi[\mathcal{H}]$ are "the same". In general, things depend on the representation one has in mind for Rochberg spaces. If one considers $\mathfrak{R}_n = \mathcal{H}/\ker\langle\Phi_n, \dots, \Phi_1\rangle$ it is clear that replacing (Φ_n) by $(\lambda_n\Phi_n)$ yields the same spaces. If, however, one considers, say, $\mathfrak{R}_2 = \{(\Phi_2 x, \Phi_1 x) : x \in \mathcal{H}\} = \{(w, x) \in \Sigma \times \Phi_1[\mathcal{H}] : w - \Phi_2 B_{\Phi_1} x \in \Phi_2[\ker\Phi_1]\}$ then (Φ_2, Φ_1) and $(\Phi_2, -\Phi_1)$ can provide different Rochberg spaces since it can perfectly occur that $(w, x) \in \mathfrak{R}_2$ but $(w, -x) \notin \mathfrak{R}_2$. The Rochberg spaces are, however, clearly isomorphic and the isomorphisms are "natural":

Proposition 4.9. *Let $(\lambda_n)_n$ be a sequence of non null scalars. The Rochberg spaces associated to a sequence (Φ_n) of interpolators and those associated to the sequence of interpolators $(\lambda_n \Phi_n)$ are isomorphic.*

Proof. Let us show first that the Rochberg space $\mathfrak{R}_2$ associated to the pair (Φ_2, Φ_1) and the Rochberg space $\mathfrak{R}_2(\lambda_2, \lambda_1)$ associated to $(\lambda_2\Phi_2, \lambda_1\Phi_1)$ are isomorphic. If B_1 is a homogeneous bounded selection for Φ_1 then $\lambda_1^{-1}B_1$ is a homogeneous bounded selection for $\lambda_1\Phi_1$. Therefore, if the differential associated to the couple (Φ_2, Φ_1) is $\Omega = \Phi_2 B_1$, the associated differential corresponding to the pair $(\lambda_2\Phi_2, \lambda_1\Phi_1)$ is $\lambda_2\lambda_1^{-1}\Omega$. Since there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Phi_2[\ker \Phi_1] & \longrightarrow & \Phi_2[\ker \Phi_1] \oplus_\Omega X_{\Phi_1} & \longrightarrow & X_{\Phi_1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda_2} & & \downarrow & & \downarrow{\scriptstyle \lambda_1} & & \\
0 & \longrightarrow & \lambda_2\Phi_2[\ker \lambda_1\Phi_1] & \longrightarrow & \lambda_2\Phi_2[\ker \lambda_1\Phi_1] \oplus_{\lambda_2\lambda_1^{-1}\Omega} X_{\lambda_1\Phi_1} & \longrightarrow & X_{\Phi_1} & \longrightarrow & 0
\end{array}$$

and both the left and right downward arrows are isomorphisms, the 3-lemma implies that the middle arrow is also an isomorphism, and thus

$$\mathfrak{R}_2 = \Phi_2[\ker \Phi_1] \oplus_\Omega X_{\Phi_1} \simeq \lambda_2\Phi_2[\ker \lambda_1\Phi_1] \oplus_{\lambda_2\lambda_1^{-1}\Omega} X_{\lambda_1\Phi_1} = \mathfrak{R}_2(\lambda_2, \lambda_1).$$

We will call (λ_2, λ_1) that middle arrow isomorphism $(w, x) \to (\lambda_2 w, \lambda_1 x)$. Let us show next that the Rochberg space $\mathfrak{R}_3$ associated to (Φ_3, Φ_2, Φ_1) and the Rochberg space $\mathfrak{R}_3(\lambda_3, \lambda_2, \lambda_1)$ associated to $(\lambda_3\Phi_3, \lambda_2\Phi_2, \lambda_1\Phi_1)$ are isomorphic. Recall from diagram (18) that $\mathfrak{R}_3$ is generated in the exact sequence
$0 \longrightarrow \langle\Phi_3, \Phi_2\rangle[\ker \Phi_1] \longrightarrow \mathfrak{R}_3 \longrightarrow \mathfrak{R}_1 \longrightarrow 0$, therefore the commutativity of the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \langle\Phi_3, \Phi_2\rangle[\ker \Phi_1] & \longrightarrow & \mathfrak{R}_3 & \longrightarrow & \mathfrak{R}_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\lambda_3,\lambda_2)} & & \downarrow & & \downarrow{\scriptstyle \lambda_1} & & \\
0 & \longrightarrow & \langle\lambda_3\Phi_3, \lambda_2\Phi_2\rangle[\ker \lambda_1\Phi_1] & \longrightarrow & \mathfrak{R}_3(\lambda_3, \lambda_2, \lambda_1) & \longrightarrow & \mathfrak{R}_1(\lambda_1) & \longrightarrow & 0
\end{array}$$

plus the 3-lemma makes the middle vertical arrow an isomorphism. The proof goes on inductively on n. $\square$

These considerations are necessary to see that reiteration for the complex method yields isomorphic Rochberg spaces. More precisely, pick a pair (X_0, X_1). Let us set $X_a = (X_0, X_1)_a$ and $X_b = (X_0, X_1)_b$. Assume that $X_\theta = (X_a, X_b)_\eta$. It follows from Propositions 3.4 and 4.9 that

Proposition 4.10. *The Rochberg spaces, $\mathfrak{R}_n(X_\theta)$ and $\mathfrak{R}_n((X_a, X_b)_\eta)$ are isomorphic.*

Let us make it explicit: the sequences of interpolators are $(\Delta_k)_{k=0}^{n-1}$ and $(\lambda_k \Delta_k)_{k=0}^{n-1}$ yield isomorphic spaces $\mathfrak{R}_n(X_\theta)$ and $\mathfrak{R}_n((X_a, X_b)_\eta)$ and if T is an operator acting on the scale it follows from the Commutator Theorem the commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{R}_n(X_\theta) & \xrightarrow{(\lambda_n,\ldots,\lambda_1)} & \mathfrak{R}_n((X_a, X_b)_\eta) \\ {\scriptstyle T_n}\downarrow & & \downarrow{\scriptstyle T_n} \\ \mathfrak{R}_n(X_\theta) & \xrightarrow[(\lambda_n,\ldots,\lambda_1)]{} & \mathfrak{R}_n((X_a, X_b)_\eta) \end{array}$$

since $(\lambda_n T x_n, \ldots, \lambda_1 T x_1) = T_n(\lambda_n x_n, \ldots, \lambda_1 x_1)$.

There is a third way: looking at $\mathfrak{R}_n$ as pushout spaces. It can be shown by plain categorical arguments that one still obtains isomorphic spaces. The fact that the Rochberg spaces obtained from, say, $(\Phi_n, \ldots, \Phi_1)$ and from $(\lambda_n\Phi_n, \ldots, \lambda_1\Phi_1)$ are isomorphic, or even isometric, but not the same spaces is however lethal for some considerations since making pushout respects equivalence of sequences but not isomorphic equivalence. See in [9] the muddy waters into which the confusion of equal and isometric spaces disembogue.

5. Rochberg spaces for the scale of weighted ℓ_2-spaces

Let w be a weight sequence, a term to describe [79, 4.e.1] a non-increasing sequence of positive numbers such that $\lim w_n = 0$ and $\sum w_n = \infty$. We denote by $\ell_2(w)$ the space of all sequences x such that $w \cdot x \in \ell_2$ endowed with the norm $\|x\|_w = \|wx\|_2$. We set $w_0 = w^{-1}$ and $w_1 = w$ and let us consider the interpolation pair $(\ell_2(w^{-1}), \ell_2(w))$, for which it is well known [6, Chapter 5] that $(\ell_2(w_0), \ell_2(w_1))_\theta = \ell_2(w_0^{1-\theta} w_1^{\theta})$. To simplify the notation we will focus when $\theta = 1/2$ because in this case we have ℓ_2 isometrically as interpolated space. A homogeneous bounded selector for Δ_0 is given by $B(x)(z) = w^{2z-1}x$ since $\Delta_0 B(x) = x$, and therefore $B(x)'(z) = 2w^{2z-1}\log w \cdot x$. It follows that the differential map is given by

$$\Omega_{1,1}x = \Delta_1 B(x) = B(x)'(1/2) = 2\log w \cdot x,$$

which is a linear map. The associated Rochberg space $\mathfrak{R}_n(\ell_2(w^{-1}), \ell_2(w))_{1/2}$ will be called $\mathfrak{R}_n(w)$. Thus, $\mathfrak{R}_2(w) = \{(y, x) : x \in \ell_2, \quad y - 2\log w \cdot x \in \ell_2\}$. In this case, $\mathrm{Dom}\,\Omega_{1,1} = \{x \in \ell_2 : 2\log w \cdot x \in \ell_2\} = \ell_2(\log w) = \{(0, x) \in \mathfrak{R}_2(w)\}$ and $\mathrm{Ran}\,\Omega_{1,1} = \ell_2((\log w)^{-1})$. Therefore, $(\Omega_{1,1})^{-1}x = \frac{1}{2\log w}x$ and $\mathrm{Dom}\,(\Omega_{1,0})^{-1} = \{x \in \ell_2((\log w)^{-1}) : (\log w)^{-1} \cdot x \in \ell_2(\log w)\} = \ell_2 = \mathrm{Ran}\,(\Omega_{1,0})^{-1}$ obtaining the following two inverse representations for $\mathfrak{R}_2(w)$

$$\begin{array}{ccccc} & & \ell_2((\log w)^{-1}) & & \\ & & \uparrow & & \\ \ell_2 & \longrightarrow & \mathfrak{R}_2(w) & \longrightarrow & \ell_2 \\ & & \uparrow & & \\ & & \ell_2(\log w) & & \end{array}$$

Let us present now a complete description of their higher order Rochberg spaces. In [31, Section 8] the reader can find a thorough description of the six diagrams that the first three Schecheter interpolators generate. Since $B(x)^k(z) = 2^k w^{2z-1} \log^k w \cdot x$ one obtains $\Delta_k B(x) = \frac{2^k}{k!} \log^k w \cdot x$ and therefore

$$\begin{aligned}\Omega_{1,k-1}x &= \left(\frac{2^{k-1}}{(k-1)!} \log^{k-1} w \cdot x, \dots, 2 \log w \cdot x\right) \\ &= \left(\frac{2^{k-1}}{(k-1)!} \log^{k-1} w, \dots, 2 \log w\right) \cdot x\end{aligned}$$

is also a linear map and

$$\mathfrak{R}_n(w) = \{(x_n, \dots, x_1) \colon x_1 \in \ell_2,\ (x_n, \dots, x_2) - \Omega_{1,n-1}x_1 \in \mathfrak{R}_{n-1}(w)\}.$$

We have

Proposition 5.1.

$$\mathfrak{R}_n(w) = \{(x_n, \dots, x_1) \colon x_n - \sum_{k=1}^{n-1} c_{n-k} \log^{n-k} w x_k \in \ell_2\},$$

where

$$c_k = \sum_{(i_1,\dots,i_m)}^{m\le k} \Bigg(\prod_{\substack{l=1,\dots,m \\ i_1+\dots+i_m=k}} (-1)^m \Big(\frac{2^{i_l}}{i_l!}\Big)\Bigg) = \sum_{\substack{(i_1,\dots,i_m) \\ i_1+\dots+i_m=k}}^{m\le k} \Bigg((-1)^m \Big(\frac{2^{i_1}}{i_1!} \cdots \frac{2^{i_m}}{i_m!}\Big)\Bigg).$$

Sketch of the Proof. Observe two facts:

(1) If we call ω_m the mth entry of $\Omega_{1,n}$, namely $\omega_m(x) = \frac{2^m}{m!} \log^m w \cdot x$ then

$$\omega_k(\omega_m(x)) = \frac{2^k}{k!} \frac{2^m}{m!} \log^{k+m} w \cdot x.$$

(2) Whenever $(x_n, \dots, x_1) \in \mathfrak{R}_n(w)$, the last component x_1 is shifted by $\Omega_{1,n-1}$ to the other $n-1$ entries to form the element $\big(x_n - \omega_{n-1}(x_1), \dots, x_2 - \omega_2(x_1)\big) \in \mathfrak{R}_{n-1}(w)$. Iterating this process one gets

$$\big(x_n - \omega_{n-1}(x_1) - \omega_{n-2}(x_2 - \omega_1(x_1)), \dots, x_3 - \omega_2(x_1) - \omega_1(x_2 - \omega_1(x_1))\big) \in \mathfrak{R}_{n-2}(w)$$

and since ω_k is linear, we obtain

$$x_n - \omega_{n-1}(x_1) - \omega_{n-2}(x_2 - \omega_1(x_1)) = x_n - \omega_{n-1}x_1 - \omega_{n-2}(x_2) + (-1)^2\, \omega_{n-2}\omega_1(x_1).$$

Therefore, we only have to worry about the coefficients of compositions like (1).

