AFFINE FUNCTORS AND DUALITY

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Abstract. A functor of sets $\mathbb{X}$ over the category of K-commutative algebras is said to be an affine functor if its functor of functions, $\mathbb{A}_{\mathbb{X}}$, is reflexive and $\mathbb{X} = \operatorname{Spec} \mathbb{A}_{\mathbb{X}}$. We prove that affine functors are equal to a direct limit of affine schemes and that affine schemes, formal schemes, the completion of affine schemes along a closed subscheme, etc., are affine functors.

Endowing an affine functor $\mathbb{X}$ with a functor of monoids structure is equivalent to endowing $\mathbb{A}_{\mathbb{X}}$ with a functor of bialgebras structure. If $\mathbb{G}$ is an affine functor of monoids, then $\mathbb{A}^*_{\mathbb{G}}$ is the enveloping functor of algebras of $\mathbb{G}$ and the category of $\mathbb{G}$-modules is equivalent to the category of $\mathbb{A}^*_{\mathbb{G}}$-modules. Applications of these results include Cartier duality, neutral Tannakian duality for affine group schemes, the equivalence between formal groups and Lie algebras in characteristic zero, etc.

Introduction

Let K be a (unital associative) commutative ring. It is well-known that K-schemes can be treated as mere "abstract sets" by means of their functor of points, which are functors of sets defined over the category of commutative K-algebras. This functorial point of view is particularly useful to study K-group schemes and their linear representations ([5], [6]). Nevertheless, to obtain reasonable results in the case of formal groups, the functors of points have to be endowed with a topology: they are defined over the category of linearly compact K-algebras, where K is a pseudo-compact ring ([7]).

In this paper, that develops ideas introduced in [1] and [10], we show that, when considering K-modules, linear representations of group schemes, formal groups, etc., as functors over the category of commutative K-algebras since the beginning, then the concepts of affine functor and reflexive functor arise naturally, allowing to prove many results from Algebraic Geometry, as obvious consequences of the reflexivity of the functors of modules considered.

All functors considered in this paper are covariant functors defined over the category of commutative K-algebras.

Given an K-module M, we denote by $\mathcal{M}$ the functor $\mathcal{M}(S) := M \otimes_K S$, for all commutative K-algebras S. We say that $\mathcal{M}$ is the quasi-coherent module associated with M. If $\mathbb{M}$ and $\mathbb{M}'$ are functors of $\mathcal{K}$-modules, then $\mathbb{H}om_{\mathcal{K}}(\mathbb{M}, \mathbb{M}')$ will denote the functor of $\mathcal{K}$-modules

$$\mathbb{H}om_{\mathcal{K}}(\mathbb{M}, \mathbb{M}')(S) := \operatorname{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathbb{M}'_{|S})$$

where $\mathbb{M}_{|S}$ is the functor $\mathbb{M}$ restricted to the category of commutative S-algebras. We write $\mathbb{M}^* := \mathbb{H}om_{\mathcal{K}}(\mathbb{M}, \mathcal{K})$ and say that this is a dual functor.

The fundamental results on which is based this paper are the reflexivity theorems:

Theorem **([1, 1.10]):** Let M be an K-module, then $\mathcal{M}^{**} = \mathcal{M}$.

Theorem ([10, 4.4]): Assume that K is a field. A functor of $\mathcal{K}$-modules $\mathbb{M}$ is reflexive if and only if it is the inverse limit of its quasi-coherent quotients.

Given a functor of commutative $\mathcal{K}$-algebras $\mathbb{A}$, let $\operatorname{Spec}\mathbb{A}$ be the functor:

$$\operatorname{Spec}\mathbb{A}(S) := \operatorname{Hom}_{\mathcal{K}-alg}(\mathbb{A}, \mathcal{S}).$$

In case $X = \operatorname{Spec} A$ is an affine K-scheme and $X^{\cdot}$ stands for its functor of points ($X^{\cdot}(S) := \operatorname{Hom}_{K-alg}(A, S)$), then $X^{\cdot} = \operatorname{Spec}\mathcal{A}$. Let C be a K-module, if $\mathcal{C}^*$ is a functor of commutative algebras we say that $\operatorname{Spec}\mathcal{C}^*$ is a formal scheme. If K is a field, $\operatorname{Spec}\mathcal{C}^*$ is a formal scheme if and only if it is a direct limit of finite K-schemes (see Note 4.2). This is the definition of formal scheme, when K is a field, that can be found in [5] and [13].

Let $\mathbb{X}$ be a functor of sets and $\mathbb{A}_{\mathbb{X}} := \mathbb{H}om(\mathbb{X}, \mathcal{K})$. We say that $\mathbb{X}$ is an affine functor if $\mathbb{X} = \operatorname{Spec}\mathbb{A}_{\mathbb{X}}$ and $\mathbb{A}_{\mathbb{X}}$ is reflexive. We warn the reader that in the literature affine functors are sometimes defined to be functors of points of affine schemes.

From now on we will assume, for simplicity, that K is a field. We prove the following theorems.

Theorem 0.1. *If $\mathbb{A}$ is a reflexive functor of commutative algebras, $\operatorname{Spec}\mathbb{A}$ is a direct limit of closed immersions of affine schemes.*

Let A be a K-algebra, we say that $\mathcal{A}$ is a quasi-coherent algebra. If $\mathbb{A}$ is the inverse limit of its quasi-coherent algebra quotients we say that $\mathbb{A}$ is a proquasi-coherent algebra.

Theorem 0.2. *Let $\mathbb{A}$ be a reflexive functor of commutative algebras. If $\mathbb{A}$ is a proquasi-coherent algebra, then $\operatorname{Spec}\mathbb{A}$ is an affine functor and $\mathbb{A}_{\operatorname{Spec}\mathbb{A}} = \mathbb{A}$. If $\mathbb{X}$ is an affine functor, then $\mathbb{A}_{\mathbb{X}}$ is a proquasi-coherent algebra.*

Theorem 0.3. *Affine schemes, formal schemes, the completion of an affine scheme along a closed subscheme are affine functors.*

Theorem 0.4. *An affine functor $\mathbb{G}$ is a functor of monoids if and only if $\mathbb{A}_{\mathbb{G}}$ is a functor of bialgebras, and given two affine functors of monoids $\mathbb{G}_1$ and $\mathbb{G}_2$, then*

$$\operatorname{Hom}_{mon}(\mathbb{G}_1, \mathbb{G}_2) = \operatorname{Hom}_{bialg}(\mathbb{A}_{\mathbb{G}_2}, \mathbb{A}_{\mathbb{G}_1})$$

In particular, the category of formal monoids is equivalent to the category of cocommutative bialgebras (see 6.8).

In Section 9, we prove the categorical equivalence of the category of infinitesimal formal groups with the category of Lie algebras, and the Poincaré-Birkhoff-Witt Theorem, in characteristic zero (see [12]). Let us speak loosely. Let $\mathbb{G} = \operatorname{Spec}\mathcal{A}^*$ be an infinitesimal formal group and $\mathcal{I}^*$ the ideal of functions of $\mathbb{G}$ vanishing at the unit element. Then, the natural morphism $S^n(\mathcal{I}^*/\mathcal{I}^{*2}) \to \mathcal{I}^{*n}/\mathcal{I}^{*n+1}$ is an isomorphism, and we prove that the inverse morphism is the morphism induced by the comultiplication morphism $\mathcal{A}^* \to \mathcal{A}^* \tilde{\otimes} \overset{n}{\cdots} \tilde{\otimes} \mathcal{A}^*$. Dually, if $L = (\mathcal{I}^*/\mathcal{I}^{*2})^* = T_e\mathbb{G}$ is the Lie algebra of $\mathbb{G}$, $U(L)$ is the universal algebra associated with L and $U(L)_n := L \cdot \overset{n}{\cdots} \cdot L$, we obtain that $U(L)_n/U(L)_{n+1} = S^n L$ and as a consequence $A = U(L)$. Given another infinitesimal formal group $\mathbb{G}' = \operatorname{Spec}\mathcal{B}^*$, then

$$\begin{aligned} \operatorname{Hom}_{grp}(\mathbb{G}, \mathbb{G}') &= \operatorname{Hom}_{bialg}(\mathcal{B}^*, \mathcal{A}^*) = \operatorname{Hom}_{bialg}(A, B) \\ &= \operatorname{Hom}_{bialg}(U(T_e\mathbb{G}), U(T_e\mathbb{G}')) = \operatorname{Hom}_{Lie}(T_e\mathbb{G}, T_e\mathbb{G}') \end{aligned}$$

It is well-known that, for a finite monoid G, the category of K-linear representations of G is equivalent to the category of KG-modules. In [1, 5.4] we extended this result to affine group schemes. Now, let $\mathbb{G}$ be a functor of monoids, such that $\mathbb{A}_{\mathbb{G}}$ is reflexive and let $\mathbb{G} \to \mathbb{A}_{\mathbb{G}}^*$ be the natural morphism. In this paper we prove that the enveloping functor of algebras of $\mathbb{G}$ is $\mathbb{A}_{\mathbb{G}}^*$, that is,

$$\mathrm{Hom}_{mon}(\mathbb{G}, \mathbb{B}) = \mathrm{Hom}_{\mathcal{K}-alg}(\mathbb{A}_{\mathbb{G}}^*, \mathbb{B}) \tag{1}$$

for all dual functors of $\mathcal{K}$-algebras $\mathbb{B}$. As consequences of Equality 1, we obtain the following two theorems.

