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SHORT COMMUNICATION

A student preconception on physical geodesy: the "best" reference ellipsoid

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Abstract
We show in this note a simple exercise to overcome a common preconception among Geodesy students about the "best" reference ellipsoid. This helps students to overcome their previous ideas based on a purely geometric vision of the reference ellipsoids and advance to a more physical point of view, where the Earth's gravity plays a fundamental role.

Keywords Ellipsoid · Gravity · Geodesy · Geophysics

1 Introduction

The search for a mathematical model that adequately represents the surface of the Earth and its gravity field has been an intellectual adventure that has occupied the most important geodesists of all time (Greenberg 1995; Hoare 2004). The ellipsoid of revolution has been identified as a suitable mathematical surface for this purpose and geodesy students from around the world are working on this topic.

All students know that an ellipsoid of revolution is determined by two parameters. These can be the major and minor semi-axis (a and b, respectively) or the flattening (f) and the major semi-axis. However, a reference ellipsoid is not only a representation of the Earth's surface, but it must also generate a representation of the Earth's gravity field. Therefore, a reference ellipsoid is determined by four parameters: the geocentric gravitational constant of the Earth including the atmosphere (GM), the dynamical form factor (J_2), the mean angular velocity (ω) and the geopotential on the surface (W_0). From these four parameters, the

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major semi-axis and the flattening can be calculated, determining a particular revolution ellipsoid.

In the teaching and learning process, it is important to pay attention to preconceptions, that is, those previous ideas that students have about certain aspects of the subjects they study (see, for example, the classical work by Clement 1982). Our Geodesy or Geophysics students also have preconceptions about these subjects, as in all branches of Earth sciences (DeLaughter et al. 1998).

After many years of teaching in Geodesy, we have noticed a very common misconception that affects most of our students about the "best ellipsoid". If we ask them what would happen if we represented the values of the semi-major axis of several reference ellipsoids against the respective values of the flattening, the most usual answer is that we will see a cloud of points centered on one point with the "optimal" values for *a* and *f* (Fig. 1). Our students are forgetting the gravitational part of the reference ellipsoids and are thinking only in geometric terms (surface of the Earth). Let's take a closer look at this example.

2 A linear relationship between *a* and *f*

It is considered that the best representation of the Earth by an ellipsoid is the "mean Earth ellipsoid". It is defined as the ellipsoid that has (*i*) the same potential W_0 as the geoid, (*ii*) the same mass as the Earth *M*, (*iii*) the same difference of moments of inertia $C - \overline{A}$ (where C and $\overline{A}$ are the polar and the mean equatorial moment of inertia, respectively), and (*iv*) the same angular velocity ω as the Earth. Namely, the "mean Earth ellipsoid" is defined completely by the four mentioned constants (or other similar set of constants). This particular ellipsoid has very interesting properties and can be considered the best representation of the Earth by an ellipsoid.

Heiskanen and Moritz (1993) studied the "mean Earth ellipsoid" showing that there is a linear relationship between the parameter *a* and *f*:

$$a = \frac{GM}{W_0}\left(1 + \frac{1}{3}f + \frac{1}{3}m\right) \quad (1)$$

where *m* is the ratio between the centrifugal force and the gravity at equator. This equation can be re-written as:

$$a = \frac{GM}{3W_0}f + \frac{GM}{W_0}\left(1 + \frac{m}{3}\right) \quad (2)$$

Thus, there is a linear relationship between *a* and *f*, where the slope is equal to $GM/3W_0$ and the independent term is equal to $(GM/W_0)(1 + m/3)$. Taking the values for these constants from WGS84 ellipsoid (see Table 1), we obtain the following results for the slope and the independent term:

$$\frac{GM}{3W_0} = 2.12 \times 10^6 \mathrm{m} \quad (3)$$

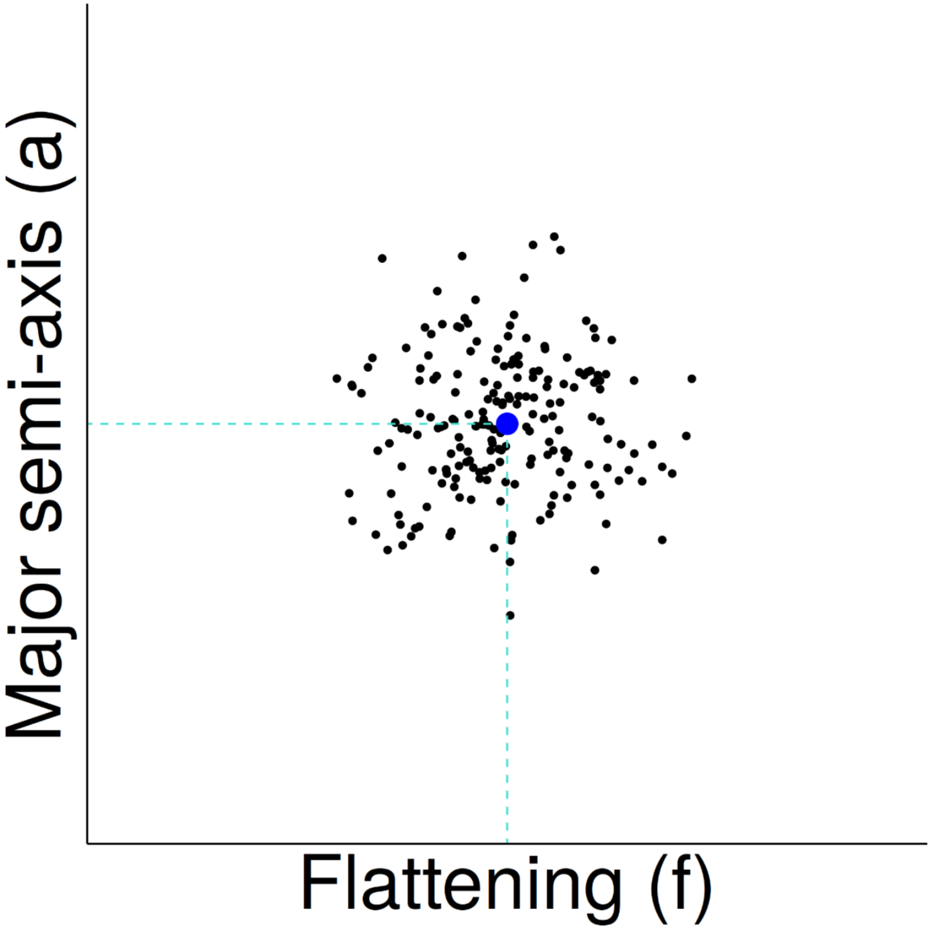

Fig. 1 Most usual answer of the student if we ask them what would happen if we represented the values of the semi-major axis of several reference ellipsoids against the respective values of the flattening: a cloud of points (in black) centered on one point with the "optimal" values for *a* and *f* (light blue)

Table 1 WGS84 ellipsoid parameters

a (m)	6378137
GM (m^3/s^2)	$3986004.418 \cdot 10^8$
W_0 (m^2/s^2)	62636851.7146
m	0.00344978650684

$$\frac{GM}{W_0}\left(1+\frac{m}{3}\right)=6.37\times10^6\mathrm{m}\;(4)$$

We observe in Eqs. (1) and (2) that there is a relationship between a, a geometrical parameter, and the physical magnitudes GM, m, and W_0 that are related with the gravity field.

3 Discussion

Now, we can verify the above values by doing a least squares analysis of the values of a and f from various determinations of reference ellipsoids. A selection of reference ellipsoids is listed in Table 2. Note that the values of the ellipsoids determined in 1738 and 1799 are quite far from the typical values of the rest of the ellipsoids. In the late 1700s and the first half of the 1800s, an enormous number of ellipsoids were published as better approximations of the shape of the Earth. Most of these ellipsoids were determined to fit the geoid very well in a relatively small region of the Earth surface. For this reason, they often offered global values far removed from the values accepted today (see, for example, the discussion in Chap. 3 of the book by Timár and Molnár 2013).