To handle them, observe that the sum $k+m$ in $\omega_k\omega_m x$ indicates how many positions to the left we have shifted the elements from their initial position. Moreover, the coefficients appearing keep the track of the succession of shifted positions. For example, $\frac{2^1}{1!} \frac{2^2}{2!} \frac{2^{n-4}}{(n-4)!}$ denotes that first we shifted $n-4$ positions, later 2 positions and lastly 1 position until the end. Since the process can be iterated $n-1$ times until reaching an element of $\mathfrak{R}_1(w) = \ell_2$, we deduce that the coefficients of a given coordinate describe all the possible ways of shifting such element to the first coordinate. One can deduce the formula for c_k from here. □

In this case, all Rochberg spaces $\mathfrak{R}_n(w)$ are isomorphic to ℓ_2 because all differentials are linear and thus all exact sequences involved split. The coefficients c_{n-1} indicate their growing complexity. A more elementary and more general construction for Köthe spaces X over a measurable space (S, μ) was presented in [33, Prop. 4.1]:

Proposition 5.2. *Let X be a Köthe function space with the Radon–Nikodym property and w_0, w_1 two weights. Then $(X(w_0), X(w_1))_\theta = X(w_0^{1-\theta} w_1^\theta)$ for every $0 < \theta < 1$ with associated linear derivation $\Omega_{1,1}x = w_0^{1-\theta} w_1^\theta \log \frac{w_1}{w_0} \cdot x$.*

6. Rochberg spaces for the scale of ℓ_p-spaces

We study now the, arguably, most important case: Rochberg spaces associated to the interpolation scale of ℓ_p spaces. As we have explained in Section 2.3, while we work inside Köthe spaces, complex interpolation and Lozanovskii factorization yield the same spaces so we will work with any of those approaches indistinctly. If we consider the pair (ℓ_∞, ℓ_1) for which the celebrated Riesz–Thorin Theorem [6] yields $(\ell_\infty, \ell_1)_\theta = \ell_{\theta^{-1}}$ for $0 \le \theta \le 1$, a bounded homogeneous selection for the evaluation map $\delta_{\frac{1}{p}} : \mathcal{C} \to \ell_p$ is given for normalized positive $x \in \ell_p$ by $B_p(x)(z) = x^{pz}$. Therefore $B'_p(x)(z) = p x^{pz} \log x$ and thus, after homogenization, we obtain the differential

$$\mathsf{KP}(x) = \mathsf{KP}_{1,1}x = \Delta_1 B_p(x)(p^{-1}) = p\, x \log\Big(\frac{|x|}{\|x\|_p}\Big),$$

usually called the Kalton–Peck map and denoted, if no confusion arises, simply by KP. The associated Rochberg space $\mathfrak{R}_2(\ell_p) = \mathfrak{R}_2(\ell_\infty, \ell_1)_{1/p}$ is the usually called Kalton–Peck space [75]

$$\mathfrak{R}_2(\ell_p) = Z_p = \{(w, x) \in \ell_\infty \times \ell_p : w - px \log \frac{|x|}{\|x\|} \in \ell_p\}.$$

The domain and range of KP acting on ℓ_2 have been already calculated. A few modifications, left to the reader, yield the corresponding domain and range spaces of KP on ℓ_p. However, basic questions such as the following are open:

Problem 10. Provide an explicit description for KP^{-1}.

Some discussion regarding this problem appears in [32, Section 5.3].

We describe now the higher order Rochberg spaces $\mathfrak{R}_n(\ell_2)$ that we will call from now on $\mathfrak{Z}_n$. For the rest of ℓ_p spaces, the construction is entirely analogous, although the properties of the spaces may be not. To that end, we compute the higher order derivatives of $B_2(x)$: for normalized x we have

$$B_2^{(k)}(x)(z) = 2^k x^{pz} \log^k x. \tag{24}$$

Therefore, the kth component of $\mathsf{KP}_{1,n}$ is

$$\mathsf{kp}_k(x) = \Delta_k B_2(x)(1/p) = \frac{2^k}{k!}\, x \log^k\Big(\frac{|x|}{\|x\|_2}\Big)$$

so that the differential $\mathsf{KP}_{1,n} : \ell_2 \curvearrowright \mathfrak{Z}_n$ is

$$\mathsf{KP}_{1,n}x = \Big(\frac{2^n}{n!}\, x \log^n\Big(\frac{|x|}{\|x\|_2}\Big), \ldots, 2\, x \log\Big(\frac{|x|}{\|x\|_2}\Big)\Big).$$

The map $\mathsf{KP}_{1,n}$ generates the exact sequence $0 \longrightarrow \mathfrak{Z}_n \longrightarrow \mathfrak{Z}_{n+1} \longrightarrow \ell_2 \longrightarrow 0$ and one therefore has

$$\mathfrak{Z}_{n+1} = \{(x_{n+1}, \ldots, x_1) \in \ell_\infty^n \times \ell_2 : (x_{n+1}, \ldots, x_2) - \mathsf{KP}_{1,n}(x_1) \in \mathfrak{Z}_n\}$$

endowed with the quasinorm

$$\|(x_{n+1}, \ldots, x_1)\|_{\mathfrak{Z}_{n+1}} = \|(x_{n+1}, \ldots, x_2) - \mathsf{KP}_{1,n}x_1\|_{\mathfrak{Z}_n} + \|x_1\|_{\ell_2}.$$

As we strived to explain, there are $2(n-1)$ "canonical representations" (except for $n = 1$ that there is just one) of $\mathfrak{Z}_n$ as a twisted sum of two lower order Rochberg spaces: for any $k, m \in \mathbb{N}$ so that $n = k+m$ one also has $0 \longrightarrow \mathfrak{Z}_m \longrightarrow \mathfrak{Z}_n \longrightarrow \mathfrak{Z}_k \longrightarrow 0$ generated by $\mathsf{KP}_{k,m}$ (which only depends on k and m and so that $\mathfrak{R}_n$ is isomorphic to the Banach space

$$\mathfrak{Z}_n = \{(x_n, \ldots, x_1) \in \ell_\infty^m \times \mathfrak{Z}_k : (x_n, \ldots, x_{k+1}) - \mathsf{KP}_{k,m}(x_k, \ldots, x_1) \in \mathfrak{Z}_m\}$$

under the quasinorm

$$\|(x_n, \ldots, x_1)\|_{\mathfrak{Z}_n} = \|(x_n, \ldots, x_1) - \mathsf{KP}_{k,m}(x_k, \ldots, x_1)\|_{\mathfrak{Z}_m} + \|(x_k, \ldots, x_1)\|_{\mathfrak{Z}_k}.$$

The problem is that it is not as easy as it seems to obtain the explicit description of $\mathsf{KP}_{k,m}$ out of thin air. Of course it *can* be done [9,17,31] and *has been* done. It is however unquestionable that the explicit description of higher Rochberg spaces turns out increasingly difficult. Even so, many interesting properties of $\mathfrak{Z}_n$ have been uncovered, and the rest of this Section is devoted to them.

The Rochberg spaces $\mathfrak{Z}_n$ possess all 3-space properties that Hilbert spaces enjoy. However, none of the space are Hilbert spaces [17, Corollary 6] or possess unconditional basis (see Theorem 6.8). Therefore, they behave simultaneously very differently than Hilbert spaces and very much like Hilbert spaces. Duality is one of those similar aspects:

6.1. Duality

The following result is due to Cabello [10]:

Theorem 6.1. *Let X be a Köthe superreflexive function space such that $(X, X^*)_{1/2} = L_2$ with associated differential Ω. Then $L_2 \oplus_\Omega L_2$ is isomorphic to its dual.*

Duality of twisted Hilbert spaces was studied first by Kalton and Peck, who showed [75, 5.1] that $Z_2^* \simeq Z_2$ and, in general, $Z_p^* \simeq Z_q$ whenever $1 = 1/p + 1/q$, under a "twisted duality": given (x, y) and (z, w) finitely supported elements of Z_p and Z_{p^*}, respectively, the duality map is given by

$$\langle (x, y), (z, w)\rangle = \langle x, w\rangle + \langle y, z\rangle.$$

Duality for higher order Rochberg spaces obtained from families of *finite dimensional* Banach spaces was studied by Rochberg himself in [97]. The study has been extended in [9] to cover the infinite dimensional case for the scale ℓ_p-spaces. For operative reasons, let us use in this section the notation $\mathfrak{R}_n(\theta)$ instead of the so far standard $\mathfrak{R}_n(\ell_{1/\theta})$. We have [9, Prop. 5.5]:

Theorem 6.2. *The linear map $D_n : \mathfrak{R}_n(1-\theta) \to \mathfrak{R}_n(\theta)^*$ given by*

$$D_n\big((x_n, \dots, x_1)\big)(y_n, \dots, y_1) = \sum_{k=1}^{n} (-1)^{k+1} \langle x_k, y_{n-k+1} \rangle \tag{25}$$

is an isomorphism. In particular, each space $\mathfrak{R}_n(1/2)$ is isomorphic to its dual.

In [43] we present an interpolation-free proof for this last result. If we focus on the case $\theta = 1/2$, so that $\mathfrak{Z}_n = \mathfrak{R}_n(1/2)$, the result is much deeper of what it sounds at first hearing: not only $\mathfrak{Z}_n$ is isomorphic to its dual in an explicit (non canonical) way, but the isomorphism also preserves the representation as twisted sum of lower order Rochberg spaces. Precisely (see [9]):

Theorem 6.3. *The following diagrams are commutative*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{Z}_m & \longrightarrow & \mathfrak{Z}_n & \longrightarrow & \mathfrak{Z}_k & \longrightarrow & 0 \\
 & & \Big\downarrow D_k & & \Big\downarrow D_n & & \Big\downarrow D_l & & \\
0 & \longrightarrow & \mathfrak{Z}_m^* & \longrightarrow & \mathfrak{Z}_n^* & \longrightarrow & \mathfrak{Z}_k^* & \longrightarrow & 0
\end{array}$$

Recall that, by Hahn–Banach, the dual of an exact sequence is an exact sequence. The combination of this last fact and Theorem 6.3 implies [40, Th. 5.1] that if $\mathsf{KP}^*_{m,k}$ denotes the quasilinear map that generates the dual sequence of
$0 \longrightarrow \mathfrak{Z}_k \longrightarrow \mathfrak{Z}_n \longrightarrow \mathfrak{Z}_m \longrightarrow 0$ then

$$D_m \mathsf{KP}_{k,m} \equiv \mathsf{KP}^*_{m,k} D_k$$

See [9,31,40,45] or else [8, Section 3.8] for a detailed explanation of this construction. In particular, it follows that Ran $\mathsf{KP}_{k,m}$ = Dom $(\mathsf{KP}_{m,k})^*$ and Dom $\mathsf{KP}_{k,m}$ = Ran $(\mathsf{KP}_{m,k})^*$.