Theorem 0.5. *The category of dual functors of $\mathbb{G}$-modules is equivalent to the category of dual functors of $\mathbb{A}_{\mathbb{G}}^*$-modules. In particular, the category of quasi-coherent $\mathbb{G}$-modules is equivalent to the category of quasi-coherent $\mathbb{A}_{\mathbb{G}}^*$-modules.*

Corollary 0.6. *Let $\mathbb{M}, \mathbb{M}'$ be reflexive functors of $\mathbb{G}$-modules. Then, a morphism of $\mathcal{K}$-modules $\mathbb{M} \to \mathbb{M}'$ is a morphism of $\mathbb{G}$-modules if and only if $\mathbb{M}(K) \to \mathbb{M}'(K)$ is a morphism of $\mathbb{A}_{\mathbb{G}}^*(K)$-modules. Let $\mathcal{M}$ be a $\mathbb{G}$-module, then the set of quasi-coherent $\mathbb{G}$-submodules of $\mathcal{M}$ is equal to the set of $\mathbb{A}_{\mathbb{G}}^*(K)$-submodules of M.*

In Section 9, we deduce the Tannaka's characterization of the category of linear representations of an affine group scheme: Let us talk loosely. If a category of finite-dimensional vector spaces with an extra structure is generated by a unique object X, then it is (weak) equivalent to the category of finitely generated A_X-modules, for some finite-dimensional algebra A_X. If the category is generated by a set of objects $\{X_i\}$ then it is equivalent to the category of finite generated $\lim\limits_{\overleftarrow{i}} \mathcal{A}_{X_i}$-modules, where $\lim\limits_{\overleftarrow{i}} \mathcal{A}_{X_i} = (\lim\limits_{\overrightarrow{i}} \mathcal{A}_{X_i}^*)^* =: \mathcal{C}^*$ is a scheme of algebras. If in addition, the tensor product of the objects of the category are objects of the category then $\mathcal{C}^*$ has a comultiplication, that is, $\mathcal{C}^*$ is a bialgebra. Therefore, the category is equivalent to the category of finite linear representations of $\operatorname{Spec} C$.

Theorem 0.7. *Assume $\mathbb{G}$ is commutative. Then,*

$$\mathbb{G}^\vee := \mathbb{H}om_{mon}(\mathbb{G}, \mathcal{K}) = \mathbb{H}om_{\mathcal{K}-alg}(\mathbb{A}_{\mathbb{G}}^*, \mathcal{K}) = \operatorname{Spec} \mathbb{A}_{\mathbb{G}}^*$$

As immediate application of Theorem 0.7 and the reflexivity theorem, we deduce the Cartier duality over commutative rings (also see [7, Ch. I, §2, 14], where formal schemes are certain functors over the category of commutative linearly compact algebras over a field).

If $G_a = \operatorname{Spec} K[x]$ is the additive group, the category of $G_a^\vee$ modules is equivalent to the category of $K[x]$-modules. Then,

$$\mathrm{Hom}_{mon}(G_a^\vee, \mathbb{E}nd_K(V)) = \mathbb{E}nd_K(V)$$

If V is a K-algebra, we prove that

$$\mathrm{Hom}_{mon}(G_a^\vee, \mathbb{E}nd_{K-alg}(V)) = \mathrm{Der}_K(V, V)$$

More generally, we prove (8.12) that if G is K-group scheme, then

$$\mathrm{Hom}_{mon}(G_a^\vee, G) = T_e G$$

and if X is a K-scheme, then $\mathrm{Hom}_{mon}(G_a^\vee, \mathbb{E}nd\, X) = \mathrm{Der}\, X$ (8.10).

That is, giving a vector field D on a K-scheme X is equivalent to giving a morphism of functors of groups $exp_D\colon G_a^\vee \to \mathbb{E}nd\, X$; and giving an invariant vector field on a K-group scheme G is equivalent to giving a morphism of functors of groups $exp_D\colon G_a^\vee \to G$.

Every morphism $\operatorname{Spec}\mathcal{C}^* \to X$, from a infinitesimal formal scheme (i.e., C^* is a local algebra) to a K-scheme uniquely factors via $\operatorname{Spec} C^*$, that is,

$$\operatorname{Hom}(\operatorname{Spec}\mathcal{C}^*, X) = \operatorname{Hom}_{K-sch}(\operatorname{Spec} C^*, X).$$

Then, exp_D uniquely factors through $\operatorname{Spec} K[x]^*$. In characteristic zero, $K[x]^* \simeq K[[z]]$ and one has the classical exponential map $exp_D\colon \operatorname{Spec} K[[z]] \to G$, associated with D ([4, II 6 3]). In characteristic $p > 0$, one has an isomorphism of K-schemes $\operatorname{Spec} K[x]^* = \operatorname{Spec} K[[x_0, \dots, x_n, \dots]]/(x_0^p, \dots, x_n^p, \dots)$ (see 7.9). We apply this construction to prove the existence and uniqueness of "analytic" solutions of an algebraic differential equation in arbitrary characteristic (see 8.18):

Let δ be the invariant field on $\operatorname{Spec} K[x]^*$. That is, in characteristic zero $\delta = \partial_z$, in characteristic $p > 0$, $\delta = \partial_{x_0} + x_0^{p-1}/(p-1)!\partial_{x_1} + x_0^{p-1}x_1^{p-1}/(p-1)!^2\partial_{x_2} + \cdots$.

Theorem 0.8. *Let X be a K-scheme, let $y \in X$ be a rational point and let D be a vector field on X. Then, $exp_{D,y}\colon \operatorname{Spec} K[x]^* \to X$, $exp_{D,y}(x) := exp_D(x)$ is the only morphism $f\colon \operatorname{Spec} K[x]^* \to X$ such that $f(0) = y$ and $f(\delta_\mu) = D_{f(\mu)}$, for every point $\mu \in (\operatorname{Spec} K[x]^*)(S)$.*

If X is a complete algebraic variety, then the scheme-theoretic image of exp_D is a commutative algebraic group, that we define to be the algebraic group associated with D (8.22). The minimal subvariety tangent to D passing through a point is the orbit of the point under the action of the algebraic group associated with D.

Let $\operatorname{Spec}\mathcal{C}^*$ be a formal monoid and $D_{\mathcal{C}} = \{w \in C^*\colon w(I) = 0$ for some bilateral ideal $I \subset C$ of finite codimension$\} \subset C^*$. Then, $\operatorname{Spec} D_{\mathcal{C}}$ is an affine monoid scheme, because $\mathcal{D}_{\mathcal{C}}^* = \lim\limits_{\substack{\leftarrow \\ I}} \mathcal{C}/\mathcal{I}$ is a scheme of bialgebras, and

$$\begin{aligned}
\operatorname{Hom}_{mon}(\operatorname{Spec}\mathcal{C}^*, \operatorname{Spec} A) &= \operatorname{Hom}_{bialg}(\mathcal{A}, \mathcal{C}^*) = \operatorname{Hom}_{bialg}(\mathcal{C}, \mathcal{A}^*) \\
&= \operatorname{Hom}_{bialg}(\lim_{\substack{\leftarrow \\ I}} \mathcal{C}/\mathcal{I}, \mathcal{A}^*) = \operatorname{Hom}_{bialg}(\mathcal{A}, \mathcal{D}_{\mathcal{C}}) \\
&= \operatorname{Hom}_{mon}(\operatorname{Spec} D_{\mathcal{C}}, \operatorname{Spec} A)
\end{aligned} \tag{2}$$

In the case the algebraic group associated with D is an affine algebraic group, then it is isomorphic to a quotient of $\operatorname{Spec} D_{\mathcal{K}[x]}$, by Equation 2. Then, by Theorem 7.11, it is isomorphic to

$$\begin{array}{ll} G_a^\delta \times G_m^n,\ \delta = 0, 1, & \text{if } \operatorname{char} k = 0 \\ \alpha_r^\vee \times \mu_1^n, & \text{if } \operatorname{char} k = p > 0 \end{array}$$

Finally, in Section 8.5, we calculate the algebraic group associated with a field on $\mathbb{P}^n(k)$, where k is an algebraically closed field of arbitrary characteristic, recovering in this way the results from [11] for the case of characteristic zero.

Theorem 0.9. *Let K be an algebraically closed field and let $\pi\colon \mathbb{A}^n(K)\backslash 0 \to \mathbb{P}^{n-1}(K)$ be the projectivization map. Let $D = \pi(\sum_{ij} \lambda_{ij} x_i \partial_{x_j})$ be a vector field on $\mathbb{P}^{n-1}(K)$ and let G be its associated algebraic group.*

(1) *If* $\operatorname{char} K = 0$

$$G \simeq G_m^r \times G_a^\delta$$

where r is the dimension of the $\mathbb{Q}$-affine space generated by the eigenvalues of the matrix (λ_{ij}) in K, $\delta = 0$ in case the matrix is diagonalizable and $\delta = 1$ otherwise.

(2) *If* $\operatorname{char} K = p > 0$

$$G \simeq \mu_1^r \times \alpha_{m+1}^\vee$$

where r *is the dimension of the* $\mathbb{Z}/p\mathbb{Z}$*-affine space generated by the eigenvalues of the matrix* (λ_{ij}) *in* K*, and* m *is such that, if* s *is the greatest of the orders of the Jordan boxes, then* $p^m \leq s-1 < p^{m+1}$ *(if* $s=1$ *we say that* $m=-1$*).*

1. Main notations

R is a (unital associative) commutative ring, S is a commutative R-algebra. A, B are R-algebras (or R-bialgebras). M, N are R-modules. K is a field, V is a K-vector space or a free R-module.

