

Energy-based theory of autoresonance phenomena: Application to Duffing-like systems

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Abstract

A general energy-based theory of autoresonance (self-sustained resonance) in low-dimensional nonautonomous systems is presented. The equations that together govern the autoresonance solutions and excitations are derived with the aid of a variational principle concerning the power functional. These equations provide a feedback autoresonance-controlling mechanism. The theory is applied to Duffing-like systems to obtain exact analytical expressions for autoresonance excitations and solutions which explain *all* the phenomenological and approximate results arising from a previous (adiabatic) approach to autoresonance phenomena in such systems. The theory is also applied to obtain new, general, and exact properties concerning autoresonance phenomena in a broad class of dissipative and Hamiltonian systems, including (as a *particular* case) Duffing-like systems.

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It has been well known for about half a century that autoresonance (AR) phenomena occur when a system continuously adjusts its amplitude so that its instantaneous nonlinear period matches the driving period, the effect being a growth of the system's energy. Autoresonant effects were first observed in particle accelerators [1,2], and have since been noted in nonlinear waves [3,4], fluid dynamics [5,6], atomic and molecular physics [7,8], plasmas [9-11], nonlinear oscillators [12,13], and planetary dynamics [14-17]. Apparently, the first mention of the notion of resonance (“risonanza”) was by Galileo [18]. Remarkably, this linear-system-based concept has survived up to now: resonance (nonlinear resonance) is identified with how well the driving period fits (a rational fraction of) a natural period of the underlying conservative system [19]. However, the genuine effect of the frequency (Galilean) resonance (FR) (i.e., the secular growth of the oscillation amplitude) can no longer be observed in a periodically driven nonlinear system. As is well known, the reason is simple: a

linear oscillator has a single period which is energy-independent, while nonlinear oscillators generally present an infinity of energy-dependent periods. This means that, although an FR can still be momentarily induced in a nonlinear system by exciting it with a driving period that exactly matches the intrinsic period of the current motion, the subsequent growth of the nonlinear oscillations changes the intrinsic period of the motion, which no longer matches the excitation period and thus takes the system out of FR. Since linear oscillations represent a limiting *degenerate* (energy) case of the more general nonlinear oscillations, it seems that any truly nonlinear generalization of the notion of resonance, in its early (etymological) sense of *resonare* (i.e., awaken an echo of some underlying nonlinear oscillation), should be based on energy (or action) considerations. A case has been provided by the notion of geometrical resonance [20]. Thus, if one is interested in obtaining a nonlinear equivalent of the secular maintained growth intrinsic to the FR, it is clear that the system must not be driven by a strictly periodic excitation. In this regard, a previous theoretical approach to autoresonance phenomena [3,7-11] provided an early explanation of the mechanism inducing the growth of the oscillation (without the use of feedback) for particular classes of resonantly driven nonlinear systems which stay locked with an *adiabatically* varying perturbing oscillation (the drive). The adiabatic excitation yields the autoresonant effect by automatically adjusting the system's amplitude so that the instantaneous nonlinear period matches the driving period. It should be stressed that a fundamental part (hereafter referred to as adiabatic autoresonance (AAR) theory, cf. refs. [9-11,21,22]) of the aforementioned previous theoretical approach to AR phenomena presents severe limitations of applicability and insight: essentially (see ref. [22] for a review), it was developed for nonlinear oscillators that reduce to a Duffing oscillator

$$\ddot{x} + \omega_0^2(x + bx^3) = -\delta \dot{x} + \varepsilon \cos(\omega_0 t + \alpha t^2/2) \quad (1)$$

for small amplitudes, where α is the *linear* sweep rate and $\delta > 0$. In the context of AAR theory, it has been found *numerically* that AR solutions only occur if (i) the damping coefficient δ is not too large, and (ii) the amplitude of the AR oscillations grows on the