6.2. Basic sequences

Let us start with an obvious fact: $\mathfrak{Z}_1 = \ell_2$ is also the Orlicz space ℓ_{f_0} generated by the function $f_0(t) = t^2$. Then, as we have already explained, Kalton and Peck showed in [75] $\mathfrak{Z}_2 = Z_2$ contains ℓ_2 and the Orlicz space ℓ_{f_1} generated by the Orlicz function $f_1(t) = t^2 \log^2 t$. In [31] it is shown that $\mathfrak{R}_3$ contains, in addition to that, the Orlicz space ℓ_{f_2} generated by the function $f_2 = t^2 \log^4 t$. The general case is studied in [43]:

Proposition 6.4. *The space $\mathfrak{Z}_n$ contains all the Orlicz spaces ℓ_{f_k} generated by the Orlicz functions $f_k(t) = t^2 \log^{2k} t$ for $0 \le k \le n-1$.*

More precisely,

$$\ell_{f_k} = \Delta_{n-1-k}\left(\bigcap_{\substack{0\le j\le n-1\\ j\neq n-k}} \ker \Delta_j\right)$$

namely, $\ell_{f_0} = \left[(e_j, 0, \ldots, 0) : j \in \mathbb{N}\right]$, $\ell_{f_k} = \left[(0, 0, \ldots, e_j, 0, \ldots, 0) : j \in \mathbb{N}\right]$ and $\ell_{f_{n-1}} = \left[(0, 0, \ldots, e_j) : j \in \mathbb{N}\right]$. Moreover $\ell_{f_n} = \mathsf{Dom}\mathsf{KP}_{1,n}$. From that, in [43] it is obtained a remarkable result regarding the structure of $\mathfrak{Z}_n$:

Theorem 6.5. *Every normalized basic sequence in $\mathfrak{Z}_n$ contains a subsequence equivalent to the basis of one of the spaces ℓ_{f_k}, $0 \le k \le n-1$.*

This result immediately yields that $\mathfrak{Z}_n$ are not Banach lattices, do not contain complemented Banach lattices and are not complemented in any Banach lattice. In particular, they do not have Gordon–Lewis unconditional structure (GL-l.u.st.) and, therefore, they do not have unconditional basis. Let us present here another proof for this last fact shaped upon the original proof of Johnson, Lindenstrauss and Schechtman [65] that Z_2 fails to have GL-l.u.st. Recall that a Banach space X has GL-l.u.st. (Local Unconditional Structure of Gordon–Lewis) if there exists a constant K such that for every finite dimensional Banach space $F \subset X$, the inclusion map $i_F : F \hookrightarrow X$ factorizes through a Banach space Y with unconditional basis so that $\|U\|\,\|V\|\, uc(Y) \le K$, where $UV = i_F$ and $uc(Y)$ is the unconditionality constant of Y. The GL-l.u.st. property is a rather weak form of unconditionality, as any Banach lattice has GL-l.u.st. Moreover, any complemented subspace of a Banach space with GL-l.u.st. also has GL-l.u.st. The key of the Johnson, Lindenstrauss, Schechtman approach is:

Lemma 6.6. *A superreflexive Banach space with an UFDD $(E_n)_{n\in\mathbb{N}}$ has GL-l.u.st. if and only if there exists a Banach space Y with an unconditional basis $((y_{i,n})_{i=1}^{k_n})_{n=1}^{\infty}$ such that:*

(i) *$E_n \subset span\{y_{i,n}\}_{i=1}^{k_n}$ for each $n \in \mathbb{N}$;*
(ii) *there exists a bounded projection $P : Y \to X$ such that $P(span\{y_{i,n}\}_{i=1}^{k_n}) = E_n$ for each $n \in \mathbb{N}$.*

We extend now their argument from $\mathfrak{Z}_2$ to $\mathfrak{Z}_n$. We begin with a technical result of independent interest. Observe that the spaces $\{E_k : k \in \mathbb{N}\}$ with $E_k = \{(e_k, 0, \ldots, 0), (0, e_k, 0, \ldots, 0), \ldots, (0, 0, \ldots, e_k)\}$ form an UFDD for $\mathfrak{Z}_n$. Moreover, operators $\mathfrak{Z}_n \longrightarrow \mathfrak{Z}_n$ can be interpreted as matrices with linear maps as entries (see [42]). When these entries are scalars, things are simpler: Johnson, Lindenstrauss and Schechtman [65, Prop. 3] show (see also [5]) that a scalar matrix extends to an operator on $Z_2 = \mathfrak{Z}_2$ if and only if has the form $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$. We extend this result to $\mathfrak{Z}_n$.

Lemma 6.7. *A $n \times n$ scalar matrix*

$$\begin{pmatrix} \alpha_1 & \beta_1 & \delta_1 & \cdots & \varepsilon_1 & \pi \\ \gamma_{21} & \alpha_2 & \beta_2 & \delta_2 & \cdots & \varepsilon_2 \\ \gamma_{31} & \gamma_{32} & \alpha_3 & \beta_{n-2} & \delta_{n-3} & \cdots \\ \vdots & \gamma_{n-2\,n-4} & \gamma_{n-2\,n-3} & \alpha_{n-2} & \beta_{n-2} & \delta_{n-2} \\ \gamma_{n-1\,1} & \vdots & \gamma_{n-1\,n-3} & \gamma_{n-1\,n-2} & \alpha_{n-1} & \beta_{n-1} \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{n\,n-2} & \gamma_{n\,n-1} & \alpha_n \end{pmatrix} \tag{26}$$

extends to a bounded operator on $\mathfrak{Z}_n$ if and only if it has the form

$$\begin{pmatrix} \alpha & \beta & \delta & \cdots & \varepsilon & \pi \\ 0 & \alpha & \beta & \delta & \cdots & \varepsilon \\ 0 & 0 & \alpha & \beta & \delta & \cdots \\ \vdots & 0 & 0 & \alpha & \beta & \delta \\ 0 & \cdots & 0 & 0 & \alpha & \beta \\ 0 & \cdots & 0 & 0 & 0 & \alpha \end{pmatrix}. \tag{27}$$

Proof. Fix $m = 0, \dots, n$. The operator having all entries 0 except those at places $(k, k+m)$ for $k = 1, \dots, n-m$ which are 1 is bounded on $\mathfrak{Z}_n$: indeed, one only has to represent $\mathfrak{Z}_n$ as a twisted sum of $\mathfrak{Z}_k$ and $\mathfrak{Z}_m$ for $n = k + m$ and work inductively. This proves the sufficiency. To prove the necessity, work again by induction on n: restricting ourselves to vectors of the form $(x_1, x_2, \dots, x_{n-1}, 0) \in \mathfrak{Z}_n$ we may assume that the thesis is verified on the elements of the upper-right $(n-1) \times (n-1)$ submatrix. Since $\mathfrak{Z}_n^* \simeq \mathfrak{Z}_n$, one can apply induction once again to deduce the thesis for all elements except γ_{n1} and the pair ε_1 and ε_2.

Assume that $\gamma_{n1} \neq 0$. Taking the sequence $u_n = \frac{1}{\sqrt{n}}(\sum_k^n e_k, 0, 0, \dots, 0)$ we deduce that $\|u_n\| = 1$ for all $n \in \mathbb{N}$; however,

$$\begin{aligned}\|Tu_n\| &\geq C\Big[\frac{1}{\sqrt{n}}\|(\alpha \sum_k^n e_k, 0, \dots, 0)\| + \frac{1}{\sqrt{n}}\|(0, 0, \dots, \gamma_{n1} \sum_k^n e_k)\|\Big] \\ &\geq C\frac{|\gamma_{n1}|}{\sqrt{n}}\sqrt{n}\log^n n\end{aligned}$$

for some constant $C > 0$, so T is not bounded. On the other hand, if $\varepsilon_1 \neq \varepsilon_2$, then we take the sequence

$$u_n = \frac{1}{\sqrt{n}}(\log^{n-1}\sqrt{n}\sum_k^n e_k, \log^{n-2}\sqrt{n}\sum_k^n e_k, \dots, \log\sqrt{n}\sum_k^n e_k, \sum_k^n e_k).$$

Thus we have

$$\|Tu_n\| \geq c\|(\varepsilon_1 \log\sqrt{n}\sum_k^n e_k, \varepsilon_2 \sum_k^n e_k)\|_{Z_2} \geq c|\varepsilon_1 - \varepsilon_2|\log\sqrt{n}$$

for some constant $c > 0$ and T is not bounded again. □

Given a sequence $(A_k)_k$ of $n \times n$ matrices of the form (26), let us denote by $\sum_k A_k$ the operator on $\mathfrak{R}_n$ defined by $\sum_k A_k|_{E_k} = A_k$ for each $k \in \mathbb{N}$. Observe that the previous

Lemma 6.7 can be restated as: consider a constant sequence of $n \times n$ matrices $A_k = A$. Then the operator $\sum_k A_k$ is bounded on $\mathfrak{R}_n$ if and only if A is of the form (27). If $A_k = A + K_k$, where K_k are small perturbations (for instance satisfying $\| \sum K_k \| < \infty$), then $\sum_k (A + K_k)$ is still bounded if and only if A is of the form (27).

We are ready to show how the Johnson–Lindenstrauss–Schechtman arguments in [65] for Z_2 can be translated to higher order Rochberg spaces.

Theorem 6.8. *$\mathfrak{Z}_n$ does not have GL-l.u.st.*

Proof. If $\mathfrak{Z}_n$ has GL-l.u.st. then it is a complemented subspace of a Banach space with an unconditional basis with the properties stated in Lemma 6.6. That should allow us to define a bounded operator $\widehat{T} : \mathfrak{Z}_n \to \mathfrak{Z}_n$ such that

$$\widehat{T}(E_n) \subset E_n \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \inf_n \operatorname{dist}(\widehat{T}|_{E_n}, T) > 0,$$

where the T represents any operator of the form (27).

The operator $\hat{T}$ can be obtained as follows: suppose we are in the conditions stated in Lemma 6.6 and denote by $y^*_{i,n}$ the biorthogonal vectors of the unconditional basis $((y_{i,n})_{i=1}^{k_n})_{n=1}^{\infty}$. Taking into account that such basis is unconditional, given any set of indices J, the operator

$$T_J(x) = P\Big(\sum_n \sum_{i \in J} (y_{i,n} \otimes x) y_{i,n}\Big)$$

is bounded on $\mathfrak{Z}_n$. It suffices then to obtain for each $n \in \mathbb{N}$ a subset $J_n \subset \{1, \ldots, k_n\}$ such that $\operatorname{dist}\big((T_{J_n})|_{E_n}, T\big) \geq \mu$ for all $n \in \mathbb{N}$ and some absolute constant $\mu > 0$. Our operator will be then $\hat{T} = T_J$ where $J = \bigcup_n J_n$.

To find J_n, fix $n \in \mathbb{N}$ and define for each $1 \leq i \leq k_n$ the following operator on E_n

$$T_i(x) = P\big((y^*_{i,n} \otimes x) y_{i,n}\big)$$

This is a rank one operator that must have the form of one of the following matrices (to simplify the notation, we can assume that, in each case, the first non-null row has normalized coefficients equal to 1)

$$\begin{pmatrix} a_i & b_i & \cdots & c_i \\ \alpha_i a_i & \alpha_i b_i & \cdots & \alpha_i c_i \\ \vdots & \vdots & \ddots & \vdots \\ \beta_i a_i & \beta_i b_i & \cdots & \beta_i c_i \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_i & b_i & \cdots & c_i \\ \vdots & \vdots & \ddots & \vdots \\ \beta_i a_i & \beta_i b_i & \cdots & \beta_i c_i \end{pmatrix}, \quad \cdots \quad \text{or} \quad \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_i & b_i & \cdots & c_i \end{pmatrix}.$$

Taking into account that P is linear and that $(y_{i,n})_{i=1}^{k_n}$ is a basis it follows that $\sum_i^{k_n} T_i = I$. Thus, summing up the last row of the matrix, we obtain by the unconditionality of the basis that

$$\sum_i^{k_n} \beta_i c_i = 1 \quad \text{and} \quad \sum_i^{k_n} |c_i| \leq K,$$

where K equals the unconditional constant of the basis multiplied by $\|P\|$. If we consider the set $I_n = \{i \in \{1, \ldots, k_n\} : |\beta_i| \geq \frac{1}{2K}\}$, then we have that $\sum_{i \in I_n} \beta_i c_i \geq 1/2$. Fix $k \in I_n$

and suppose that the associated matrix to this k is of the first form described; then one of the following cases occurs:

(1) $\beta_k a_k \geq \frac{1}{1+2K}|\beta_k c_k|$;
(2) $-\beta_k a_k \geq \frac{1}{1+2K}|\beta_k c_k|$;
(3) $a_k - \beta_k c_k \geq \frac{1}{1+2K}|\beta_k c_k|$;
(4) $\beta_k c_k - a_k \geq \frac{1}{1+2K}|\beta_k c_k|$.