$\mathbb{X}, \mathbb{Y}$ are (covariant) functors of sets. $\mathbb{G}$ is a functor of monoids (or semigroups, or groups). $\mathbb{M}$, $\mathcal{M}, \mathcal{N}$ are functors of $\mathcal{R}$-modules. $\mathcal{M}, \mathcal{N}$ are called quasi-coherent modules, $\mathcal{M}(S) := M \otimes_R S$.

$\operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')$ is the set of morphisms of functors of $\mathcal{R}$-modules from $\mathbb{M}$ to $\mathbb{M}'$. $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{N})$ is the $\mathcal{R}$-module functor of morphisms of $\mathcal{R}$-modules. $\mathbb{M}^* := \mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathcal{R})$ is the dual functor (of $\mathbb{M}$). The dual functor of $\mathcal{M}$, $\mathcal{M}^*$ is called module scheme.

$\mathbb{A}, \mathbb{B}, \mathcal{A}, \mathcal{B}, \mathcal{C}^*$ are functors of $\mathcal{R}$-algebras or $\mathcal{R}$-bialgebras ($\mathcal{A}, \mathcal{B}$ are quasi-coherent $\mathcal{R}$-modules and $\mathcal{C}^*$ is an $\mathcal{R}$-module scheme). $\operatorname{Spec} A = \operatorname{Spec} \mathcal{A}$ is an affine scheme. $\operatorname{Spec} \mathcal{C}^*$ is called formal scheme.

$\mathbb{A}_{\mathbb{X}} := \mathbb{H}om(\mathbb{X}, \mathcal{R})$ is the functor of functions of the functor of sets $\mathbb{X}$.

$\mathfrak{F}$ is a wide family of reflexive $\mathcal{R}$-modules, which contains free quasi-coherent $\mathcal{R}$-modules and it is closed by the functor $\mathbb{H}om_{\mathcal{R}}(-,-)$ and "essentially" closed by the functor $- \otimes_{\mathcal{R}} -$.

2. Preliminaries

Let R be a commutative ring (associative with a unit). All functors considered in this paper are covariant functors over the category of commutative R-algebras (always assumed to be associative with a unit). A functor $\mathbb{X}$ is said to be a functor of sets (resp. monoids, etc.) if $\mathbb{X}$ is a functor from the category of commutative R-algebras to the category of sets (resp. monoids, etc.).

Notation 2.1. *For simplicity, given a functor of sets* $\mathbb{X}$*, we sometimes use* $x \in \mathbb{X}$ *to denote* $x \in \mathbb{X}(S)$*. Given* $x \in \mathbb{X}(S)$ *and a morphism of commutative* R*-algebras* $S \to S'$*, we still denote by* x *its image by the morphism* $\mathbb{X}(S) \to \mathbb{X}(S')$*.*

Let $\mathcal{R}$ be the functor of rings defined by $\mathcal{R}(S) := S$, for all commutative R-algebras S. A functor of sets $\mathbb{M}$ is said to be a functor of $\mathcal{R}$-modules if we have morphisms of functors of sets, $\mathbb{M} \times \mathbb{M} \to \mathbb{M}$ and $\mathcal{R} \times \mathbb{M} \to \mathbb{M}$, so that $\mathbb{M}(S)$ is an S-module, for every commutative R-algebra S. A functor of algebras (associative with a unit), $\mathbb{A}$, is said to be a functor of $\mathcal{R}$-algebras if we have a morphism of functors of algebras $\mathcal{R} \to \mathbb{A}$ (and $\mathcal{R}(S) = S$ commutes with all the elements of $\mathbb{A}(S)$, for every commutative R-algebra S).

Given a commutative R-algebra S, we denote by $\mathbb{M}_{|S}$ the functor $\mathbb{M}$ restricted to the category of commutative S-algebras.

Let $\mathbb{M}$ and $\mathbb{M}'$ be functors of $\mathcal{R}$-modules. A morphism of functors of $\mathcal{R}$-modules $f\colon \mathbb{M} \to \mathbb{M}'$ is a morphism of functors such that the defined morphisms $f_S\colon \mathbb{M}(S) \to \mathbb{M}'(S)$ are morphisms of S-modules, for all commutative R-algebras S. We will denote by $\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')$ the set of all the morphisms of $\mathcal{R}$-modules from $\mathbb{M}$ to $\mathbb{M}'$. We will denote by $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')$[1] the functor of $\mathcal{R}$-modules

$$\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')(S) := \mathrm{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathbb{M}'_{|S})$$

Obviously,

$$(\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{M}'))_{|S} = \mathbb{H}om_{\mathcal{S}}(\mathbb{M}_{|S}, \mathbb{M}'_{|S})$$

Notation 2.2. *We denote* $\mathbb{M}^* = \mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathcal{R})$.

Notation 2.3. *Tensor products, direct limits, inverse limits, kernels, cokernels, images, etc., of functors of $\mathcal{R}$-modules are regarded in the category of functors of $\mathcal{R}$-modules.*

Definition 2.4. *Given an R-module M, the functor of $\mathcal{R}$-modules $\mathcal{M}$ defined by $\mathcal{M}(S) := M \otimes_R S$ is called a quasi-coherent $\mathcal{R}$-module.*

Proposition 2.5. [1, 1.3] *For every functor of $\mathcal{R}$-modules $\mathbb{M}$ and every R-module M, it holds that*

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathbb{M}) = \mathrm{Hom}_R(M, \mathbb{M}(R))$$

The functors $M \rightsquigarrow \mathcal{M}$, $\mathcal{M} \rightsquigarrow \mathcal{M}(R) = M$ establish an equivalence between the category of $\mathcal{R}$-modules and the category of quasi-coherent $\mathcal{R}$-modules ([1, 1.12]). In particular, $\mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{M}') = \mathrm{Hom}_R(M, M')$. For any pair of R-modules M and N, the quasi-coherent module associated with $M \otimes_R N$ is $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N}$. $\mathcal{M}_{|S}$ is the quasi-coherent $\mathcal{S}$-module associated with $M \otimes_R S$

The functor $\mathcal{M}^* = \mathbb{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{R})$ is called an $\mathcal{R}$-module scheme. Moreover, $\mathcal{M}^*(S) = \mathrm{Hom}_S(M \otimes_R S, S) = \mathrm{Hom}_R(M, S)$ and it is easy to check that $(\mathcal{M}^*)_{|S}$ is an $\mathcal{S}$-module scheme.

Proposition 2.6. [1, 1.8] *Let M, M' be R-modules. Then*

$$\mathbb{H}om_{\mathcal{R}}(\mathcal{M}^*, \mathcal{M}') = \mathcal{M} \otimes_{\mathcal{R}} \mathcal{M}'$$

As a corollary we obtain the following theorem.

Theorem 2.7. [1, 1.10] *Let M be an R-module. Then*

$$\mathcal{M}^{**} = \mathcal{M}$$

The functors $\mathcal{M} \rightsquigarrow \mathcal{M}^*$ and $\mathcal{M}^* \rightsquigarrow \mathcal{M}^{**} = \mathcal{M}$ establish an anti-equivalence between the categories of quasi-coherent modules and module schemes.

Let us recall the Formula of adjoint functors.

Definition 2.8. *Let $i^*\colon R \to S$ be a commutative R-algebra. Given a functor of $\mathcal{R}$-modules, $\mathbb{M}$, let $i^*\mathbb{M}$ be the functor of $\mathcal{S}$-modules defined by $(i^*\mathbb{M})(S') := \mathbb{M}(S')$. Given a functor of $\mathcal{S}$-modules, $\mathbb{N}$, let $i_*\mathbb{N}$ be the functor of $\mathcal{R}$-modules defined by $(i_*\mathbb{N})(R') := \mathbb{N}(S \otimes_R R')$.*

Formula of adjoint functors 2.9. [10, 2.11] *Let $\mathbb{M}$ be a functor of $\mathcal{R}$-modules and let $\mathbb{N}$ be a functor of $\mathcal{S}$-modules. Then, it holds that*

$$\mathrm{Hom}_{\mathcal{S}}(i^*\mathbb{M}, \mathbb{N}) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, i_*\mathbb{N})$$

[1] In this paper, we will only consider functors $\mathbb{M}$ and $\mathbb{M}'$ such that $\mathrm{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, {\mathbb{M}'}_{|S})$ are sets, for all S.

Corollary 2.10. [10, 2.12] *Let $\mathbb{M}$ be a functor of $\mathcal{R}$-modules. Then*

$$\mathbb{M}^*(S) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{S})$$

for all commutative R-algebras S.

Definition 2.11. *Let $\mathbb{M}$ be a functor of $\mathcal{R}$-modules. We will say that $\mathbb{M}^*$ is a dual functor. We will say that a functor of $\mathcal{R}$-modules $\mathbb{M}$ is reflexive if $\mathbb{M} = \mathbb{M}^{**}$.*

Examples 2.12. *Quasi-coherent modules and module schemes are reflexive functors of $\mathcal{R}$-modules.*

Proposition 2.13. [10, 2.16] *Let $\mathbb{A}$ be a functor of $\mathcal{R}$-algebras such that $\mathbb{A}^*$ is a reflexive functor of $\mathcal{R}$-modules. The closure of dual functors of $\mathcal{R}$-algebras of $\mathbb{A}$ is $\mathbb{A}^{**}$, that is, it holds the functorial equality*

$$\mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{A}, \mathbb{B}) = \mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{A}^{**}, \mathbb{B})$$

for every dual functor of $\mathcal{R}$-algebras $\mathbb{B}$.

Moreover, endowing a dual functor of $\mathcal{R}$-modules $\mathbb{M}^*$ with a structure of $\mathbb{A}$-module is equivalent to endowing $\mathbb{M}^*$ with a structure of $\mathbb{A}^{**}$-module.