Using the values of Table 2, the best linear fit is obtained with a value for the slope equal to $(10.1 \pm 0.2) \cdot 10^6$ m and a value for the independent term equal to $(6.3440 \pm 0.0006) \cdot 10^6$ m. If the ellipsoids determined in 1738 and 1799 are excluded (because their values are very different from the rest of the set), the results for the slope and independent term are $(13 \pm 1) \cdot 10^6$ m and $(6.333 \pm 0.004) \cdot 10^6$ m, respectively, with a correlation coefficient equal to 0.9. Figure 2 shows these last results. The values of the constants of the different ellipsoids differ in the precision with which they were calculated and, therefore, there is some scatter around the linear regression. This is consistent with the fact that the ellipsoid is a first order approximation to the geoid.

If we compare the values obtained from the linear regression with the ones calculated by applying the Eqs. (3) and (4), we observe that the slope is slightly higher than expected. We also observe it excluding the 1738 and 1799 ellipsoids. The values of the independent terms are very similar. In any case, the results agree with the expected order of magnitude.

4 Conclusions

We have found a simple test to show a preconception of Geodesy students. We asked our students for the expected result when they plotted the semi-major axis of a set of reference ellipsoids against their respective flats. They usually respond that they expect a point cloud centered on the "optimal values" of semi-major axis and flattening, corresponding to the "best" reference ellipsoid. However, this answer is incorrect, as we have seen.

We have analyzed the major semi-axis and the flattening values for different ellipsoids (Table 2). Although one could think that there is no relation between the two parameters, a linear regression is plausible. This linear relationship agrees with Eq. (2) because the GM, W_0 and m values are similar for all the reference ellipsoids. From the linear regression, we have obtained that the values of slope and independent term present a magnitude order that is equal to the values obtained from Eqs. (3) and (4). In particular, the independent term value is very close to the one calculated for WGS84 ellipsoid.

It is important to note that we have found this preconception mainly in undergraduate students in Geomatics. In Spain, the organization of these degrees contemplates that the students receive a geometrical geodesy course in the first years. Only later, in the last years of those degrees, the students receive a geophysics course that includes a part devoted to physical geodesy. This may be one reason that explains the prevalence of this preconception

Table 2 A selection of reference ellipsoids, including the major semi-axis (a) and the reciprocal flattening ($1/f$)

Ellipsoid	a (m)	$1/f$
Airy 1830	6377563.396	299.3249613
Andrae 1876 (Denmark included)	6377104.43	300
Applied Physics 1965	6378137	298.25
Australian National Spheroid	6378160	298.25
Bessel 1841	6377397.155	299.1528128
Bessel 1841 (Namibia)	6377483.865	299.1528128
Clarke 1866	6378206.4	294.9786982
Clarke 1880 modified	6378249.145	293.465
Commission des Poids et Mesures 1799	**6375738.7**	**334.29**
Delambre 1810 (Belgium)	6376428	311.5
Engelis 1985	6378136.05	298.2566
Everest (Sabah and Sarawak)	6377298.556	300.8017
Everest 1830	6377276.345	300.8017
Everest 1948	6377304.063	300.8017
Everest 1956	6377301.243	300.8017
Everest 1969	6377295.664	300.8017
Fischer (Mercury Datum) 1960	6378166	298.3
Fischer 1968	6378150	298.3
GRS 80 (IUGG 1980)	6378137	298.2572221
GRS 67 (IUGG 1967)	6378160	298.2471674
Hayford 1909	6378388	297
Heiskanen 1919 (*)	6378400.00	298.20
Helmert 1906	6378200	298.3
Hough 1960	6378270	297
IAU 1976	6378140	298.257
Indonesian 1974	6378160	298.247
International 1924	6378388	297
Jeffreys 1948 (*)	6378099.00	297.10
Kaula 1961	6378163	298.24
Krassovsky 1940	6378245	298.3
Lerch 1979	6378139	298.257
Maupertuis 1738	**6397300**	**191**
Modified Airy	6377340.189	299.3249655
Modified Fischer 1960	6378155	298.3
MERIT 1983	6378137	298.257
Naval Weapons Laboratory 1965	6378145	298.25
New International 1967	6378157.5	298.2496154
Oxford 1959 (*)	6378201.00	297.65
Plessis 1817 (France)	6376523	308.6409971
South America 1969 Spheroid	6378160	298.25
Southeast Asia	6378155	298.3000002
Soviet Geodetic System 1985 (SGS 85)	6378136	298.257
Struve 1860 (*)	6378297.00	294.73
Walbeck	6376896	302.7800002
WGS 60	6378165	298.3