If none of those possibilities occurred, then we can sum the following inequalities

$$|a_k| < \frac{1}{1+2K}|c_k| \leq \frac{2K}{1+2K}|\beta_k c_k|$$

and

$$|a_k - \beta_k c_k| < \frac{1}{1+2K}|\beta_k c_k|,$$

thus forcing $|c_k| < |c_k|$, which is absurd. If the matrix associated to k is any of the other matrices, then they satisfy the analogue of $|a_k - \beta_k c_k| \geq \frac{1}{1+2K}|\beta_k c_k|$, i.e., that $|\beta_k c_k| \geq \frac{1}{1+2K}|\beta_k c_k|$. This represents that there exist both a non-null entry and a null entry in the diagonal.

We deduce that there exists a subset $J_n \subset I_n$ such that $\sum_{i\in J_n} \beta_i c_i \geq 1/8$ and one of the four previous inequalities holds for each $i \in J_n$. This choice for the set J_n satisfies that $\operatorname{dist}\big((T_{J_n})|_{E_n}, T\big) \geq \mu > 0$, so we may define $\hat{T} = T_J$ for $J = \bigcup_n J_n$. Once that has been settled, we may choose a subsequence k_j of the integers such that we can define an operator on $\mathfrak{Z}_n$ extending the operators $\widehat{T}|_{E_{k_j}}$, each of which is a small perturbation of a fixed matrix that's not of the form described in the previous Lemma 6.7, which is a contradiction by the comments before the proof (see also [5, pp.383]). □

Immediate consequences of this result are:

Corollary 6.9. *$\mathfrak{Z}_n$ is not complemented in a Banach space with GL-l.u.st. In particular, in a Banach lattice.*

Moreover, using forthcoming Theorem 6.18 (1) it follows:

Corollary 6.10. *$\mathfrak{Z}_n$ does not contain any infinite dimensional complemented Banach space with GL-l.u.st. In particular, a Banach lattice.*

6.3. Natural subspaces of $\mathfrak{Z}_n$

The topic of subspaces of twisted sum spaces is complicated in itself. Think about the simplest (non trivial) space in this scale: Z_2. One can currently identify only a small number of subspaces of this space: ℓ_2, ℓ_{f_1}, Z_2 and a recently identified new subspace [78], plus their products. That's all.

In $\mathfrak{Z}_n$ spaces we encounter a family of distinguished subspaces. Given a subset $A = \{i_1, \ldots, i_k\} \subset \{0, \ldots, n-1\}$ with $i_1 < \cdots < i_k$ we denote by Δ_A the interpolator

$(\Delta_{i_k}, \ldots, \Delta_{i_1})$ and define the subspace $X_A \subset \mathfrak{Z}_n$ as

$$X_A = \Delta_A \left(\bigcap_{0 \le k \le n-1, k \notin A} \ker \Delta_k \right).$$

We have already encountered subspaces of this type: for $k = 0, \ldots, n-1$ we already know that the choice $A = \{k\}$ yields $X_A = \ell_{f_{n-k-1}} = \mathrm{Dom}\mathsf{KP}_{1,n-k-1}$. If we call $X_{[l,k]}$ the space generated by sets $A = [l,k] = \{l, l+1, l+2, \ldots, k\}$ when $l < k$ then $X_{[l,k]} = \mathrm{Dom}(\mathsf{KP}_{k-l+1,n-k-1})$ if $0 < l < k < n-1$ and $X_{[0,k]} = \mathrm{Dom}(\mathsf{KP}_{k+1,n-k-1})$. This suggests that X_A spaces should be regarded as domains or ranges of "some" differentials.

Problem 11. Identify the natural subspaces of $\mathfrak{Z}_n$ as Domain or Range spaces.

As we said before, the Rochberg spaces associated to a scale of weighted ℓ_2 spaces are isomorphic. However, that is no longer true for the scale of ℓ_p spaces due to the following result from [9]:

Theorem 6.11. *$\mathfrak{Z}_n$ is not isomorphic to a subspace of $\mathfrak{Z}_m$ when $m < n$.*

The result is consequence of the following estimate for the sequence $a_{n,2}(\mathfrak{Z}_n)$ of type 2 constants of the spaces:

$$a_{m,2}(\mathfrak{Z}_n) \sim a_{m,2}(\ell_{f_n}) \sim \log^n m$$

In particular $\mathfrak{Z}_n$ is not isomorphic to $\mathfrak{Z}_m$ when $n \neq m$. In combination with Theorem 6.18 we obtain that any operator $\mathfrak{Z}_m \longrightarrow \mathfrak{Z}_n$ is strictly singular when $m > n$.

Problem 12 (*A bit vague, we must admit*)**.** Generalize the previous results to spaces X_A with $|A| \ge 2$.

6.4. Rochberg spaces are symplectic without Lagrangian subspaces

In this section we need to work with the real versions of the spaces $\mathfrak{Z}_n$: i.e., the spaces of real sequences in $\mathfrak{Z}_n$. Recall from the introduction that a real Banach space X is said to be *symplectic* if there is a continuous alternating bilinear map $\omega : X \times X \to \mathbb{R}$ such that the induced map $L_\omega : X \to X^*$ given by $L_\omega(x)(y) = \omega(x, y)$ is an isomorphism onto. A symplectic Banach space is necessarily isomorphic to its dual and reflexive [36, Lemma 2.2]. During the decade of 1970's several authors drew the attention to the importance of the study of symplectic forms on Banach spaces and, more broadly, on Banach manifolds. For instance, in the proof of Weinstein [111] of an infinite dimensional version of the classical Darboux theorem for symplectic geometry, or in the Hamiltonian formulation of infinite dimensional mechanics due to Chernoff and Marsden [48]. See also Swanson [107,108] for various results about symplectic structures on Banach spaces. A subspace $F \subset X$ of a symplectic Banach space (X, ω) is called *isotropic* is $\omega(x, y) = 0$ for all $x, y \in F$. A *Lagrangian subspace* of (X, ω) is an isotropic complemented subspace whose complement is also isotropic.

Kalton and Swanson [76] solved in the negative the question raised by Weinstein [111] of whether every infinite dimensional symplectic Banach space is trivial, in the sense that there does exist a reflexive Banach space Y and an isomorphism $T : X \to Y \oplus Y^*$ such that $\omega(x, y) = \Omega_Y(Tx, Ty)$ for every $x, y \in X$, where

$$\Omega_Y[(z, z^*), (w, w^*)] = w^*(z) - z^*(w).$$

Observe that if so then $T^{-1}(Y \times \{0\})$ is a *Lagrangian* subspace of (X, ω). Thus, triviality means that the space does not contain (nontrivial) Lagrangian subspaces. Kalton and Swanson showed that Z_2 is a symplectic space with no Lagrangian subspaces. Moreover, it is essentially the only one known so far.

In [36] it is shown that the higher order Rochberg spaces $\mathfrak{R}_n$ can be added to this restricted list.

Theorem 6.12. *For all $n > 1$ the space $\mathfrak{Z}_n$ is symplectic and contains no Lagrangian subspace.*

What is remarkable in the proof is that while the symplectic structure of Rochberg spaces of even order is the one induced by the natural duality, the symplectic structure on Rochberg spaces of odd order requires to introduce a perturbation of the duality with a suitable complex structure. Precisely, given $n \geq 1$ one can consider the continuous bilinear map $\omega_n : \mathfrak{Z}_n \times \mathfrak{Z}_n \to \mathbb{R}$ given by

$$\omega_n\big((x_{n-1}, \ldots, x_0), (y_{n-1}, \ldots, y_0)\big) = \sum_{i+j=n-1} (-1)^i \langle x_i, y_j \rangle$$

for which the induced operator $D_n : \mathfrak{Z}_n \to \mathfrak{Z}_n^*$ given by $D_n(x)(y) = \omega_n(x, y)$ is an isomorphism onto. Since ω_n is alternating if (and only if) n is even, the result follows for even n.

For n odd, one has to additionally consider a complex structure σ on ℓ_2 acting on the scale; say, $\sigma(x) = (-x_2, x_1, -x_4, x_3, \ldots)$. The Generalized Commutator Theorem 4.7 shows that the $n \times n$ diagonal matrix operator τ_σ with σ at its entries is bounded on $\mathfrak{Z}_n$. It turns out that the bilinear map

$$\begin{aligned}\overline{\omega_n}\big((x_{n-1}, \ldots, x_0), (y_{n-1}, \ldots, y_0)\big) &= \omega_n\big((x_{n-1}, \ldots, x_0), \tau_\sigma(y_{n-1}, \ldots, y_0)\big)\\ &= \sum_{i+j=n-1} (-1)^i \langle x_i, \sigma y_j \rangle.\end{aligned}$$

is alternating because $\sigma^* = -\sigma$, is bounded because ω_n and τ_σ are and the induced linear map $L_{\overline{\omega_n}} : \mathfrak{Z}_n \to {\mathfrak{Z}_n}^*$ is an isomorphism onto, but proving that is a technical matter, see [36]. The proof that those symplectic structures are non-trivial when $n > 1$ requires new knowledge on the behaviour of operators on Rochberg spaces, something we will display in the next section. A topic not solved in [36] is whether the symplectic structure of $\mathfrak{R}_n$ is unique, something that could be true since the complex structure of ℓ_2 is unique, up to equivalence (see [59] for a larger background on this topic):

Problem 13. Does $\mathfrak{R}_n$ admit a unique, up to equivalence, non-trivial symplectic structure?

6.5. Operators on $\mathfrak{Z}_n$

Following Pietsch [91], an *operator ideal* $\mathfrak{A}$ is a subclass of the class $\mathfrak{L}$ of bounded operators between Banach spaces such that finite range operators belong to $\mathfrak{A}$, $\mathfrak{A}+\mathfrak{A} \subset \mathfrak{A}$ and $\mathfrak{L}\mathfrak{A}\mathfrak{L} \subset \mathfrak{A}$. The operator ideals we will mainly work with are those of compact (resp. strictly singular) operators, that will be denoted by $\mathfrak{K}$ (resp. $\mathfrak{S}$) [91, 1.4, 1.9 and 1.10]. Recall that an operator is called *compact* (resp.*strictly singular*) [91] if the closure of the image of the unit ball is a norm compact set (rep. its restriction to any infinite dimensional subspace is not an isomorphism). It is a standard fact that $\mathfrak{K} \subset \mathfrak{S}$. Given an operator ideal $\mathfrak{A}$ and $k \in \mathbb{N}$ we will denote by $\mathfrak{A}^k$ the class of operators formed as the composition of k operators of $\mathfrak{A}$.

In its own peculiar way, operators on $\mathfrak{Z}_n$ behave quite similarly as they do in Hilbert spaces:

- An operator $T : \ell_2 \to X$ is either strictly singular or an isomorphism on a complemented copy of ℓ_2 (see [79, Chapter 2]).
- An operator $T : \ell_2 \to \ell_2$ is either strictly singular or an isomorphism on a complemented copy E of ℓ_2 such that $T[E]$ is complemented in ℓ_2.

Kalton and Peck [75] and Kalton [69] extended these results to Z_2:

- Every operator $\tau : Z_2 \to X$ is either strictly singular or an isomorphism on a complemented copy of Z_2.
- Every operator $\tau : Z_2 \to Z_2$ is either strictly singular or an isomorphism on a complemented subspace $E \cong Z_2$ such that $\tau(E)$ is also complemented.

with a twist:

- The quotient map $Z_2 \longrightarrow \ell_2$ is strictly singular.
- An operator $T \in \mathfrak{L}(Z_2)$ is strictly singular if and only if its restriction to the canonical copy of ℓ_2 is strictly singular.