Definition 2.14. [10, 5.2] *Let $\mathfrak{F}$ be the family of dual functors of $\mathcal{R}$-modules, $\mathbb{M}$, such that there exist a set J (which depends on $\mathbb{M}$), a structure of functor of $\prod_J \mathcal{R}$-modules on $\mathbb{M}$ and inclusions of functors of $\prod_J \mathcal{R}$-modules*

$$\oplus_J \mathcal{R} \subseteq \mathbb{M} \subseteq \prod_J \mathcal{R}$$

Proposition 2.15. [10, 5.3,5.8,5.9] *Every $\mathbb{M} \in \mathfrak{F}$ is a functor of $\mathcal{R}$-modules reflexive. If M is a free R-module then $\mathcal{M}, \mathcal{M}^* \in \mathfrak{F}$. If $\mathbb{M}, \mathbb{M}' \in \mathfrak{F}$, then $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') \in \mathfrak{F}$ and $(\mathbb{M} \otimes_{\mathcal{R}} \mathbb{M}')^{**} \in \mathfrak{F}$, which satisfies*

$$\mathrm{Hom}_{\mathcal{R}}((\mathbb{M} \otimes_{\mathcal{R}} \mathbb{M}')^{**}, \mathbb{M}'') = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M} \otimes_{\mathcal{R}} \mathbb{M}', \mathbb{M}'')$$

for every reflexive functor of $\mathcal{R}$-modules, $\mathbb{M}''$.

Proposition 2.16. [10, Section 1] *Let $\mathbb{A}, \mathbb{M}, \mathbb{M}'$ be reflexive functors of $\mathcal{R}$-modules. Assume that $\mathbb{A}, \mathbb{M}, \mathbb{M}' \in \mathfrak{F}$ or that $R = K$ is a field. If $\mathbb{A}$ is a functor of $\mathcal{R}$-algebras and $\mathbb{M}, \mathbb{M}'$ are functors of $\mathbb{A}$-modules, then a morphism of $\mathcal{R}$-modules $\mathbb{M} \to \mathbb{M}'$ is a morphism of $\mathbb{A}$-modules if and only if $\mathbb{M}(R) \to \mathbb{M}'(R)$ is a morphism of $\mathbb{A}(R)$-modules. Let $\mathcal{M}$ be an $\mathbb{A}$-module, then the set of quasi-coherent $\mathbb{A}$-submodules of $\mathcal{M}$ is equal to the set of $\mathbb{A}(R)$-submodules of M.*

Definition 2.17. *Let A be a R-algebra, we say that $\mathcal{A}$ is a quasi-coherent algebra. If $\mathbb{A}$ is the inverse limit of its quasi-coherent algebra quotients we say that $\mathbb{A}$ is a proquasi-coherent algebra.*

Proposition 2.18. [10, 3.18, 5.17] *Let $\mathbb{A}$ be a reflexive functor of $\mathcal{R}$-algebras. Assume that $\mathbb{A} \in \mathfrak{F}$ or that $R = K$ is a field. Every morphism of $\mathcal{R}$-algebras $\phi\colon \mathbb{A} \to \mathcal{B}$ uniquely factors through an epimorphism of functors of algebras onto the quasi-coherent algebra associated with $\mathrm{Im}\,\phi_R$. Then, if $\{\mathcal{A}_i\}_i$ is the set of the quasi-coherent algebra quotients of $\mathbb{A}$,*

$$\mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{A}, \mathbb{B}) = \lim_{\substack{\to \\ i}} \mathrm{Hom}_{\mathcal{R}-alg}(\mathcal{A}_i, \mathbb{B})$$

for every functor of proquasi-coherent algebras $\mathbb{B}$.

Note 2.19. *Let $\mathcal{C}^* \in \mathfrak{F}$ be a functor of $\mathcal{R}$-algebras. $\mathcal{C}^*$ is a proquasi-coherent algebra and the quasi-coherent algebra quotients of $\mathcal{C}^*$ are R-modules of finite type (see proof of Proposition* [10, 5.20]*).*

Notation 2.20. *Given two $\mathcal{R}$-modules, $\mathbb{M}$ and $\mathbb{M}'$, we denote $\mathbb{M}\tilde{\otimes}\mathbb{M}' := (\mathbb{M}^* \otimes \mathbb{M}'^*)^*$.*

Proposition 2.21. *Let $\mathbb{M}, \mathbb{M}' \in \mathfrak{F}$. By Proposition 2.15, $\mathbb{M}\tilde{\otimes}\mathbb{M}' = (\mathbb{M}^* \otimes \mathbb{M}'^*)^* = \mathbb{H}om_{\mathcal{R}}(\mathbb{M}^*, \mathbb{M}') \in \mathfrak{F}$. Given two modules M and N,*

$$\mathcal{M}\tilde{\otimes}\mathcal{N} = (\mathcal{M}^* \otimes \mathcal{N}^*)^* = \mathbb{H}om_{\mathcal{R}}(\mathcal{M}^*, \mathcal{N}) \overset{2.6}{=} \mathcal{M} \otimes \mathcal{N}$$

Finally, $\mathcal{M}^*\tilde{\otimes}\mathcal{N}^* = (\mathcal{M} \otimes \mathcal{N})^*$ is a module scheme and

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{M}^*\tilde{\otimes}\mathcal{N}^*, \mathbb{P}^*) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{P}, \mathcal{M} \otimes \mathcal{N}) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}^* \otimes \mathcal{N}^*, \mathbb{P}^*)$$

for all dual modules $\mathbb{P}^*$.

Proposition 2.22. [10, 5.22] *Let $\mathbb{A}, \mathbb{B} \in \mathfrak{F}$ be functors of proquasi-coherent algebras. Then, $\mathbb{A}\tilde{\otimes}\mathbb{B} := (\mathbb{A}^* \otimes \mathbb{B}^*)^* \in \mathfrak{F}$ is a proquasi-coherent algebra such that*

$$\mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{A} \otimes \mathbb{B}, \mathbb{C}) = \mathrm{Hom}_{\mathcal{R}-alg}((\mathbb{A}^* \otimes \mathbb{B}^*)^*, \mathbb{C})$$

for every functor of proquasi-coherent algebras $\mathbb{C}$.

Definition 2.23. *A functor of proquasi-coherent algebras $\mathbb{B} \in \mathfrak{F}$ is said to be a functor of bialgebras (resp. a functor of proquasi-coherent bialgebras) if $\mathbb{B}^*$ is a functor of $\mathcal{R}$-algebras (resp. a functor of proquasi-coherent $\mathcal{R}$-algebras) such that the dual morphisms of the multiplication morphism $m\colon \mathbb{B}^* \otimes \mathbb{B}^* \to \mathbb{B}^*$ and the unit morphism $u\colon \mathcal{R} \to \mathbb{B}^*$ are morphisms of functors of $\mathcal{R}$-algebras.*

Let $\mathbb{B}, \mathbb{B}'$ be two functors of bialgebras. We will say that a morphism of $\mathcal{R}$-modules, $f\colon \mathbb{B} \to \mathbb{B}'$ is a morphism of functors of bialgebras if f and $f^*\colon \mathbb{B}'^* \to \mathbb{B}^*$ are morphisms of functors of $\mathcal{R}$-algebras.

Theorem 2.24. [10, 5.27] *Let $\mathcal{C}_{\mathfrak{F}-Bialg.}$ be the category of functors $\mathbb{B} \in \mathfrak{F}$ of proquasi-coherent bialgebras. The functor $\mathcal{C}_{\mathfrak{F}-Bialg.} \rightsquigarrow \mathcal{C}_{\mathfrak{F}-Bialg.}$, $\mathbb{B} \rightsquigarrow \mathbb{B}^*$ is a categorical anti-equivalence.*

Notation 2.25. *Let $\mathbb{A}$ be a reflexive functor of $\mathcal{K}$-algebras and let $\{\mathcal{A}_i\}$ the set of quasi-coherent quotients of $\mathbb{A}$ such that $\dim_K A_i < \infty$. We denote $\bar{\mathbb{A}} := \lim\limits_{\underset{i}{\leftarrow}} \mathcal{A}_i$ which is an algebra scheme because $\mathcal{A}_i^*$ is quasi-coherent and $\lim\limits_{\underset{i}{\leftarrow}} \mathcal{A}_i = (\lim\limits_{\underset{i}{\rightarrow}} \mathcal{A}_i^*)^*$.*

Proposition 2.26. [1, 5.9] *Let $\mathbb{A}$ be a reflexive functor of $\mathcal{K}$-algebras. Then,*

$$\mathrm{Hom}_{\mathcal{K}-alg}(\mathbb{A}, \mathcal{C}^*) = \mathrm{Hom}_{\mathcal{K}-alg}(\bar{\mathbb{A}}, \mathcal{C}^*)$$

for all algebra schemes $\mathcal{C}^*$.

Theorem 2.27. [10, 5.30] *Let $\mathbb{B} \in \mathfrak{F}$ be a functor of proquasi-coherent $\mathcal{K}$-bialgebras. Then, $\bar{\mathbb{B}}$ is a scheme of bialgebras and*

$$\mathrm{Hom}_{bialg}(\mathbb{B}, \mathcal{C}^*) = \mathrm{Hom}_{bialg}(\bar{\mathbb{B}}, \mathcal{C}^*)$$

for all bialgebra schemes $\mathcal{C}^*$.