average, but also oscillates around the average growth. Also, AAR theory predicts that (iii) there exists a threshold for AR, in particular, if the normalized excitation amplitude $\varepsilon/\omega_0^{1/2}$ exceeds a threshold proportional to $\alpha^{3/4}$, the system will follow the excitation to high amplitude, while the amplitude will stay very low otherwise, (iv) that the threshold sweep rate α_{th} scales as δ^2 , (v) that the AR effect is *solely* expected for the case with initial conditions near some equilibrium of the (unperturbed) nonlinear system, and (vi) that there exists a breaking time for AR, t_b . Properties (ii), (iii), (v), (vi) also hold in (vii) the case with no dissipation (cf. refs. [9, 10, 21]), but there has as yet been no theoretical explanation of that fact. It is worth mentioning that, to the best of the author's knowledge, the case of weak dissipation has only been considered in a single previous work (cf. ref. [11]).

In this Letter a new, general, and energy-based theory for AR phenomena in nonautonomous systems is presented and applied to the above Duffing oscillators to explain conjointly points (i)-(vii) as well as to deduce new properties concerning AR phenomena in generic systems (including Duffing-like systems). The theory arises from the question as to whether there exists an *upper* limit for the growth rate of the system's amplitude when a *small*-amplitude force acts on the system. Consider the general family of systems

$$\ddot{x} = g(x) - d(x, \dot{x}) + p(x, \dot{x})F(t), \quad (2)$$

where $g(x) \equiv -dV(x)/dx$ [$V(x)$ being an arbitrary time-independent potential], $-d(x, \dot{x})$ is a general damping force, and $p(x, \dot{x})F(t)$ is an as yet undetermined suitable AR-inducing force. Clearly, the corresponding equation for the energy is $\dot{E} = \dot{x} [-d(x, \dot{x}) + p(x, \dot{x})F(t)] \equiv P(x, \dot{x}, t)$, where $E(t) \equiv (1/2) \dot{x}^2(t) + V[x(t)]$ and $P(x, \dot{x}, t)$ are the energy and power, respectively. In the spirit of the aforementioned energy-based approach to resonance phenomena, the AR solutions are defined by imposing that the energy variation $\Delta E = \int_{t_1}^{t_2} P(x, \dot{x}, t) dt$ is a maximum (with t_1, t_2 arbitrary but fixed instants), where the power is considered as a functional. This implies a necessary condition (hereafter referred to as the *AR condition*) to be fulfilled by AR solutions and excitations, which is the Euler equation [23]

$$\frac{\partial P}{\partial x} - \frac{d}{dt} \left(\frac{\partial P}{\partial \dot{x}} \right) = 0. \quad (3)$$

From eq. (3), a relationship between x , \dot{x} , and F can be deduced such that the solutions of the system given by eqs. (2) and (3) together provide the AR excitations, $F_{AR}(t)$, and the AR solutions, $x_{AR}(t)$. It is worth noting that the AR condition (3) represents a *feedback* AR-controlling mechanism, which is absent in the aforementioned previous approach to AR phenomena [3,7-11] where an explicit, coordinate-independent, and adiabatic force is used from the beginning. In this regard, autoresonant control has been previously discussed in the context of vibro-impact systems [24] on the basis of the analysis of nearly sinusoidal self-oscillations [25] where the term self-resonance was introduced to indicate “resonance under the action of a force generated by the motion of the system itself” (cf. ref. [25], p.166). The corresponding AR equations for the multidimensional case can be straightforwardly obtained from the same principle and they will be discussed elsewhere [26]. To compare the present approach with the previous one [9,11,21,22] (cf. eq. (1)), consider the power functional $P(x, \dot{x}, t) = \dot{x} [-\delta \dot{x} + F(t)]$. For the *particular* case of Duffing oscillators, the system (2), (3) reduces to

$$\ddot{x}_{AR} + \omega_0^2 (x_{AR} + bx_{AR}^3) = \delta \dot{x}_{AR}, \quad (4a)$$