The first of these last assertions was proved by Kalton and Peck [75, Th. 6.4]. A different approach can be given [8, Chapter 9]: Let us call from now on a short exact sequence $0 \to Y \to Z \to X \to 0$ *singular* when its quotient is a strictly singular operator. It is not hard to prove that an exact sequence generated by a quasilinear map Ω is singular (we shall also say that Ω is singular) if and only if the restriction of Ω to an infinite dimensional subspace is never trivial. Using the following *Transfer Principle* [8, Chapter 9]

Lemma 6.13. *Let X be a Banach space with unconditional basis and $Q : Z \to X$ a quotient map. Then either Q is strictly singular or it is invertible on an infinite dimensional subspace of X generated by a block sequence.*

The singularity the Kalton–Peck map KP is immediate since given a block subspace U generated by the sequence $(u_n)_n$ of blocks, the restriction $\mathsf{KP}|_U$ to U has the form

$$\mathsf{KP}(\sum \lambda_n u_n) = \sum_n \lambda_n \log \frac{|\lambda_n|}{\|u\|} u_n + L(\sum \lambda_n u_n)$$

where $L : U \to \ell_2$ is a linear map.

A careful look at diagrams (15) plus the ideal property of $\mathfrak{S}$ shows [17] that all Rochberg sequences $0 \longrightarrow \mathfrak{Z}_n \longrightarrow \mathfrak{Z}_{n+m} \longrightarrow \mathfrak{Z}_m \longrightarrow 0$ are singular. To prove the other twisted assertion just keep in mind the following well known characterization (see [47, Prop. 3.2]):

Proposition 6.14. *A quotient map $Q : Z \to Z/Y$ is strictly singular if and only if for every infinite dimensional $Z' \subset Z$ there exist an infinite dimensional subspace $Y' \subset Y$ and a compact operator $K : Y' \to Z'$ such that $I + K : Y' \to Z'$ is an isomorphic embedding.*

It is now simple to obtain

• An operator $T \in \mathfrak{L}(\mathfrak{Z}_n)$ is strictly singular if and only if for some $m < n$ its restriction to some $\mathfrak{Z}_m$ is strictly singular

The proofs of the first two assertions are much more delicate and require to introduce the existence in $\mathfrak{L}(\mathfrak{Z}_n)$ of a large family of well behaved isometries called *block operators*. Let $(u_n)_n$ be a sequence of disjointly supported consecutive normalized blocks in ℓ_2. The operator $u : \ell_2 \to \ell_2$ given by $x \mapsto x \cdot u$ (seen as the product on each coordinate) is bounded since $\| \sum_n x_n u_n \| = \| \sum_n x_n e_n \|$.

Definition 6.15. Let u be a block sequence in ℓ_2. We inductively define the block operator $T_u^n \in \mathfrak{L}(\mathfrak{Z}_n)$ as: $T_u^1 = u$, and for $n > 1$, T_u^n is the upper triangular operator making the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{Z}_n & \longrightarrow & \mathfrak{Z}_{n+m} & \longrightarrow & \mathfrak{Z}_m & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle T_u^n} & & \downarrow {\scriptstyle T_u^{n+m}} & & \downarrow {\scriptstyle T_u^m} & & \\ 0 & \longrightarrow & \mathfrak{Z}_n & \longrightarrow & \mathfrak{Z}_{n+m} & \longrightarrow & \mathfrak{Z}_m & \longrightarrow & 0 \end{array}$$

commutative.

Let us see now how these block operators are generated and where does Definition 6.15 come from. The canonical example of block operators in $\mathfrak{Z}_2 = Z_2$ are the ones originally given by Kalton [69]: $T_u(e_n, 0) = (u_n, 0)$ and $T_u(0, e_n) = (\mathsf{KP}u_n, u_n)$. This is an injective isometry yielding a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle u} & & \downarrow {\scriptstyle T_u} & & \downarrow {\scriptstyle u} & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0 \end{array}$$

Recall from the Commutator Theorem that when τ is an operator acting on the scale then $T = \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix}$ is a bounded operator on $\mathfrak{R}_2$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_2 & \longrightarrow & \ell_2 & \longrightarrow & 0 \end{array}$$

commute. What occurs when τ is not an operator acting on the scale? To make the diagram commutative T has to have the form $\begin{pmatrix} \tau & \square \\ 0 & \tau \end{pmatrix}$, and since $\begin{pmatrix} \tau & -[\tau, \mathsf{KP}] \\ 0 & \tau \end{pmatrix}$ makes the diagram commutative (although it is not linear), $\square$ will have to be a linear map at finite distance from the commutator $-[\tau, \mathsf{KP}]$ (see [42] for details). We have a situation here: how to find out such $\square$ linear map? Moreover, there might exist many valid choices for $\square$. Observe that when u is the operator induced by a sequence (u_n) of blocks of disjointly supported normalized blocks in ℓ_2, the operator u is not compatible with the scale generated by (ℓ_∞, ℓ_1) because the u_n are normalized blocks in ℓ_2 only and thus something having the form $\begin{pmatrix} u & \square \\ 0 & u \end{pmatrix}$ is needed. The proposal of Kalton was to set $\begin{pmatrix} u & \mathsf{KP}u \\ 0 & u \end{pmatrix}$ where $\mathsf{KP}u$ is the linear map induced by the (not necessarily uniformly bounded) sequence of blocks $(\mathsf{KP}u_n)_n$. In [37] the authors developed a theory to somehow unify the commutator theorem and the existence of block operators through the study of actions of a semigroup G compatible with the structure of exact sequences. What is interesting for us here is that given a normalized sequence $u = (u_n)_n$ of blocks on ℓ_2, we can define, for each $z \in \mathbb{S}$, the sequence of blocks $u^{2z} = (u_n^{2z})_n$ normalized in $\ell_{1/z}$ and that therefore defines a bounded operator on $\ell_{1/z}$. The theory of analytic semigroups acting on interpolation scales as developed in [21,37] shows that the role of $\square$ can be played by the *derivative* of the action u^{2z} (with respect to z) at $1/2$ [37, Theorem 4.7], namely $2u \log u = \mathsf{KP}u$. And this is indeed Kalton's block operator [37, Section 6]. Some variations in the argument (not simple iteration) [37, Section 7] yield that the matrices

$$T_U^n = \begin{pmatrix} u & \frac{2^1}{1!} u \log u & \frac{2^2}{2!} u \log^2 u & \cdots & \frac{2^{n-1}}{(n-1)!} u \log^{n-1} u \\ 0 & u & \frac{2^1}{1!} u \log u & \frac{2^2}{2!} u \log^2 u & \cdots \\ 0 & 0 & u & \frac{2^1}{1!} u \log u & \frac{2^2}{2!} u \log^2 u \\ 0 & 0 & 0 & u & \frac{2^1}{1!} u \log u \\ 0 & 0 & 0 & 0 & u \end{pmatrix}$$

define bounded block operators on $\mathfrak{Z}_n$. To prove the assertion about the complemented range we need to come back to the symplectic structure. Block operators are *symplectic*, in the sense that

$$\omega_n(T_U^n x, T_U^n y) = \omega_n(x, y), \tag{28}$$

or, equivalently, that $(T_U^n)^+ T_U^n = I$ [36, Prop. 6.2]. This can be put in a broader context: if we define the symplectic dual $T^+ : \mathfrak{Z}_n \to \mathfrak{Z}_n$ of an operator $T : \mathfrak{Z}_n \to \mathfrak{Z}_n$ [36,37] as the operator $T^+ \in \mathfrak{L}(\mathfrak{Z}_n)$ such that

$$\omega_n(T^+x, y) = \omega_n(x, Ty)$$

it turns out that the symplectic dual can be seen as the translation to higher order Rochberg spaces of the Hilbert dual of an operator on ℓ_2. In fact, symplectic and classical duals are connected by a commutative diagram:

$$\begin{array}{ccc} \mathfrak{Z}_n & \xrightarrow{D_n} & \mathfrak{Z}_n^* \\ {\scriptstyle T^+}\downarrow & & \downarrow{\scriptstyle T^*} \\ \mathfrak{Z}_n & \xrightarrow[D_n]{} & \mathfrak{Z}_n^* \end{array}$$

In other words, $T^+ = D_n^{-1}T^*D_n$. Thus, since D_n is an isomorphism, it is not strange that most of the properties of T^+ coincide with those of T^*: for instance, $\|T\| \sim \|T^+\|$, ImT is closed if and only if ImT^+ is closed, and, therefore, T is an isomorphism into if and only if T^+ is surjective.

Still one more digression before we return to our main topic in [108] Swanson showed that the symplectic group, formed by those operators that preserve a given symplectic form, is a contractible subgroup of $GL(\ell_2)$.

Problem 14. Is the symplectic group of $\mathfrak{Z}_n$ contractible. Is it path connected? It is not even known if $GL(Z_2)$ is path connected [42, Question 4].

OK, we are back: why block operators are important in the analysis of operators in Rochberg spaces? Because of the following result [36,69]:

Lemma 6.16. *If $T : \mathfrak{Z}_n \to \mathfrak{Z}_n$ is not strictly singular then there exist $\alpha \neq 0$ and block operators T_U^n and T_V^n such that $TT_W^n - \alpha T_V^n$ is strictly singular.*

The result says that if $T : \mathfrak{Z}_n \to \mathfrak{Z}_n$ is not strictly singular then there is a complemented isometric copy W of $\mathfrak{Z}_n$ so that $T|_W$ is, up a strictly singular operator, a multiple of a block operator. From here, we can obtain [36,42,69]:

Proposition 6.17. *Let $T : \mathfrak{Z}_n \to \mathfrak{Z}_n$ be any operator. If T^+T is strictly singular then T is strictly singular.*

Proof. We present the proof due to its simplicity: if T is not strictly singular, by Lemma 6.16 there exist $\alpha \neq 0$ and block operators T_U^n, T_V^n such that $TT_U^n = \alpha T_V^n + S$ with $S \in \mathfrak{S}(\mathfrak{Z}_n)$. Therefore

$$\begin{aligned}(T_U^n)^+T^+TT_U^n &= (TT_U^n)^+TT_U = (\alpha(T_V^n)^+ + S^+)(\alpha T_V + S) = \alpha'(T_V^n)^+T_V^n + S' \\ &= \alpha' I + S',\end{aligned}$$

where $S' \in \mathfrak{S}(\mathfrak{Z}_n)$. Since T^+T is strictly singular, this implies that the identity I is strictly singular, a contradiction. □

And from here we finally obtain several extensions of Kalton's results from Z_2 to $\mathfrak{Z}_n$ [43]:

Theorem 6.18.

- *Every operator* $\tau : \mathfrak{Z}_n \to X$ *is either strictly singular or an isomorphism on a complemented copy of* $\mathfrak{Z}_n$.
- *Every operator* $\tau : \mathfrak{Z}_n \to \mathfrak{Z}_n$ *is either strictly singular or an isomorphism on a complemented subspace* $E \cong \mathfrak{Z}_n$ *such that* $\tau[E]$ *is also complemented.*
- $\mathfrak{S}^l(\mathfrak{Z}_n) \neq \mathfrak{K}(\mathfrak{Z}_n)$ *for all* $1 \leq l \leq n-1$.
- $\mathfrak{S}^n(\mathfrak{Z}_n) = \mathfrak{K}(\mathfrak{Z}_n)$.
- $\mathfrak{S}(\mathfrak{Z}_n) = \mathfrak{C}(\mathfrak{Z}_n)$ *is the only nontrivial maximal ideal of* $\mathfrak{L}(\mathfrak{Z}_n)$.

6.6. Complemented subspaces of $\mathfrak{Z}_n$

Theorem 6.18 implies that any complemented subspace of $\mathfrak{Z}_n$ contains a complemented copy of $\mathfrak{Z}_n$, but it is not known if every complemented subspace of $\mathfrak{Z}_n$ is isomorphic to $\mathfrak{Z}_n$. Actually, the result is an extension of a famous long-standing open problem:

Problem 15. Is Z_2 is isomorphic to its hyperplanes?