3. Affine functors

Let X be an R-scheme and let $X^{\cdot}$ be the functor of points of X; i.e., $X^{\cdot}$ is the functor of sets

$$X^{\cdot}(S) = \mathrm{Hom}_{R-\mathrm{sch}}(\mathrm{Spec}\, S, X)$$

For any other scheme Y, Yoneda's lemma proves that

$$\mathrm{Hom}_{R-sch}(X, Y) = \mathrm{Hom}(X^{\cdot}, Y^{\cdot}),$$

so $X^{\cdot} \simeq Y^{\cdot}$ if and only if $X \simeq Y$. We will sometimes denote $X^{\cdot} = X$. If $X = \mathrm{Spec}\, A$ is an affine scheme, then

$$(\mathrm{Spec}\, A)^{\cdot}(S) = \mathrm{Hom}_{R-\mathrm{sch}}(\mathrm{Spec}\, S, \mathrm{Spec}\, A) = \mathrm{Hom}_{R-alg}(A, S)$$

Definition 3.1. *Given a functor of commutative $\mathcal{R}$-algebras $\mathbb{A}$, the functor* $\mathrm{Spec}\, \mathbb{A}$*, "spectrum of $\mathbb{A}$", is defined to be*

$$(\mathrm{Spec}\, \mathbb{A})(S) := \mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{A}, \mathcal{S})$$

for every commutative R-algebra S.

Proposition 3.2. *Let $\mathbb{A}$ be a functor of commutative algebras. Then,*

$$\mathrm{Spec}\, \mathbb{A} = \mathbb{H}om_{\mathcal{R}-alg}(\mathbb{A}, \mathcal{R}).$$

Proof. By Adjoint Formula ([10, 2.11]), restricted to the morphisms of algebras, it holds that

$$\mathbb{H}om_{\mathcal{R}-alg}(\mathbb{A}, \mathcal{R})(S) = \mathrm{Hom}_{\mathcal{S}-alg}(\mathbb{A}_{|S}, \mathcal{S}) = \mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{A}, \mathcal{S}) = (\mathrm{Spec}\, \mathbb{A})(S).$$

□

Therefore, $\mathrm{Spec}\, \mathbb{A} = \mathbb{H}om_{\mathcal{R}-alg}(\mathbb{A}, \mathcal{R}) \subset \mathbb{H}om_{\mathcal{R}}(\mathbb{A}, \mathcal{R}) = \mathbb{A}^*$.

Notation 3.3. *Given a functor of sets $\mathbb{X}$, the functor $\mathbb{A}_{\mathbb{X}} := \mathbb{H}om(\mathbb{X}, \mathcal{R})$ is said to be the functor of functions of $\mathbb{X}$.*

Proposition 3.4. *Let $\mathbb{X}$ be a functor of sets and $\mathbb{A}_{\mathbb{X}}$ its functor of functions. Then,*

$$\mathrm{Hom}(\mathbb{X}, \mathrm{Spec}\, \mathbb{B}) = \mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{B}, \mathbb{A}_{\mathbb{X}}),$$

for every functor of commutative algebras, $\mathbb{B}$.

Proof. Given $f \colon \mathbb{X} \to \mathrm{Spec}\, \mathbb{B}$, let $f^* \colon \mathbb{B} \to \mathbb{A}_{\mathbb{X}}$ be defined by $f^*(b)(x) := f(x)(b)$, for every $x \in \mathbb{X}$. Given $\phi \colon \mathbb{B} \to \mathbb{A}_{\mathbb{X}}$, let $\phi^* \colon \mathbb{X} \to \mathrm{Spec}\, \mathbb{B}$ be defined by $\phi^*(x)(b) := \phi(b)(x)$, for all $b \in \mathbb{B}$. It is easy to check that $f = f^{**}$ and $\phi = \phi^{**}$.

□

Example 3.5. *If A is a commutative R-algebra, then* $\mathrm{Spec}\, \mathcal{A} = (\mathrm{Spec}\, A)^{\cdot}$ *and* $\mathbb{A}_{\mathrm{Spec}\, \mathcal{A}} = \mathbb{H}om(\mathrm{Spec}\, \mathcal{A}, \mathcal{R}) = \mathbb{H}om(\mathrm{Spec}\, \mathcal{A}, \mathrm{Spec}\, R[x]) = \mathbb{H}om_{\mathcal{R}-alg}(\mathcal{R}[x], \mathcal{A}) = \mathcal{A}$.

If $R = K$ is a field and X is a noetherian K-scheme, then the functor of functions of $X^{\cdot}$ is a quasi-coherent $\mathcal{R}$-module.

Definition 3.6. *We will say that a functor of sets $\mathbb{X}$ is affine when* $\mathbb{X} = \mathrm{Spec}\, \mathbb{A}_{\mathbb{X}}$ *and $\mathbb{A}_{\mathbb{X}}$ is reflexive.*

Let $\mathbb{X}$ and $\mathbb{Y}$ be affine functors. By Proposition 3.4,

$$\mathrm{Hom}(\mathbb{X}, \mathbb{Y}) = \mathrm{Hom}_{\mathcal{R}-alg}(\mathbb{A}_{\mathbb{Y}}, \mathbb{A}_{\mathbb{X}})$$

Example 3.7. *Affine schemes,* $\mathrm{Spec}\, \mathcal{A}$*, are affine functors, by Example 3.5.*

Proposition 3.8. *Let* $\mathbb{X} = \lim\limits_{\substack{\rightarrow \\ i}} \operatorname{Spec} \mathcal{A}_i$. *Then,* $\mathbb{A}_{\mathbb{X}} = \lim\limits_{\substack{\leftarrow \\ i}} \mathcal{A}_i$.

Proof. It holds that $\operatorname{Hom}((\operatorname{Spec} A)^{\cdot}, \mathbb{Y}) = \mathbb{Y}(A)$ for every functor of sets $\mathbb{Y}$, by Yoneda's lemma. Then,

$$\mathbb{A}_{\mathbb{X}} = \mathbb{H}om(\mathbb{X}, \mathcal{R}) = \mathbb{H}om(\lim_{\substack{\longrightarrow \\ i}} \operatorname{Spec} \mathcal{A}_i, \mathcal{R}) = \lim_{\substack{\longleftarrow \\ i}} \mathbb{H}om(\operatorname{Spec} \mathcal{A}_i, \mathcal{R}) = \lim_{\substack{\leftarrow \\ i}} \mathcal{A}_i$$

□

Theorem 3.9. *Let* $\mathbb{A}$ *be a reflexive functor of commutative algebras. Assume* $R = K$ *is a field or* $\mathbb{A} \in \mathfrak{F}$. *Then,* $\operatorname{Spec} \mathbb{A}$ *is a direct limit of closed immersions of affine schemes.*

Proof. Let $\{\mathcal{A}_i\}_i$ be the set of of commutative quasi-coherent algebra quotients of $\mathbb{A}$. $\operatorname{Spec} \mathbb{A} = \mathbb{H}om_{\mathcal{R}-alg}(\mathbb{A}, \mathcal{R}) \overset{2.18}{=} \lim\limits_{\substack{\rightarrow \\ i}} \mathbb{H}om_{\mathcal{R}-alg}(\mathcal{A}_i, \mathcal{R}) = \lim\limits_{\substack{\rightarrow \\ i}} \operatorname{Spec} \mathcal{A}_i$. □

Theorem 3.10. *Let* $\mathbb{A}$ *be a reflexive functor of commutative algebras. Assume* $R = K$ *is a field or* $\mathbb{A} \in \mathfrak{F}$. *If* $\mathbb{A}$ *is a proquasi-coherent algebra then* $\operatorname{Spec} \mathbb{A}$ *is an affine functor and* $\mathbb{A} = \mathbb{A}_{\operatorname{Spec} \mathbb{A}}$. *If* $\operatorname{Spec} \mathbb{A}$ *is an affine functor then* $\mathbb{A}_{\operatorname{Spec} \mathbb{A}}$ *is a proquasi-coherent algebra.*

Proof. Let $\{\mathcal{A}_i\}_i$ be the set of of commutative quasi-coherent algebra quotients of $\mathbb{A}$. Then, $\mathbb{A}_{\operatorname{Spec} \mathbb{A}} = \lim\limits_{\substack{\leftarrow \\ i}} \mathcal{A}_i$, by Theorem 3.9 and Proposition 3.8.

If $\mathbb{A}$ is a proquasi-coherent algebra, then $\mathbb{A} = \lim\limits_{\substack{\leftarrow \\ i}} \mathcal{A}_i = \mathbb{A}_{\operatorname{Spec} \mathbb{A}}$, and $\operatorname{Spec} \mathbb{A}$ is an affine functor.

Suppose that $\operatorname{Spec} \mathbb{A}$ is an affine functor. Let $f \colon \mathbb{A}_{\operatorname{Spec} \mathbb{A}} \to \mathcal{B}$ be a morphism of functors of algebras. The composition morphism $\mathbb{A} \to \mathbb{A}_{\operatorname{Spec} \mathbb{A}} \to \mathcal{B}$ factors through some $\mathcal{A}_i$. As $\operatorname{Spec} \mathbb{A} = \operatorname{Spec} \mathbb{A}_{\operatorname{Spec} \mathbb{A}}$, f factors through $\mathcal{A}_i$. In conclusion, the set of quasi-coherent algebra quotients of $\mathbb{A}_{\operatorname{Spec} \mathbb{A}}$ is $\{\mathcal{A}_i\}_{i \in I}$, and $\mathbb{A}_{\operatorname{Spec} \mathbb{A}}$ is a proquasi-coherent algebra.

□

Definition 3.11. *Let* $\mathbb{X}$ *be a functor of sets. Let* $\mathcal{R}\mathbb{X}$ *be the functor of* $\mathcal{R}$*-modules defined by*

$\mathcal{R}\mathbb{X}(S) := \oplus_{\mathbb{X}(S)} S = \{$*formal finite* S*-linear combinations of elements of* $\mathbb{X}(S)\}$

Clearly, $\mathbb{H}om(\mathbb{X}, \mathbb{M}) = \mathbb{H}om_{\mathcal{R}}(\mathcal{R}\mathbb{X}, \mathbb{M})$, for all functors of $\mathcal{R}$-modules, $\mathbb{M}$.