$$F_{AR} = 2\delta \dot{x}_{AR}. \quad (4b)$$

Note that eq. (4b) gives the AR condition, i.e., the AR excitations and the (corresponding) AR solutions have the same instantaneous nonlinear period, at all instants, but without the adiabaticity requirement of the AAR theory. Generally, the AR condition (3) means that the instantaneous period of the AR solution fits a rational fraction of the instantaneous period of the AR excitation. This property generalizes (and contains as a particular case) the persistent phase-locking condition of the AAR theory. To obtain AR solutions (and hence AR excitations, cf. eq. (4b)) consider the ansatz $x_{AR}(t) = \gamma f(t) \text{cn}[\beta g(t) + \phi; m]$, where cn is the Jacobian elliptic function of parameter m , and where the constants β, m , and the functions $f(t), g(t)$ have to be determined for the ansatz to satisfy eq. (4a), while γ, ϕ are arbitrary constants. After substituting this ansatz into eq. (4a), one finds the exact general AR solution

$$\begin{aligned}
x_{AR}(t) &= \gamma_0 e^{\delta t/3} \operatorname{cn}[\varphi(t); 1/2], \\
\varphi(t) &\equiv 3\gamma_0 \omega_0 \sqrt{b} (e^{\delta t/3} - 1) / \delta + \varphi_0,
\end{aligned} \tag{5}$$

with the constraint $\omega_0^2 = 2\delta^2/9$ and where $\varphi_0 \equiv \phi + 3\gamma_0 \omega_0 \sqrt{b}/\delta$, $\gamma_0 \equiv \gamma$. Clearly, the exact AR excitation corresponding to solution (5) is

$$F_{AR}(t) = \frac{2}{3} \gamma_0 \delta^2 e^{\delta t/3} \operatorname{cn}[\varphi(t); 1/2] - 2\gamma_0^2 \delta \omega_0 \sqrt{b} e^{2\delta t/3} \operatorname{sn}[\varphi(t); 1/2] \operatorname{dn}[\varphi(t); 1/2], \tag{6}$$

where sn and dn are the Jacobian elliptic functions. Observe that the particular time-dependence of the AR solution (5) directly explains the above point (ii) (see fig. 1). In comparing the present predictions with those from AAR theory, recall that the latter *solely* exist for the case with $x(0) \simeq 0$, $\dot{x}(0) \simeq 0$, for $b > 0$ (point (v)). Thus, for this case $\gamma_0 \simeq 0$ and hence eq. (6) can be approximated by $F_{AR}(t) \simeq \frac{2}{3} \gamma_0 \delta^2 (1 + \frac{1}{3} \delta t + \dots) \operatorname{cn}[\gamma_0 \sqrt{b} (\omega_0 t + \frac{1}{6} \omega_0 \delta t^2 + \dots); 1/2]$, and, using the Fourier expansion of cn [27], one finally obtains

$$F_{AR}(t) \simeq \frac{2}{3} \kappa \gamma_0 \delta^2 \left(1 + \frac{1}{3} \delta t + \dots\right) \cos \left[\kappa' \gamma_0 \sqrt{b} \left(\omega_0 t + \frac{1}{6} \omega_0 \delta t^2 + \dots \right) \right], \tag{7}$$

where $\kappa \equiv \pi \sqrt{2} \operatorname{csch}(\pi/2) / K(1/2) \simeq 1$, $\kappa' \equiv \pi / (2K(1/2)) \simeq 1$. Now, one sees that to consider the excitation $\varepsilon \cos(\omega_0 t + \alpha t^2/2)$ (cf. eq. (1)) as a reliable approximation to $F_{AR}(t)$ (cf. eq. (7)) implies that the damping coefficient has to be sufficiently small (point (i)) so as to have a sufficiently large breaking time, $t_b \sim \delta^{-1}$ (point (vi)). Thus, for $t \lesssim t_b$, one obtains $\varepsilon_{th} \sim \delta^2$, $\alpha_{th} \sim \omega_0 \delta$ (cf. eqs. (1), (7)). When $\omega_0 \sim \delta$ (recall that $\omega_0^2 = 2\delta^2/9$ for the exact AR solution (5)), one finds $\alpha_{th} \sim \delta^2$ (point (iv)), *which explains the adiabaticity requirement of AAR theory for dissipative systems*, $\varepsilon_{th}/\omega_0^{1/2} \sim \alpha_{th}/\alpha_{th}^{1/4} \equiv \alpha_{th}^{3/4}$ (point (iii)), and the cosine's argument in eq. (7) can be reliably approximated by the first two terms, as in AAR theory (cf. eq. (1)). Figure 1 shows an illustrative comparison between the AR responses yielded by AR excitations given by $\varepsilon \cos(\omega_0 t + \alpha t^2/2)$, where in all cases $\varepsilon > \varepsilon_{th}$, and $F_{AR}(t)$ (cf. eq. (6)), respectively, for the cases $\omega_0 \sim \delta$ (fig. 1a) and $\omega_0 \gg \delta$ (fig. 1b). Point (vii) is rather striking in view of the very different properties of Hamiltonian and dissipative systems, and its explanation is a little more subtle. Firstly, note that current AR