Here is what is currently known [43]:

Theorem 6.19.

- *Every infinite dimensional complemented subspace of* $\mathfrak{Z}_n$ *contains a further complemented subspace isomorphic to* $\mathfrak{Z}_n$.
- *Every subspace of* $\mathfrak{R}_n$ *isomorphic to* $\mathfrak{Z}_n$ *is complemented.*
- *The complement of two infinite codimensional copies of* $\mathfrak{Z}_n$ *in* $\mathfrak{Z}_n$ *are isomorphic.*
- $\mathfrak{Z}_m$ *is not complemented in* $\mathfrak{Z}_n$ *for* $m \neq n$.

7. Rochberg spaces for scales of L_p-spaces

A large part of our analysis about Rochberg spaces associated to the scale of ℓ_p spaces can be translated to the scale of $L_p(\mu)$ spaces, but there are important issues to consider. On the mimicry side we have that, independently on the base measure space, the interpolation formula $(L_1, L_\infty)_{1/2} = L_2$ yields the L_∞-centralizer $\mathsf{KP}(f) = 2f \log \frac{|f|}{\|f\|}$ as associated differential on L_2 with corresponding Rochberg spaces $\mathfrak{R}_n(L_2)$ obtained following the general construction and, in particular,

$$\mathfrak{R}_2(L_2) = \{(f, g) \in L_\infty \times L_2 : f - \mathsf{KP}(g) \in L_2\}$$

endowed with the corresponding quasinorm. The spaces $\mathfrak{R}_2(\ell_2)$ and $\mathfrak{R}_2(L_2)$ are not isomorphic when the underlying measure space is not atomic, even if L_2 and ℓ_2 are isometric Banach spaces: observe that the sequence

$$0 \longrightarrow L_p \longrightarrow \mathfrak{R}_2(L_p) \longrightarrow L_p \longrightarrow 0 \tag{29}$$

is not singular. This was first proved in [104] for $0 < p < \infty$ and shortly thereafter, Cabello [13] extended the result to any L_∞-centralizer on L_p for $0 < p < \infty$. The result was subsequently extended in [35] to superreflexive Köthe (nonatomic) spaces. All these results essentially consist in showing that centralizers become bounded on the subspace spanned by the Rademacher functions, isomorphic to ℓ_2. Of course that such a distinct behaviour of $\mathfrak{R}_2(L_2)$ and $\mathfrak{R}_2(\ell_2)$ is possible because the isomorphism between L_2 and ℓ_2 is not induced by operators acting on the scales. Let us consider now two interesting cases in which the Rochberg spaces become isomorphic.

- *Hardy spaces* Let $\mathbb{T}$ be the unit circle and by m the normalized Lebesgue measure on $\mathbb{T}$. If $H(\mathbb{D})$ stands for the space of complex valued analytic functions on the unit disk and $1 \le p < \infty$ then the Hardy space H_p is the Banach space
$$H_p = \{f \in H(\mathbb{D}) \colon \sup_{0<r<1} M_p(f, r) < \infty\}$$
where
$$M_p(f, r) = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dm(t)\Big)^{1/p}, \quad \text{for each } 0 < r < 1.$$
For $p = \infty$ just consider the usual modification $M_\infty(f, r) = \sup_{0<t<2\pi} |f(re^{it})|$. The space H_p can be identified with the closed subspace of $L_p(\mathbb{T}, m)$ generated by the polynomials, i.e., $H_p = \{f \in L_p(\mathbb{T}, m)\colon \widehat{f}(n) = 0,\ n < 0\}$, where $\widehat{f}(n)$ is the nth Fourier coefficient of f (see [112, Th. 8] or [88]).
Complex interpolation of Hardy spaces highly resembles the case of Lebesgue spaces [67] and one has: If $1 < p < \infty$ and $1/p = 1 - \theta$, then $(H_1, H_\infty)_\theta = H_p$. See [113] for a generalization to rearrangement invariant spaces and [94] for an alternative proof and an extension to the noncommutative realm. What is important for our purposes is that the scale of Hardy spaces is, excluding the endpoints $p = 1$ and $p = \infty$, isomorphic to the scale of Lebesgue spaces $L_p(\mathbb{T}, m)$ (see [88, Theorem 0.3] and [112, Corollary 11]) as it was originally proved by Boas [7]:

Proposition 7.1. *The map $B : L_p(\mathbb{T}, m) \to H_p$ given by*
$$B\Big(\sum_{n\in\mathbb{Z}} a_n e^{int}\Big) = a_0 + \sum_{n\ge1} a_n e^{i(2n)t} + \sum_{n\ge1} a_{-n} e^{i(2n-1)t},$$
is a bounded isomorphism for any $1 < p < \infty$.

The extremal cases are known to be false: H_∞ is not isomorphic to a quotient of a $\mathcal{C}(K)$ space [88, Prop. 4.1] and H_1 is a separable dual space [88, Th. 1.2].

- *Sobolev spaces* For $1 \le p \le \infty$ and $k \in \mathbb{N}$, the Sobolev space $W_p^k(\mathbb{R}^n)$ (for which general references are [1,82], although we follow here the description given in [90]) is the Banach space of all functions $f : \mathbb{R}^n \to \mathbb{R}$ whose distributional derivatives up to order k are in L_p. Precisely, given a finite sequence $\alpha = (\alpha_i)_{i=1}^n \in \mathbb{N}^n$ denote by $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1}\,\partial x_2^{\alpha_2}\ldots\partial {x_n}^{\alpha_n}}$ the usual partial derivative associated to the multi-index α, where $|\alpha| = \sum \alpha_i$ is the order of the derivative. A function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be the α-th distributional partial derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$, denoted by $g = D^\alpha f$, if satisfies the equation

$$\int_{\mathbb{R}^n} g\varphi\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f D^\alpha \varphi\, dx,$$

when tested against all infinitely differentiable functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ with compact support. The Sobolev space $W_p^k(\mathbb{R}^n)$ is, for $k \in \mathbb{N}$ and $1 \le p \le \infty$, the Banach space

$$W_p^k(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} \colon D^\alpha f \text{ exist and } D^\alpha f \in L_p(\mathbb{R}^n) \text{ for all } |\alpha| \le k\}$$

endowed with the norm

$$\|f\|_{k,p} = \begin{cases} \left(\sum_{|\alpha|\le k} \int_{\mathbb{R}^n} |D^\alpha f(x)|^p\, dx\right)^{1/p}, & \text{for } 1 \le p < \infty \\ \max_{|\alpha|\le k} \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |D^\alpha f(x)|, & \text{if } p = \infty \end{cases}$$

One has [83] that $(W_1^k(\mathbb{R}^n), W_p^k(\mathbb{R}^n))_\theta = W_{p_\theta}^k(\mathbb{R}^n)$ for $k \in \mathbb{N}$ and $1 < p < \infty$ and where $\frac{1}{p_\theta} = 1 - \theta + \frac{\theta}{p}$. Moreover, $W_p^k(\mathbb{R}^n)$ is isomorphic to $L_p(\mathbb{R}^n)$ for each $1 < p < \infty$ through an isomorphism on the scale obtained using the theory of Fourier multipliers [90, Th. 6] (see also [89]):

Proposition 7.2. *For each $k \in \mathbb{N}$ and $n \in \mathbb{N}$ there exists an isomorphism $T : L_p(\mathbb{R}^n) \to W_p^k(\mathbb{R}^n)$ for all $1 < p < \infty$.*

The operator T does not depend on p (see [90, pp. 1372], the comments before Proposition 8).

We are thus ready to show that the corresponding Rochberg spaces $\mathfrak{R}_n(L_2)$, $\mathfrak{R}_n(H_2)$ and $\mathfrak{R}_n(W_2^k)$ are all isomorphic for all $k \in \mathbb{N}$. Indeed, by general arguments it suffices to work with the Rochberg spaces associated to the scale (L_p, L_{p^*}). The Commutator theorem and the existence of isomorphisms $B : L_p \to H_p$ and $T : L_p \to W_p$ for all $1 < p < \infty$ yields a “diagonal” isomorphism between the corresponding Rochberg spaces.

8. Rochberg spaces for scales of Orlicz spaces

We follow the same notation as in Section 2.3. In [38] the authors identify the differential maps corresponding to an interpolation scale (L_{ϕ_0}, L_{ϕ_1}) of Orlicz spaces:

Proposition 8.1. *Let ϕ_0 and ϕ_1 be two N-functions that satisfy the Δ_2-property such that $t = \phi_0^{-1}(t) \cdot \phi_1^{-1}(t)$. Then $(L_{\phi_0}(\mu), L_{\phi_1}(\mu))_{1/2} = L_2(\mu)$ with associated derivation*

$$\Omega_{1/2} f = f \log \frac{\phi_1^{-1}(f^2)}{\phi_0^{-1}(f^2)} = 2f \log \frac{\phi_1^{-1}(f^2)}{f},$$

for $f \in L_2(\mu)$ a normalized positive element.

This follows from the observation that, in the hypothesis of Proposition 8.1, a bounded homogeneous selection for the evaluation map $\delta_{1/2}$ is $B(f)(z) = (\phi_0^{-1}(f^2))^{1-z}(\phi_1^{-1}(f^2))^z$, from where it follows

$$B'(f)(z) = B(f)(z)\log\frac{\phi_1^{-1}(f^2)}{\phi_0^{-1}(f^2)},$$

which yields the result. Also, it follows by reiteration that

$$B^{(n)}(f)(z) = B(f)(z)\log^n\frac{\phi_1^{-1}(f^2)}{\phi_0^{-1}(f^2)}.$$

Thus the associated differential map $(\Delta_n, \dots, \Delta_1)B(f)(z)$ is given on normalized positive elements $f \in L_2(\mu)$ by

$$\Omega_{1,n}(f) = \Big(\frac{2^n}{n!} f\log^n\frac{\phi_1^{-1}(f^2)}{f}, \dots, 2f\log\frac{\phi_1^{-1}(f^2)}{f}\Big).$$

Orlicz spaces appear in this theory we are formalizing from different considerations too. Kalton and Peck considered in [75] a rather general variation of the Kalton–Peck map:

$$\mathsf{KP}_\phi(x) = x\phi\Big(-\log\frac{|x|}{\|x\|}\Big)$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an unbounded Lipschitz map such that $\phi(t) = 0$ for $t \le 0$. The maps KP_ϕ are non-trivial centralizers on ℓ_2 [70,75] and therefore define non-trivial twisted Hilbert spaces $\ell_2 \oplus_{\mathsf{KP}_\phi} \ell_2$, usually denoted $\ell_2(\phi)$. Note that the Kalton–Peck space Z_2 corresponds (up to a sign) to the simple choice of $\phi(t) = t$. Moreover, Kalton and Peck show in [75, Corollary 5.5] that, setting for $0 < r < 1$ the Lipschitz map

$$\phi_r = \begin{cases} t, & \text{for } 0 \le t \le 1 \\ t^r, & \text{for } t > 1 \end{cases}$$

then $\ell_2(\phi_r)$ and $\ell_2(\phi_s)$ are not isomorphic for $r \ne s$. Moreover, one has [38, Prop. 6.11]:

Proposition 8.2. *Let $0 < r < 1$ and let ϕ_0 and ϕ_1 be the maps from $[0, \infty)$ to itself defined by*

$$\phi_0(t) = t^{1/2+1/4(-\log t)^{r-1}} \quad \textit{and} \quad \phi_1(t) = t^{1/2-1/4(-\log t)^{r-1}}$$

on a neighbourhood of 0, and extended to N-functions on $[0, \infty)$ satisfying the Δ_2-property. Then $(\ell_{\phi_0}, \ell_{\phi_1})_{1/2} = \ell_2$ and the induced differential is boundedly equivalent to KP_{ϕ_r}. In particular $\ell_2(\phi_r)$ and $\mathfrak{R}_2(\ell_{\phi_0}, \ell_{\phi_1})_{1/2}$ are isomorphic.