Observe that $\mathbb{A}_{\mathbb{X}} = \mathbb{H}om(\mathbb{X}, \mathcal{R}) = (\mathcal{R}\mathbb{X})^*$ is a dual functor.

Proposition 3.12. *Let* $\mathbb{X}$ *be a functor of sets. Let* $\mathbb{B}_{\mathbb{X}}$ *be a functor of* $\mathcal{R}$*-modules such that* $\mathbb{A}_{\mathbb{X}} = \mathbb{B}_{\mathbb{X}}^*$. *Then,*

$$\operatorname{Hom}(\mathbb{X}, \mathbb{M}^*) = \operatorname{Hom}_{\mathcal{R}}(\mathbb{B}_{\mathbb{X}}, \mathbb{M}^*)$$

for every dual functor of $\mathcal{R}$*-modules* $\mathbb{M}^*$.

Proof. It holds that

$$\begin{aligned} \operatorname{Hom}(\mathbb{X}, \mathbb{M}^*) &= \operatorname{Hom}_{\mathcal{R}}(\mathcal{R}\mathbb{X}, \mathbb{M}^*) = \operatorname{Hom}_{\mathcal{R}}(\mathcal{R}\mathbb{X} \otimes \mathbb{M}, \mathcal{R}) = \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{A}_{\mathbb{X}}) \\ &= \operatorname{Hom}_{\mathcal{R}}(\mathbb{B}_{\mathbb{X}}, \mathbb{M}^*) \end{aligned}$$

□

Proposition 3.13. *Let $\mathbb{X}_i$, $i = 1, \dots, n$ be functors of sets and $\mathbb{A}_{\mathbb{X}_i} = \mathbb{B}^*_{\mathbb{X}_i}$. Then,*

$$\mathbb{H}om(\mathbb{X}_1 \times \cdots \times \mathbb{X}_n, \mathbb{M}^*) = \mathbb{H}om_{\mathcal{R}}(\mathbb{B}_{\mathbb{X}_1} \otimes \cdots \otimes \mathbb{B}_{\mathbb{X}_n}, \mathbb{M}^*)$$

for all dual functors of $\mathcal{R}$-modules $\mathbb{M}^*$. In particular, if $\mathbb{X}_i$ are affine functors, then

$$\mathbb{A}_{\mathbb{X}_1 \times \cdots \times \mathbb{X}_n} = (\mathbb{A}^*_{\mathbb{X}_1} \otimes \cdots \otimes \mathbb{A}^*_{\mathbb{X}_n})^*$$

Proof. It is a consequence of the equalities

$$\begin{aligned} \mathbb{H}om(\mathbb{X}_1 \times \cdots \times \mathbb{X}_n, \mathbb{M}^*) &= \mathbb{H}om(\mathbb{X}_1, \mathbb{H}om(\mathbb{X}_2 \times \cdots \times \mathbb{X}_n, \mathbb{M}^*)) \\ &\overset{\text{Induction}}{=} \mathbb{H}om(\mathbb{X}, \mathbb{H}om_{\mathcal{R}}(\mathbb{B}_{\mathbb{X}_2} \otimes \cdots \otimes \mathbb{B}_{\mathbb{X}_n}, \mathbb{M}^*)) \\ &\overset{3.12}{=} \mathbb{H}om_{\mathcal{R}}(\mathbb{B}_{\mathbb{X}_1}, \mathbb{H}om_{\mathcal{R}}(\mathbb{B}_{\mathbb{X}_2} \otimes \cdots \otimes \mathbb{B}_{\mathbb{X}_n}, \mathbb{M}^*)) = \mathbb{H}om_{\mathcal{R}}(\mathbb{B}_{\mathbb{X}_1} \otimes \cdots \otimes \mathbb{B}_{\mathbb{X}_n}, \mathbb{M}^*) \end{aligned}$$

□

Proposition 3.14. *Let $\mathbb{X}, \mathbb{Y}$ be affine functors such that $\mathbb{A}_{\mathbb{X}}, \mathbb{A}_{\mathbb{Y}} \in \mathfrak{F}$, then $\mathbb{X} \times \mathbb{Y}$ is an affine functor and $\mathbb{A}_{\mathbb{X}\times\mathbb{Y}} \in \mathfrak{F}$.*

Proof. $\mathbb{A}_{\mathbb{X}\times\mathbb{Y}} = (\mathbb{A}^*_{\mathbb{X}} \otimes \mathbb{A}^*_{\mathbb{Y}})^* = \mathbb{A}_{\mathbb{X}} \tilde{\otimes} \mathbb{A}_{\mathbb{Y}} \in \mathfrak{F}$ and $\operatorname{Spec} \mathbb{A}_{\mathbb{X}\times\mathbb{Y}} = \operatorname{Spec}(\mathbb{A}_{\mathbb{X}} \otimes \mathbb{A}_{\mathbb{Y}}) = \mathbb{X} \times \mathbb{Y}$, by Theorem 2.22.

□

Proposition 3.15. *If $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$ is an affine functor and $\mathbb{X}(R) \neq \emptyset$ then $\mathbb{X}_1$ is an affine functor.*

Proof. Given $(x_1, x_2) \in \mathbb{X}_1(R) \times \mathbb{X}_2(R) = \mathbb{X}(R)$, let $i \colon \mathbb{X}_1 \hookrightarrow \mathbb{X}$, $i(x) = (x, x_2)$. Let $\pi_1 \colon \mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \to \mathbb{X}_1$, $\pi_1((y_1, y_2)) = y_1$. Let $i^* \colon \mathbb{A}_{\mathbb{X}} \to \mathbb{A}_{\mathbb{X}_1}$, $\pi_1^* \colon \mathbb{A}_{\mathbb{X}_1} \to \mathbb{A}_{\mathbb{X}}$ be the morphisms induced by i and π_1 respectively. Obviously, $\pi_1^* \circ i^* = Id$ because $i \circ \pi_1 = Id$. Hence, $\mathbb{A}_{\mathbb{X}_1}$ is a direct summand of $\mathbb{A}_{\mathbb{X}}$, and it is a reflexive functor because $\mathbb{A}_{\mathbb{X}}$ is reflexive.

Given, $f \in \mathbb{H}om_{\mathcal{R}-alg}(\mathbb{A}_{\mathbb{X}_1}, \mathcal{R})$ then $f \circ i^* = (y_1, y_2) \in \mathbb{H}om_{\mathcal{R}-alg}(\mathbb{A}_{\mathbb{X}}, \mathcal{R}) = \mathbb{X}$ and $f = f \circ i^* \circ \pi_1^* = (y_1, y_2) \circ \pi_1 = y_1$, that is, the morphism $\mathbb{X}_1 \to \operatorname{Spec} \mathbb{A}_{\mathbb{X}_1}$ is surjective. Finally, since the composition

$$\mathbb{X}_1 \to \operatorname{Spec} \mathbb{A}_{\mathbb{X}_1} \overset{i^{**}}{\to} \operatorname{Spec} \mathbb{A}_{\mathbb{X}} = \mathbb{X}$$

is equal to the morphism i, then $\mathbb{X}_1 = \operatorname{Spec} \mathbb{A}_{\mathbb{X}_1}$.

□

4. Formal schemes

Definition 4.1. *Let $\mathcal{C}^* \in \mathfrak{F}$ be a scheme of commutative algebras. We will say that* $\operatorname{Spec} \mathcal{C}^*$ *is a formal scheme. If* $\operatorname{Spec} \mathcal{C}^*$ *is a functor of monoids we will say that it is a formal monoid.*

Recall that if C is a free module $\mathcal{C}^* \in \mathfrak{F}$.

Note 4.2. *By Note 2.19, formal schemes are affine functors. In fact,* $\operatorname{Spec} \mathcal{C}^*$ *is a direct limit of finite R-schemes. Reciprocally, if R is a field, a direct limit of finite R-schemes are formal schemes, by Theorem 4.4. If R is a field, Demazure ([5]) defines a formal scheme as a functor (over the R-finite dimensional rings) which is a direct limit of finite R-schemes.*

The direct product $\operatorname{Spec} \mathcal{C}_1^* \times \operatorname{Spec} \mathcal{C}_2^* = \operatorname{Spec}(\mathcal{C}_1^* \tilde{\otimes} \mathcal{C}_2^*) = \operatorname{Spec}(\mathcal{C}_1 \otimes \mathcal{C}_2)^*$ of formal schemes is a formal scheme.

Example 4.3. *Let X be a set. Let us consider the discrete topology on X. Let $\mathbf{X}$ be "the constant functor X", defined by*

$$\mathbf{X}(S) := \mathrm{Homeo}(\mathrm{Spec}\, S, X)$$

for every commutative R-algebra S. If $\mathrm{Spec}\, S$ is connected then $\mathbf{X}(S) = X$.