theory provides an unsatisfactory result for the limiting Hamiltonian case. For example, eq. (3) yields $r(x) \dot{F}(t) = 0$ for the family given by eq. (2) with $d(x, \dot{x}) \equiv 0, p(x, \dot{x}) \equiv r(x)$, i.e., including the cases of external and parametric (of a potential term) excitations. Clearly, the two possible types of corresponding particular solutions, equilibria and those yielded by a constant excitation (cf. eqs. (2), (3)), can no longer be AR solutions. Secondly, for the above Duffing oscillators one has $\ddot{x}_{AR} + \omega_0^2(x_{AR} + bx_{AR}^3) = F_{AR}/2$ (cf. eq. (4)). Therefore, it is natural to assume the ansatz $F(t) \equiv \lambda \dot{x}(t), \lambda > 0$, for the case with no dissipation, where now the AR rate, λ , is a *free* parameter which controls the initial excitation strength. Thus, the corresponding AR solutions are given by eq. (5) while AR excitations are given by the expression in eq. (6) multiplied by 1/2, both with λ instead of δ , which explains point (vii) and hence *the adiabaticity requirement of AAR theory for Hamiltonian systems* (recall point (iv)). It is worth mentioning that this valuable result holds for the broad family of dissipative systems $\ddot{x} + dV(x)/dx = -\delta \dot{x} |\dot{x}|^{n-1} + F(t)$, where $V(x)$ is a generic time-independent potential and $-\delta \dot{x} |\dot{x}|^{n-1}$ is a general dissipative force ($\delta > 0, n = 1, 2, 3, \dots$). The corresponding AR equations (cf. eqs. (2), (3)) are $\ddot{x}_{AR} + dV(x_{AR})/dx_{AR} = n\delta \dot{x}_{AR} |\dot{x}_{AR}|^{n-1}, F_{AR} = (n+1)\delta \dot{x}_{AR} |\dot{x}_{AR}|^{n-1}$, and hence one obtains $\ddot{x}_{AR} + dV(x_{AR})/dx_{AR} = nF_{AR}/(n+1)$. For the limiting Hamiltonian case ($\delta = 0$), it is therefore natural to assume the ansatz $F(t) \equiv n\lambda \dot{x} |\dot{x}|^{n-1}, \lambda > 0$. Thus, AR solutions are the same for the dissipative and Hamiltonian cases, while the AR excitations associated with the Hamiltonian case are the (corresponding) AR excitations associated with the dissipative case multiplied by $\frac{n}{n+1}$, with λ instead of δ for the Hamiltonian case [28]. In the light of the exact AR excitation (cf. eq. (6)), one can readily obtain a reliable approximation for *arbitrary* initial conditions, i.e., not just those near the equilibrium of the unperturbed Duffing oscillator:

$$F_{AR}(t) \simeq \frac{2}{3}\kappa\gamma_0\delta^2 \left(1 + \frac{1}{3}\delta t + \dots\right) \cos \left[\kappa'\gamma_0\sqrt{b} \left(\omega_0 t + \frac{1}{6}\omega_0\delta t^2 + \dots \right) \right] \\ - \kappa''\gamma_0^2\delta\omega_0\sqrt{b} \left(1 + \frac{2}{3}\delta t + \dots\right) \sin \left[\kappa'\gamma_0\sqrt{b} \left(\omega_0 t + \frac{1}{6}\omega_0\delta t^2 + \dots \right) \right], \quad (8)$$

where $\kappa'' \equiv \pi^2\sqrt{2}\operatorname{sech}(\pi/2)/K^2(1/2) \simeq 1.61819 \simeq (1 + \sqrt{5})/2$ (i.e., the golden ratio).