Specific properties of Rochberg spaces associated to interpolation scales of Orlicz sequence spaces have been studied by Corrêa in [53]. Among other results, he obtains that such Rochberg spaces are Fenchel–Orlicz spaces, generalizing the results of [3].

9. Rochberg spaces for scales of Tsirelson-like spaces

Let X be a space with a common unconditional basis with X^*. Even so, it is rather difficult, in general, to obtain the differential map $\Omega_X : \ell_2 \curvearrowright \ell_2$ associated to $\ell_2 = (X, X^*)_{1/2}$. The reason lies behind the lack of information that we have about the Lozanovskii factorization $\ell_1 = X \cdot X^*$ when the basis of X is not symmetric [29]. Despite this, Suárez [106] and Morales and Suárez [84] have been able to study the properties of the twisted Hilbert space generated by Ω_X without knowing an explicit formula for Ω_X. The core idea in Suárez's work is to study properties that can be characterized *locally* through the behaviour of the sequences $a_{n,2}(X)$ and $c_{n,2}(X)$ of *local type and cotype* 2 *constants* of X, respectively, already used in Section 6.3 to show that $\mathfrak{R}_n$ and $\mathfrak{R}_m$ are not isomorphic when $n \neq m$. For instance, Kwapień showed [77] (see also [2, Th. 7.4.7]) that $\sup_n a_{n,2}(X)c_{n,2}(X) < \infty$ if and only if X is isomorphic to a Hilbert space, while Pisier proved in [92] that $a_{n,2}(X)c_{n,2}(X)(\log n)^{-1} \longrightarrow 0$ when $n \to \infty$ implies that X is superreflexive. Suárez obtained [106] the following estimate for the local type/cotype constants for order 2 Rochberg spaces $\mathfrak{R}_2(X_\theta)$ associated to the interpolation space $(X_0, X_1)_\theta$ obtained from a pair of Banach spaces (X_0, X_1):

Lemma 9.1. *There exists a constant $C > 0$ such that*

$$a_{n,2}(\mathfrak{R}_2(X_\theta)) \leq C a_{n,2}(X_\theta)\left[a_{n,2}(X_0)^{1-\theta} a_{n,2}(X_1)^{\theta} + \left|\log \frac{a_{n,2}(X_0)}{a_{n,2}(X_1)}\right|\right]$$

This means that when $a_n(X_0)$ and $a_n(X_1)$ grow *slowly*, then $a_n(\mathfrak{R}_2)$ will also grow *slowly*. And this slowly increasing is connected with the weak Hilbert character of the space: a Banach space X is said to be weak Hilbert if there are $0 < \varepsilon < 1$ and $C > 0$ such that every finite-dimensional subspace $E \subset X$ contains a subspace $F \subset E$ with $\dim F \geq \varepsilon \dim E$ such that $\mathrm{dist}(F, \ell_2^{\dim F}) \leq C$ and there is a projection $P : X \longrightarrow F$ with $\|P\| \leq C$. This is not the original definition, but it is equivalent [93, Th. 12.2].

The typical weak Hilbert space is the 2-convexification $\mathcal{T}_2$ of Tsirelson space $\mathcal{T}$, as well as its dual, subspaces and quotients (see [64,93]). Moreover, the Hilbert space ℓ_2 is continuously and densely embedded in $\mathcal{T}_2$ and therefore $(\mathcal{T}_2, \overline{\mathcal{T}_2}^*)_{1/2} = \ell_2$. The associated Rochberg space $\mathfrak{R}_2(\mathcal{T}_2)$ is not a Hilbert space and Suárez [106] proved that it is a weak Hilbert space. His proof can be summarized in the following steps:

(i) Use Lemma 9.1 to obtain

$$a_{n,2}(\mathfrak{R}_2(\mathcal{T}_2)) \leq C \max\{a_{n,2}(\mathcal{T}_2), a_{n,2}(\mathcal{T}_2^*)\}$$

Since $\mathfrak{R}_2(\mathcal{T}_2)$ is isomorphic to its dual, the same asymptotic bound holds for the cotype 2 sequence $c_{n,2}(\mathfrak{R}_2(\mathcal{T}_2))$.

(ii) This implies that the sequence $a_{n,2}(\mathfrak{R}_2(\mathcal{T}_2))c_{n,2}(\mathfrak{R}_2(\mathcal{T}_2))$ grows slowly and, by the aforementioned result of Kwapien, we have that the n-dimensional subspaces $\mathfrak{R}_2(\mathcal{T}_2)$ are $a_{n,2}(\mathfrak{R}_2(\mathcal{T}_2))\, c_{n,2}(\mathfrak{R}_2(\mathcal{T}_2))$-isomorphic to a Hilbert space.

(iii) Replacing $\mathfrak{R}_2(\mathcal{T}_2)$ by some carefully chosen finite codimensional subspaces V_n, it follows from the same approach that the 5^{5^n}-dimensional subspaces of V_n are uniformly isomorphic to Hilbert spaces.

(iv) Use a result of W.B. Johnson [64, Lemma 1.6] to conclude that $\mathfrak{R}_2(\mathcal{T}_2)$ is weak Hilbert.

We do not know however any explicit formulation for the associated differential. It would be extremely interesting to know:

Problem 16. Provide an explicit formulation for $\Omega_{\mathcal{T}}$ or for $\Omega_{\mathcal{T}_2}$.

Problem 17. Is the differential map $\Omega_{\mathcal{T}_2}$ associated to the 2-convexification of Tsirelson space singular?

It is proved in [38] that $\Omega_{\mathcal{T}}$, the associated differential to the scale $(\mathcal{T}, \mathcal{T}^*)$ of Tsirelson space is singular. The butterfly lemma then yields

$$\Omega_{\mathcal{T}_2} = \frac{1}{2}\left(\Omega_{\mathcal{T}} + \mathsf{KP}\right)$$

but this is not enough to deduce that $\Omega_{\mathcal{T}_2}$ is singular since KP is also singular. In a forthcoming paper [46] we will show that all higher Rochberg spaces $\mathfrak{R}_n(\mathcal{T}_2)$ are also weak Hilbert spaces, which provides the existence of new more exotic weak Hilbert spaces. Additional properties of $\mathfrak{R}_2(\mathcal{T}_2)$ were studied by Suárez in [105]. It would be important to advance in the following question:

Problem 18. Identify the natural subspaces of $\mathfrak{R}_n(\mathcal{T})$ and $\mathfrak{R}_n(\mathcal{T}_2)$

Properties of Rochberg spaces associated to other exotic Banach spaces of the Tsirelson family were studied by Morales and Suárez [84]. In particular, those associated to the symmetrization of Tsirelson space. The key difference between $\mathcal{T}_2$ and $\mathcal{T}_2^s$ is that the symmetrized space is not weak Hilbert [27, Prop. 3.7]. In fact, it is not even asymptotically hilbertian (a Banach space X is asymptotically hilbertian if there is a constant C such that for every n, there exists a finite codimensional subspace of X all whose n-dimensional subspaces are C-isomorphic to some ℓ_2^n). All weak Hilbert spaces are asymptotically hilbertian [93, Prop. 14.2 and Th. 14.4]. However, $\mathcal{T}_2$ and $\mathcal{T}_2^s$ have the common property of being HAPpy spaces [66], i.e., spaces such that all of their subspaces have the Approximation Property. Weak Hilbert spaces are HAPpy [93, Th. 15.1]. The space $\mathfrak{R}_n(\mathcal{T}_2^s)$ was studied by Morales and Suárez in [84, Section 5] showing that it is an example of HAPpy space that is not asymptotically hilbertian. Moreover, using the local approach described above they prove [84, Prop. 8] that $\mathfrak{R}_n(\mathcal{T}_2^s)$ is HAPpy. The proof is a clever combination of:

(i) The using of Kwapien's bound [77], mentioned earlier, and the estimates above to get that, for any n-dimensional subspace E of $\mathfrak{R}_2(\mathcal{T}_2^s)$, the function $d(E, \ell_2^n)$ grows as the inverse of Ackerman type (see the initial discussion of [84, Section 5]).
(ii) A result of Johnson and Szankowski [66, Th. 2.1] asserting that if the function $d_n(X) = \sup\{d(E, \ell_2^n) \colon E \subset X, \dim E = n\}$ grows sufficiently slow, as the inverse of Ackerman type, then X is HAPpy.

Again, we do not have any explicit formulation for the associated differential $\Omega_{\mathcal{T}_2^s}$.

Problem 19. Is $\Omega_{\mathcal{T}_2^s}$ singular?

Morales and Suárez prove [84, Prop. 9] that $\mathfrak{R}_2(\mathcal{T}_2^s)$ is not asymptotically hilbertian because, if it were, it would contain a non-Hilbert, asymptotically hilbertian Banach space with symmetric basis, in contradiction with a result of Johnson [64, Remark 2] asserting that an asymptotically hilbertian space with a symmetric basis is ℓ_2. A stronger result is true:

Proposition 9.2. *A non-trivial twisted Hilbert obtained from the interpolation scale (X, X^*) in which X and X^* have a common symmetric basis is not asymptotically hilbertian.*

Proof. Assume that $(X, X^*)_{1/2} = \ell_2$ and denote by Ω the corresponding differential. Let $(e_n)_n$ be the common symmetric basis. If $\mathfrak{R}_2$ were asymptotically hilbertian then the domain space $\mathrm{Dom}\Omega$ would also be asymptotically hilbertian. But observe that $\mathrm{Dom}\Omega$ has a symmetric basis: indeed, if σ is a permutation of the integers, the canonical projections P_σ is an operator on the scale (X, X^*) and therefore, by Proposition 4.8, it acts on $\mathrm{Dom}\Omega$, from where we conclude that $(e_n)_n$ is a symmetric basis for $\mathrm{Dom}(\Omega)$. The aforementioned result of Johnson implies that $\mathrm{Dom}\Omega$ is isomorphic to a Hilbert space. Therefore

$$\Big\|\Big(0, \sum x_j e_j\Big)\Big\|_{\mathfrak{R}_2} = \Big\|\sum x_j e_j\Big\|_{\mathrm{Dom}\Omega} \sim \Big\|\sum x_j e_j\Big\|_{\ell_2},$$

and thus

$$\Big\|\Omega\Big(\sum x_j e_j\Big)\Big\|_{\ell_2} \le C\Big\|\sum x_j e_j\Big\|_{\ell_2},$$

implying that $\Omega : \ell_2 \longrightarrow \ell_2$ is bounded. □

Suárez and Morales also study the Rochberg spaces associated to the scale of Schreier and 2-convexified Schreier spaces. However, we do not know:

Problem 20. Provide an explicit formulation for $\Omega_{\mathcal{S}}$, the differential associated the Schreier space. Is it singular? Similar questions for the 2-convexification $\mathcal{S}_2$ of the Schreier space.

10. Advanced topics

10.1. Stability

Stability issues are important in every mathematical theory. The meaning of "stability" in the context of the study of differentials of interpolation processes and their associated Rochberg spaces is to determine to what extent small modifications of the data (the interpolation pair, the parameter at which one interpolates) produce significant variations of the properties of either the Rochberg spaces, the associated exact sequences or the differentials that generate them. Let us begin with the result that started it all in this context: Kalton's Stability Theorem.