Let $\mathbb{A}_X$ be the functor of algebras defined by

$$\mathbb{A}_X(S) := \prod_X S$$

for each commutative R-algebra S. Observe that $\mathbb{A}_X = \prod\limits_X \mathcal{R} = (\bigoplus\limits_X \mathcal{R})^*$ is a commutative algebra scheme. $\mathbf{X}$ is a formal scheme because $\mathrm{Spec}\, \mathbb{A}_X = \mathbf{X}$:

$$\begin{aligned}(\mathrm{Spec}\, \mathbb{A}_X)(S) &= \mathrm{Hom}_{\mathcal{R}-alg}(\prod_X \mathcal{R}, \mathcal{S}) \overset{2.6}{=} \lim_{\substack{\rightarrow \\ Y \subset X \\ |Y| < \infty}} \mathrm{Hom}_{\mathcal{R}-alg}(\prod_Y \mathcal{R}, \mathcal{S}) \\ &= \lim_{\substack{\rightarrow \\ Y \subset X \\ |Y| < \infty}} \mathrm{Hom}_{R-alg}(\prod_Y R, S) = \lim_{\substack{\rightarrow \\ Y \subset X \\ |Y| < \infty}} \mathrm{Homeo}(\mathrm{Spec}\, S, Y) \\ &= \mathrm{Homeo}(\mathrm{Spec}\, S, X) = \mathbf{X}(S)\end{aligned}$$

Obviously, $\mathrm{Spec}\, (\varinjlim_{i \in I} \mathbb{A}_i) = \varprojlim_{i \in I} (\mathrm{Spec}\, \mathbb{A}_i)$.

Theorem 4.4. *Let $\{\mathrm{Spec}\, \mathcal{C}_i^*\}_{i \in I}$ be a direct system of formal schemes. Then,*

$$\lim_{\substack{\rightarrow \\ i}} \mathrm{Spec}\, \mathcal{C}_i^* = \mathrm{Spec}(\lim_{\substack{\rightarrow \\ i}} \mathcal{C}_i)^*$$

and it is an affine functor.

Proof. Write $C = \lim\limits_{\substack{\rightarrow \\ i \in I}} C_i$, then $\mathcal{C}^* = \lim\limits_{\substack{\leftarrow \\ i \in I}} \mathcal{C}_i^*$. $\mathrm{Hom}_{\mathcal{R}}(\mathcal{C}^*, \mathcal{S}) = \mathrm{Hom}_{\mathcal{R}}((\lim\limits_{\substack{\rightarrow \\ i \in I}} \mathcal{C}_i)^*, \mathcal{S}) = (\lim\limits_{\substack{\rightarrow \\ i \in I}} C_i) \otimes S = \varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{R}}({\mathcal{C}_i}^*, \mathcal{S})$. Likely, $\mathrm{Hom}_{\mathcal{R}}(\mathcal{C}^* \otimes \mathcal{C}^*, \mathcal{S}) = \mathrm{Hom}_{\mathcal{R}}((\mathcal{C} \otimes \mathcal{C})^*, \mathcal{S}) = \varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{R}}((\mathcal{C}_i \otimes \mathcal{C}_i)^*, \mathcal{S}) = \varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{R}}({\mathcal{C}_i}^* \otimes {\mathcal{C}_i}^*, \mathcal{S})$. Then the kernel of the morphism $\mathrm{Hom}_{\mathcal{R}}(\mathcal{C}^*, \mathcal{S}) \to \mathrm{Hom}_{\mathcal{R}}(\mathcal{C}^* \otimes \mathcal{C}^*, \mathcal{S})$, $f \mapsto \tilde{f}$, where $\tilde{f}(c_1 \otimes c_2) = f(c_1 c_2) - f(c_1) f(c_2)$ coincides with the kernel of the morphism $\varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{R}}({\mathcal{C}_i}^*, \mathcal{S}) \to \varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{R}}({\mathcal{C}_i}^* \otimes {\mathcal{C}_i}^*, \mathcal{S})$, $(f_i) \mapsto (\tilde{f}_i)$. Then, $\mathrm{Hom}_{\mathcal{R}-alg}(\mathcal{C}^*, \mathcal{S}) = \varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{R}-alg}({\mathcal{C}_i}^*, \mathcal{S})$ and

$$(\mathrm{Spec}\, \mathcal{C}^*)(S) = (\varinjlim_{i \in I} \mathrm{Spec}\, {\mathcal{C}_i}^*)(S)$$

Finally, $\mathbb{A}_{\mathrm{Spec}\, \mathcal{C}^*} = \lim\limits_{\substack{\leftarrow \\ i}} \mathbb{A}_{\mathrm{Spec}\, {\mathcal{C}_i}^*} = \lim\limits_{\substack{\leftarrow \\ i}} \mathcal{C}_i^* = \mathcal{C}^*$.

□

From now on, in this section, we will assume that $R = K$ is a field.

Definition 4.5. *Let X be a K-scheme and I the set of finite subschemes of X. Given K-scheme Y write $A_Y := \mathcal{O}_Y(Y)$, the ring of functions of Y. Define $\bar{\mathcal{A}}_X :=$*

$\lim\limits_{\substack{\leftarrow \\ i\in I}} \mathcal{A}_i$ *and*

$$\bar{X} := \operatorname{Spec} \bar{\mathcal{A}}_X \overset{4.4}{=} \lim_{\substack{\rightarrow \\ i\in I}} \operatorname{Spec} \mathcal{A}_i$$

That is, "$\bar{X}$ is the direct limit of the set of all finite subschemes of X".

We have a natural morphism $\bar{X} \hookrightarrow X$.

Theorem 4.6. *Let X be a K-scheme. Then:*

$$\operatorname{Hom}(\operatorname{Spec} \mathcal{C}^*, X) = \operatorname{Hom}(\operatorname{Spec} \mathcal{C}^*, \bar{X})$$

for every formal scheme $\operatorname{Spec} \mathcal{C}^*$.

Proof. $\mathcal{C}^* = \lim\limits_{\substack{\leftarrow \\ i}} \mathcal{S}_i$, where the S_i are finite K-algebras. Then,

$$\begin{aligned}\operatorname{Hom}(\operatorname{Spec} \mathcal{C}^*, X) &= \operatorname{Hom}(\lim_{\substack{\rightarrow \\ i}} \operatorname{Spec} \mathcal{S}_i, X) = \lim_{\substack{\leftarrow \\ i}} \operatorname{Hom}(\operatorname{Spec} \mathcal{S}_i, X) \\ &= \lim_{\substack{\leftarrow \\ i}} \operatorname{Hom}(\operatorname{Spec} \mathcal{S}_i, \bar{X}) = \operatorname{Hom}(\lim_{\substack{\rightarrow \\ i}} \operatorname{Spec} \mathcal{S}_i, \bar{X}) = \operatorname{Hom}(\operatorname{Spec} \mathcal{C}^*, \bar{X})\end{aligned}$$

□

Let X and Y be two K-schemes. By the universal property 4.6, it can be checked that

$$\overline{X \times Y} = \bar{X} \times \bar{Y}$$

For every functor of $\mathcal{K}$-modules $\mathbb{M}$ there exists a natural morphism from the quasi-coherent module associated with $\mathbb{M}(K)$ into $\mathbb{M}$. If $\mathcal{C}^*$ is a commutative algebra scheme, then we have a natural morphism from the quasi-coherent algebra associated with C^* into $\mathcal{C}^*$ and therefore a natural morphism $\operatorname{Spec} \mathcal{C}^* \to \operatorname{Spec} C^*$. Moreover, this is injective, because $\mathcal{C}^* = \lim\limits_{\substack{\leftarrow \\ i}} \mathcal{C}_i$, (where $\mathcal{C}_i$ are the quotients of coherent algebras of $\mathcal{C}^*$) and:

$$\begin{aligned}(\operatorname{Spec} \mathcal{C}^*)(S) &= \operatorname{Hom}_{\mathcal{K}-alg}(\mathcal{C}^*, \mathcal{S}) = \lim_{\substack{\rightarrow \\ i}} \operatorname{Hom}_{\mathcal{K}-alg}(\mathcal{C}_i, \mathcal{S}) \\ &= \lim_{\substack{\rightarrow \\ i}} \operatorname{Hom}_{K-alg}(C_i, S) \subseteq \operatorname{Hom}_{K-alg}(C^*, S) = (\operatorname{Spec} C^*)(S)\end{aligned}$$

Let X be a compact and separated K-scheme and $\bar{A}_X = \bar{\mathcal{A}}_X(K) = \lim\limits_{\substack{\leftarrow \\ i}} A_i$. We can define a natural morphism $\operatorname{Spec} \bar{A}_X \to X$:

Let $X' := \{x \in X\colon$ the residual field of x is a finite extension of K, $\dim_K \mathcal{O}_{X,x}/\mathfrak{p}_x < \infty\}$. For any subset $J \subset X'$ let us denote $\bar{A}_J := \prod_{x\in J} \hat{\mathcal{O}}_{X,x}$, where

$$\hat{\mathcal{O}}_{X,x} := \lim_{\substack{\leftarrow \\ \dim_K \mathcal{O}_{X,x}/I<\infty}} \mathcal{O}_{X,x}/I$$

Write $\bar{J} := \operatorname{Spec} \bar{A}_J$. For any two disjoint subsets $J, J' \subseteq X'$, $\bar{A}_{J\coprod J'} = \bar{A}_J \times \bar{A}_{J'}$, and, in general, $\bar{A}_{J\cup J'} = \bar{A}_{J\cap J'} \times \bar{A}_{J-J\cap J'} \times \bar{A}_{J'-J\cap J'}$. $\bar{J}$ is an open and closed subset of $\overline{X'}$ and we have that $\overline{J \cup J'} = \bar{J} \cup \overline{J'}$ and $\overline{J \cap J'} = \bar{J} \cap \overline{J'}$.

For any affine open subset $U = \operatorname{Spec} A \subset X$ we have a natural morphism $A \to \prod_{x\in U'} \hat{\mathcal{O}}_{X,x} = \bar{A}_U$, and therefore a natural morphism $f_U\colon \overline{U'} \to U \subset X$. For any other affine open subset $V \subset X$ we have another morphism $f_V : \overline{V'} \to V \subset X$. As $U \cap V$ is affine and f_U is equal to f_V on $\overline{(U \cap V)'} = \overline{U'} \cap \overline{V'}$, we have a well-defined natural morphism $\operatorname{Spec} \bar{A}_X = \overline{X'} \to X$.