Thus, for $t \lesssim t_b \sim \delta^{-1}(\lambda^{-1})$ one obtains the general (i.e., valid for any initial condition) *1st-order adiabatic excitation*

$$F_{A,1}(t) = \varepsilon \cos(\omega_0 t + \alpha t^2/2) - \varepsilon' \sin(\omega_0 t + \alpha t^2/2), \quad (9)$$

with the above scalings for ε_{th} , α_{th} , and $\varepsilon'_{th} \sim \varepsilon_{th} \gamma_0 b^{1/2}$. Figure 2 shows illustrative examples for several initial conditions far from $x(0) = \dot{x}(0) = 0$. Another fundamental consequence of the present approach is the derivation of the scaling laws for the thresholds corresponding to higher-order chirps [29]. Indeed, consider the general *n*th-order adiabatic excitation $F_{A,n}(t) \equiv \varepsilon \cos[\omega(t)t] - \varepsilon' \sin[\omega(t)t]$, $\omega(t) \equiv \omega_0 + \sum_{n=1}^{\infty} \alpha_n t^n$, instead of $\varepsilon \cos(\omega_0 t + \alpha t^2/2)$ in eq. (1), where α_n is the *n*th-order sweep rate ($\alpha_1 \equiv \alpha/2$). For this general case, the above analysis straightforwardly yields the scaling law $\varepsilon_{th}/\omega_0^{1/2} \sim N(n) \alpha_{n,th}^{3/(2n+2)}$ for $t \lesssim t_b \sim \delta^{-1}(\lambda^{-1})$, where $\alpha_{n,th}$ is the threshold *n*th-order sweep rate and $N(n) \equiv [3^n (n+1)!]^{3/(2n+2)}$ is a monotonous increasing function. Thus, the 3/4 scaling law is a particular law which solely applies to a linear chirp. For the case of a single chirp term ($\omega(t) \equiv \omega_0 + \alpha_n t^n$), the dependence of the above general scaling law on *n* indicates that one can expect a similar AR effect for *ever smaller* values of α_n as *n* increases. Computer simulations confirm this point: an illustrative example is shown in fig. 3.

A further question remains to be discussed: We have seen why AAR theory requires AR excitations to be adiabatically varying perturbing oscillations, but which are the underlying adiabatic invariants? To answer this question, note that eq. (4a) (with λ instead of δ for the case with no dissipation) can be derived from a Lagrangian, which one defines as $L = e^{-\delta t} (p^2/2 - \omega_0^2 x^2/2 - \omega_0^2 b x^4/4)$, $p \equiv \dot{x}$, and whose associated Hamiltonian is $H = p^2 e^{\delta t}/2 + \omega_0^2 (x^2/2 + b x^4/4) e^{-\delta t}$. The form of this Hamiltonian suggests the following simplifying canonical transformation: $X = x e^{-\delta t/2}$, $P = p e^{\delta t/2}$. It is straightforward to see that the generating function of the canonical transformation [30] is $F_2(x, P, t) = x P e^{-\delta t/2}$. The new Hamiltonian therefore reads: $K(X, P, t) = H(x, p, t) - \partial F_2 / \partial t = P^2/2 + \omega_0^2 (X^2/2 + b e^{\delta t} X^4/4) + \delta P X/2$. In the limiting linear case ($b = 0$), one sees that K is conserved, i.e., the AR solutions corresponding to the linear system are associated (in terms of the old canoni-