Complex interpolation between Köthe function spaces over the same measure space yield centralizers, something that can be seen as a consequence of the Commutator Theorems considered in Section 4.5. Indeed, let (X_0, X_1) be a pair of Köthe spaces and note that, for any $a \in L_\infty(\mu)$ and $j = 0, 1$, the multiplication operator $\tau_a : X_j \to X_j$

given by $\tau_a(f) = af$ is bounded by $\|a\|_\infty$. Thus, by the Commutator Theorem for the case $n = 2$ (see Theorem 4.7) it follows that for any $0 < \theta < 1$,

$$\|a\Omega_\theta(f) - \Omega_\theta(af)\|_X = \|\tau_a\Omega_\theta(f) - \Omega_\theta(\tau_a(f))\|_X \leq C\|f\|_X.$$

Moreover, the centralizer Ω_θ is *real*, meaning that it sends real functions to real functions. Summing it up, complex interpolation of Köthe spaces always define real centralizers on Köthe spaces. Kalton's Stability Theorem provides the converse under some additional hypothesis: any real centralizer on a *superreflexive* Köthe function space X comes as the derivation of an interpolation scale (X_0, X_1) of Köthe spaces such that $(X_0, X_1)_{1/2} = X$. Precisely (see [29,38,71]):

Theorem 10.1. *Let X be a superreflexive Köthe space and suppose that $\Omega : X \to L_0(\mu)$ is a real centralizer. Then:*

- *There exist $\varepsilon > 0$ and Köthe spaces X_0, X_1 such that $X = (X_0, X_1)_{1/2}$ and $\varepsilon\Omega$ is boundedly equivalent to $\Omega_{1/2}$.*
- *The spaces X_0 and X_1 are determined up to equivalent renorming, i.e, if Y_0 and Y_1 are Köthe spaces such that $(Y_0, Y_1)_{1/2} = X$ and the induced centralizer $\Omega^Y_{1/2}$ is boundedly equivalent to $\Omega_{1/2}$, then $X_0 \simeq Y_0$ and $X_1 \simeq Y_1$.*

The uniqueness condition implies, very remarkably, that if (X_0, X_1) is any compatible couple of superreflexive Köthe spaces such that $\Omega_\theta : X_\theta \to L_0(\mu)$ is bounded, then $X_0 \simeq X_1$ and $\Omega_\rho \equiv 0$ for every $0 < \rho < 1$. We say in this case that complex interpolation of superreflexive Köthe spaces enjoys *bounded global stability* since boundedness of a differential at a point implies that all the remaining differentials at other points must also be bounded. Kalton's Stability Theorem opens the door to ask if a kind of bounded *local* stability, out of the Köthe space case:

Problem 21. Assume that for $\theta \in \mathbb{S}$ one has $\Omega_\theta \equiv 0$. Does there exist $\varepsilon > 0$ such that $\Omega_z \equiv 0$ for $z \in (\theta - \varepsilon, \theta + \varepsilon)$?

Another stability notion is worth consideration. We say that there is *global stability* (resp. *local stability*) for (X_0, X_1) if the triviality of Ω_θ implies that Ω_ρ is trivial for all $0 < \rho < 1$ (resp. for all $\rho \in (\theta - \varepsilon, \theta + \varepsilon)$ for some $\varepsilon > 0$). The same ideas can be transplanted for families. The paper [33] was devoted to studying the (bounded) stability of the complex method, and it provides unexpected results:

- There is bounded and global stability for pairs of Köthe space.
- There is bounded and global stability for families of up to three Köthe spaces.
- There is no bounded or local stability for families of four Köthe space spaces.

Moreover

Problem 22. Is there local stability for pairs of arbitrary Banach spaces?

In the abstract setting, one could consider an interpolation method with parameters living in a metric space P in such a way that for each $\theta \in P$ there are two interpolators $\Psi(\theta)$, $\Phi(\theta)$ obtained from the method. These interpolators generate a differential Ω_z with its associated exact sequence. The previous stability results translate in this general case to the following problem:

Problem 23. Study the stability of the map $z \to \Omega_{\Psi(z),\Phi(z)}$.

with the meaning of whether relevant properties of $\Omega_{\Psi(z),\Phi(z)}$ are maintained by small perturbations of z.

10.2. Homology

The reader is addressed to [8,28,30] for a sounder development of categorical and homological elements in Banach space theory. Let us denote, as it is usual in homology, the vector space of exact sequences $0 \longrightarrow \ell_2 \longrightarrow Z \longrightarrow \ell_2 \longrightarrow 0$ modulo equivalence by $\mathrm{Ext}(\ell_2, \ell_2)$. The 0 element is the class of trivial sequences. What the Enflo–Lindenstrauss–Pisiser example ELP shows is that $\mathrm{Ext}(\ell_2, \ell_2) \neq 0$. The space Z_2 and, in general, any non-Hilbert twisted Hilbert space, provides a new nonzero element of $\mathrm{Ext}(\ell_2, \ell_2)$. The space $\mathrm{Ext}(\ell_2, \ell_2)$, more precisely the functor Ext, is part of a hierarchy of functors Ext^n studied in homology. The reader is invited to peruse [8,16] for a comprehensive overview of the theory. These functors provide vector spaces $\mathrm{Ext}^n(\ell_2, \ell_2)$, whose study is by no means easy. For instance, the second space in the hierarchy is $\mathrm{Ext}^2(\ell_2, \ell_2)$, whose elements are equivalence classes of four-term exact sequences

$$0 \longrightarrow \ell_2 \longrightarrow Z_1 \longrightarrow Z_2 \longrightarrow \ell_2 \longrightarrow 0$$

modulo a suitable equivalence relation. Since a four-term exact sequence can be regarded as the concatenation of two short exact sequences

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \ell_2 & \longrightarrow & Z_1 & & \longrightarrow & & Z_2 & \longrightarrow & \ell_2 \longrightarrow 0 \\ & & & & & \searrow & & \nearrow & & & \\ & & & & & & V & & & & \end{array}$$

then the four-term sequence can be labelled as FG, where F represents the exact sequence

$$0 \longrightarrow \ell_2 \longrightarrow Z_1 \longrightarrow V \longrightarrow 0$$

and G the exact sequence

$$0 \longrightarrow V \longrightarrow Z_2 \longrightarrow \ell_2 \longrightarrow 0\ .$$

The final section of [17] is devoted to showing that the existence of the nets (15) of Rochberg spaces shows that given any differential Ω associated at $1/2$ to an interpolation pair (X, X^*) with $(X, X^*)_{1/2} = \ell_2$ satisfies $\Omega\Omega = 0$ in the space $\mathrm{Ext}^2(\ell_2, \ell_2)$. This result seems to come out of the blue, and turned out to be more unexpected yet when the authors proved in [15] that $\mathrm{Ext}^2(\ell_2, \ell_2) \neq 0$ by showing the existence of a differential Ω such that $\Omega\mathsf{KP} \neq 0$. These results together seem to point out some strong, still concealed, connections between nets of order Rochberg spaces and higher order spaces in the hierarchy of Ext. However, nobody has been able to prove so far that $\mathrm{Ext}^n(\ell_2, \ell_2) \neq 0$.

10.3. Nonlinear classification

A must reading for this section is [74], even if it can be followed independently. The *gap* $g(M, N)$ between two closed subspaces M, N of a Banach space Z is defined as

$$g(M, N) = \max\Big\{ \sup_{x\in B_M} \operatorname{dist}(x, B_N),\ \sup_{y\in B_N} \operatorname{dist}(y, B_M)\Big\},$$

where $\operatorname{dist}(x, B_N) = \inf\{\|x - n\| : n \in B_N\}$. The *Kadets distance* d_{K} between two Banach spaces X, Y is defined as the infimum of $g(iX, jY)$, where $i : X \to W$, $j : Y \to W$ range through linear isometric embeddings into the same Banach space W. The following result essentially is in [41], although not formulated in this way:

Lemma 10.2. *Given an exact sequence* $0 \to Y \to X \xrightarrow{\pi} Z \to 0$ *of Banach spaces,* $d_K(X, Y \oplus Z) = 0$.

Proof. $Y \oplus_1 Z$ is a subspace of $X \oplus_1 Z$ and for each positive ε the subspace $X_\varepsilon = \{(\varepsilon x, \pi x) : x \in X\}$ of $X \oplus_1 Z$ is isomorphic to X. Then, $g(X_\varepsilon, Y \oplus_1 Z) \le \varepsilon$ [87, Lemma 5.9] yields $d_K(X, Y \oplus_1 Z) = 0$. □

The combination of this result plus the existence of the exact sequences (13) imply, by an straightforward induction argument:

Proposition 10.3.

- *If* $\mathfrak{R}_n = \mathfrak{R}_n(X_0, X_1)_\theta$ *denote the Rochberg spaces obtained from the complex interpolation pair* (X_0, X_1) *at* θ *then* $d_K(\mathfrak{R}_n, (X_0, X_1)^n_\theta) = 0$.
- *In particular, if* $\mathfrak{R}_n = \mathfrak{R}_n(X, X^*)_{1/2}$ *denote the Rochberg spaces obtained from a suitable complex interpolation pair* (X, X^*) *at* $1/2$ *then* $d_K(\mathfrak{R}_n, \ell_2) = 0$.

11. List of open problems

We conclude the paper by listing most of the problems that have been mentioned in the text. The number corresponds to the numbering already used in the text, although problems have been gathered by topics.

Basic unsolved questions about twisted Hilbert spaces

Problem 2 Is every twisted Hilbert space isomorphic to its dual?
Problem 4 Does every twisted Hilbert space have the BAP?
Problem 5 Let X be a Banach space such that $(X, X^*)_{1/2} = H$ is a Hilbert space with associated differential Ω. Is $H \oplus_\Omega H$ isomorphic to its dual?

Advanced questions about interpolation methods

Problem 1 Do interpolation methods generate Rochberg spaces?
Problem 6 Obtain a generalized Butterfly Lemma for higher order differential maps.
Problem 7 Is every compatible family of interpolators strongly compatible?
Problem 8 Do non-compatible families of interpolators appear in nature?

Problem 9 Do Domains and Ranges form interpolation scales on their own? More precisely, is $(\mathrm{Dom}\mathsf{KP}_p, \mathrm{Dom}\mathsf{KP}_{p^*})_{1/2} = \mathrm{Dom}\mathsf{KP}_2$ or $(\mathrm{Dom}\mathsf{KP}_p, \mathrm{Dom}\mathsf{KP}_{p^*})_{1/2} = \ell_2$?

Problem 24 Is $\mathrm{Ext}^n(\ell_2, \ell_2) = 0$?

Questions about specific spaces

Problem 3 Is the space ELP constructed by Enflo, Lindenstrauss and Pisier [58] isomorphic to its dual?

Problem 10 Provide an explicit description for KP^{-1}.

Problem 11 Identify the natural subspaces of Rochberg spaces for the scale of ℓ_p-spaces as Domain or Range spaces.

Problem 12 Obtain manageable characterizations of the natural subspaces $X_A \subset \mathfrak{R}_n$ for $|A| \geq 2$.

Problem 13 Does $\mathfrak{R}_n$ admit a unique, up to equivalence, non-trivial symplectic structure?

Problem 14 Is the symplectic group of $\mathfrak{R}_n$ contractible. Is it path connected? It is not even known if $GL(Z_2)$ is path connected.

Problem 15 Is Z_2 isomorphic to its hyperplanes?

Problem 16 Provide an explicit formulation for $\Omega_{\mathcal{T}}$ or $\Omega_{\mathcal{T}_2}$

Problem 17 Is the differential $\Omega_{\mathcal{T}_2}$ associated to the 2-convexification of Tsirelson space singular?

Problem 18 Identify the natural subspaces of $\mathfrak{R}_n(\mathcal{T})$ and $\mathfrak{R}_n(\mathcal{T}_2)$.

Problem 19 Is the differential $\Omega_{\mathcal{T}_2^s}$ singular?

Problem 20 Provide an explicit formulation for the differential map $\Omega_{\mathcal{S}}$ associated the Schreier space. Is it singular? Similar questions for the 2-convexification $\mathcal{S}_2$ of the Schreier space.

Stability problems

Problem 21 Let (X_0, X_1) be an interpolation pair of Banach spaces. Assume that $\Omega_\theta \equiv 0$. Does there exist $\varepsilon > 0$ such that $\Omega_z \equiv 0$ for $z \in (\theta - \varepsilon, \theta + \varepsilon)$.

Problem 22 Is there local stability for pairs of arbitrary Banach spaces?

Problem 23 Study the stability of the map $z \to \Omega_{\Psi,\Phi}(z)$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgement

The authors want to thank the referee for many thoughtful comments that helped us to improve the organization of the paper.

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