Table 2 (continued)

Ellipsoid	*a* (m)	1/*f*
WGS 66	6378145	298.25
WGS 72	6378135	298.26
WGS 84	6378137	298.2572236

Data from Meyer (2010) (except those ellipsoids marked with (*) that were taken from Bomford 1985). Ellipsoid with name in bold were excluded in our final analysis

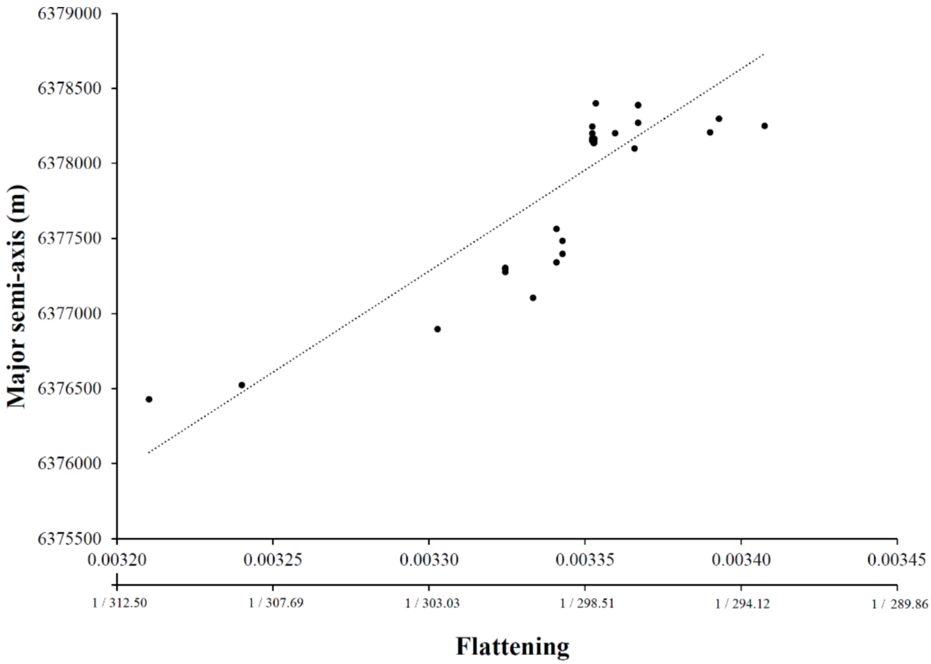

Fig. 2 Major semi-axis versus flattening of the reference ellipsoids from Table 2 (except for the two ellipsoids whose name is in bold in Table 2). The dashed line shows the best linear fit

(that only takes into account the geometry of the ellipsoid and forgets its gravitational field) among our students.

We think that this simple exercise can help students to move from a purely geometric view of the reference ellipsoids to a more physical one where the Earth's gravity is more protagonist.

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Declarations

Competing Interests The authors have no competing interests to declare that are relevant to the content of this article.

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