cal variables) with the invariant $e^{\delta t} p^2/2 + \omega_0^2 e^{-\delta t} x^2/2 + \delta x p/2$, while for the nonlinear case ($b \neq 0$) one obtains (after expanding $e^{\delta t}$) that the respective AR solutions are associated with the adiabatic invariant $p^2/2 + \omega_0^2 (x^2/2 + b x^4/4) + \delta x p/2 \equiv E + \delta x p/2$ over the time interval $0 \leq t \leq t_{AI}$, $t_{AI} \sim \delta^{-1}$ (i.e., the same scaling as for the breaking time, t_b , deduced above), where E is the energy of the underlying integrable Duffing oscillator. Observe that the adiabatic invariant reduces to E provided that δ (λ) is sufficiently small (as required in AAR theory) and that the *same* result is obtained for a general potential $V(x)$ instead of Duffing's potential.

In sum, a general energy-based theory of AR phenomena in low-dimensional nonautonomous systems has been deduced from a simple variational principle concerning the power functional. In particular, the theory explains all the phenomenological and approximate results arising from a previous adiabatic approach to AR in Duffing-like systems. For this class of systems, the present theory also explains the adiabaticity requirement as well as why the same theoretical predictions hold in the cases with and without dissipation, and yields the analytical expression for the adiabatic invariants. Additionally, new adiabatic approximations to AR excitations are derived concerning two general cases which were not considered in the previous adiabatic approach, namely, the case of arbitrary initial conditions (not just those near equilibria) and the case of arbitrary potential (not just linear) chirps, for which new general scaling laws were deduced (including the 3/4 scaling law as a particular case). Computer simulations confirmed all the theoretical predictions. In view of the generality of the present theory of AR, one can expect it to be quite readily testable by experiment (e.g., in the Diocotron system in pure-electron plasmas), and that it will find applications in different fields of physics, such as plasmas, fluids, and solar system dynamics.

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A. Figure Captions

Figure 1. Autoresonant responses (energy vs time, both variables in arbitrary units) to a linearly swept excitation (cf. eq. (1), grey lines) and to an exact AR excitation (cf. eq. (6), black lines), for the parameters $b = 5, x(0) = 10^{-3}, \dot{x}(0) = 0, \gamma_0 = 10^{-3}, \varphi_0 = 0$. (a) Case with $\omega_0 \sim \delta$, as required for an exact AR excitation, and $\varepsilon = 0.5$. (b) Case with $\omega_0 \gg \delta$, i.e., far from the exact AR excitation requirement, and $\omega_0 = 2\pi$.

Figure 2. Autoresonant responses (energy vs time, both variables in arbitrary units) to a general 1st-order adiabatic excitation (cf. eq. (9), black lines) and to a harmonic and linearly swept excitation (cf. eq. (1), grey lines), for the parameters $b = 5, \delta = 0.4, \omega_0 = 0.2, \varepsilon = 0.5 \sim \varepsilon_{th}, \alpha = 0.08 \sim \alpha_{th}$, and initial conditions $x(0) = 0.8, \dot{x}(0) = 0.107$ ($\gamma_0 = 0.8, \varepsilon' = 0.9 \sim \varepsilon'_{th}$, thick lines) and $x(0) = 0.6, \dot{x}(0) = 0.08$ ($\gamma_0 = 0.6, \varepsilon' = 0.67 \sim \varepsilon'_{th}$, thin lines).

Figure 3. Autoresonant responses (energy vs time, both variables in arbitrary units) to a harmonic excitation with a linear chirp ($\omega(t) = \omega_0 + \alpha_1 t$, cf. eq. (1), grey lines) and with a quadratic chirp ($\omega(t) = \omega_0 + \alpha_2 t^2$, black lines), for the parameters $b = 5, \delta = 0.4, \omega_0 = 0.2, \varepsilon = 0.5 \sim \varepsilon_{th}, \alpha_1 = 0.04 \sim \alpha_{1,th}, \alpha_2 = 0.003 \sim \alpha_{2,th}$, and the initial conditions $x(0) = 10^{-3}, \dot{x}(0) = 0$ (thick lines) and $x(0) = 0, \dot{x}(0) = 1$ (thin lines).