The natural morphism $\bar{X} \to X$ is equal to the composition of the natural morphisms: $\bar{X} = \operatorname{Spec} \bar{\mathcal{A}}_X \to \operatorname{Spec} \bar{A}_X \to X$, because for any finite subscheme X_i of X the composition morphism $X_i \to \operatorname{Spec} \bar{A}_X \to X$ is equal to the natural inclusion $X_i \hookrightarrow X$.

Theorem 4.7. *Every morphism $\operatorname{Spec} \mathcal{C}^* \to X$, from a formal scheme to a compact and separated K-scheme uniquely factors via $\operatorname{Spec} C^*$, that is,*

$$\operatorname{Hom}(\operatorname{Spec} \mathcal{C}^*, X) = \operatorname{Hom}_{K-sch}(\operatorname{Spec} C^*, X).$$

Proof. If $X = \operatorname{Spec} A$ is an affine scheme then

$$\begin{aligned} \operatorname{Hom}(\operatorname{Spec} \mathcal{C}^*, \operatorname{Spec} A) &= \operatorname{Hom}_{\mathcal{R}-alg}(\mathcal{A}, \mathcal{C}^*) = \operatorname{Hom}_{\mathcal{R}-alg}(A, C^*) \\ &= \operatorname{Hom}_{K-sch}(\operatorname{Spec} C^*, \operatorname{Spec} A) \end{aligned}$$

By Theorem 4.6, every morphism from $\operatorname{Spec} \mathcal{C}^*$ into X uniquely factors through $\bar{X}$. The morphism $\bar{X} \to X$ factors through $\operatorname{Spec} \bar{A}_X$ and the morphism $\operatorname{Spec} \mathcal{C}^* \to \operatorname{Spec} \bar{A}_X$ uniquely factors through $\operatorname{Spec} C^*$. So that we have the commutative diagram:

$$\begin{array}{ccccc} \operatorname{Spec} \mathcal{C}^* & \longrightarrow & \operatorname{Spec} C^* & & \\ \downarrow & & \downarrow & & \\ \bar{X} & \longrightarrow & \operatorname{Spec} \bar{A}_X & \longrightarrow & X \end{array}$$

Then, the natural morphism $\operatorname{Hom}(\operatorname{Spec} C^*, X) \to \operatorname{Hom}(\operatorname{Spec} \mathcal{C}^*, X)$ is surjective.

Let us write $\mathcal{C}^* = \lim\limits_{\substack{\leftarrow \\ i}} \mathcal{C}_i$, with $\dim_K C_i < \infty$ and $\mathcal{C}^* \to \mathcal{C}_i$ surjective. The smallest closed subscheme of $\operatorname{Spec} C^*$ containing every $\operatorname{Spec} C_i$ is $\operatorname{Spec} C^*$. Therefore,

$$\begin{aligned} \operatorname{Hom}_{K-sch}(\operatorname{Spec} C^*, X) &\subseteq \lim_{\substack{\leftarrow \\ i}} \operatorname{Hom}_{K-sch}(\operatorname{Spec} C_i, X) = \operatorname{Hom}(\lim_{\substack{\rightarrow \\ i}} \operatorname{Spec} \mathcal{C}_i, X) \\ &= \operatorname{Hom}(\operatorname{Spec} \mathcal{C}^*, X) \end{aligned}$$

□

Note 4.8. *Let X and Y be compact and separated K-schemes. Every commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{B}^* & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{C}^* & \longrightarrow & Y \end{array}$$

uniquely extends to a commutative diagram

$$\begin{array}{ccccc} \operatorname{Spec} \mathcal{B}^* & \longrightarrow & \operatorname{Spec} B^* & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{C}^* & \longrightarrow & \operatorname{Spec} C^* & \longrightarrow & Y \end{array}$$

because the composition $\operatorname{Spec} \mathcal{B}^* \to \operatorname{Spec} \mathcal{C}^* \to \operatorname{Spec} C^*$ factors through $\operatorname{Spec} B^*$.

5. Completion along a closed subscheme

Proposition 5.1. *Let $M_{n+1} \to M_n$ be epimorphisms of R-modules, for all $n \in \mathbb{N}$ and let N be an R-module. It holds that*

$$\mathrm{Hom}_{\mathcal{R}}(\lim_{\substack{\leftarrow \\ n\in\mathbb{N}}} \mathcal{M}_n, \mathcal{N}) = \lim_{\substack{\rightarrow \\ n\in\mathbb{N}}} \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}_n, \mathcal{N})$$

Hence $(\lim\limits_{\substack{\leftarrow \\ n\in\mathbb{N}}} \mathcal{M}_n)^* = \lim\limits_{\substack{\rightarrow \\ n\in\mathbb{N}}} \mathcal{M}_n^*$ *and* $\lim\limits_{\substack{\leftarrow \\ n\in\mathbb{N}}} \mathcal{M}_n$ *is a reflexive* $\mathcal{R}$*-module.*

Proof. Let $f \in \mathrm{Hom}_{\mathcal{R}}(\lim\limits_{\substack{\leftarrow \\ n\in\mathbb{N}}} \mathcal{M}_n, \mathcal{N})$. Firstly, let us prove that the morphism $f_R \colon \lim\limits_{\substack{\leftarrow \\ n\in\mathbb{N}}} M_n \to N$ induced by f factors through M_r, for some $r \in \mathbb{N}$: suppose that, for each r, there exists an element $s_r = (m_n) \in \lim\limits_{\substack{\leftarrow \\ n\in\mathbb{N}}} M_n \subset \prod_n M_n$, such that $m_r = 0$ and $f_R(s_r) \neq 0$. The morphism $g \colon \prod_{n\in\mathbb{N}} \mathcal{R} \to \mathcal{N}$, $g((a_n)_n) := f(\sum_n a_n \cdot s_n)$ satisfies that $g_{|\mathcal{R}} \neq 0$ for every factor $\mathcal{R} \subset \prod_n \mathcal{R}$ and this contradicts the fact that $\mathrm{Hom}_{\mathcal{R}}(\prod_n \mathcal{R}, \mathcal{N}) \overset{2.6}{=} (\oplus_n R) \otimes N = \oplus_n \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}, \mathcal{N})$. Then, f_R factors through a (unique) morphism $h_R \colon M_r \to N$, for some r.

Next, given a commutative R-algebra S, let us check that the morphism

$$f_S \colon \lim_{\substack{\leftarrow \\ n\in\mathbb{N}}} (M_n \otimes_R S) \to N \otimes_R S$$

induced by f factors through $h_S \colon M_r \otimes_R S \to N \otimes S$, $h_S(m_r \otimes s) := h_R(m_r) \otimes s$: there exists $r' \geq r$ such that f_S factors through a morphism $h' \colon M_{r'} \otimes_R S \to N \otimes S$. Given $m_{r'} \in M_{r'}$, let $(m'_n) \in \lim\limits_{\substack{\leftarrow \\ n\in\mathbb{N}}} \mathcal{M}_n$ such that $m_{r'} = m'_{r'}$. Then, $h'(m_{r'} \otimes 1) = f_S((m'_n \otimes 1)) = f_R((m'_n)) \otimes 1 = h_R(m'_r) \otimes 1 = h_S(m'_r \otimes 1)$ and h' factors through h_S. Hence f_S factors through h_S.

□

Definition 5.2. *Let A be a commutative R-algebra and $I \subset A$ an ideal. Let* $\hat{\mathcal{A}} := \lim\limits_{\substack{\leftarrow \\ n\in\mathbb{N}}} \mathcal{A}/\mathcal{I}^n$. *We will say that* $\mathrm{Spec}\,\hat{\mathcal{A}}$ *is the completion of* $\mathrm{Spec}\, A$ *along the closed set* $\mathrm{Spec}\, A/I$.

$\hat{\mathcal{A}}$ is a reflexive functors of R-modules by Proposition 5.1. $\hat{\mathcal{A}}$ is proquasi-coherent algebra: $\hat{\mathcal{A}}^* = \lim\limits_{\substack{\rightarrow \\ n}} (\mathcal{A}/\mathcal{I}^n)^*$, then every morphism of functors of algebras $\hat{\mathcal{A}} \to \mathcal{B}$ factors through some $\mathcal{A}/\mathcal{I}^n$. Hence the inverse limit of the proquasi-coherent algebra quotients of $\hat{\mathcal{A}}$ is equal to $\lim\limits_{\substack{\leftarrow \\ n\in\mathbb{N}}} \mathcal{A}/\mathcal{I}^n = \hat{\mathcal{A}}$.

Proposition 5.3. *The completion of* $\mathrm{Spec}\, A$ *along the closed set* $\mathrm{Spec}\, A/I$ *is an affine functor.*

Proof. $\mathrm{Hom}_{\mathcal{R}-alg}(\hat{\mathcal{A}}, \mathcal{C}) = \lim\limits_{\substack{\rightarrow \\ n\in\mathbb{N}}} \mathrm{Hom}_{\mathcal{R}-alg}(\mathcal{A}/\mathcal{I}^n, \mathcal{C})$, then $\mathrm{Spec}\,\hat{\mathcal{A}} = \lim\limits_{\substack{\rightarrow \\ n\in\mathbb{N}}} \mathrm{Spec}\,\mathcal{A}/\mathcal{I}^n$.
By 3.8, $\mathbb{A}_{\mathrm{Spec}\,\hat{\mathcal{A}}} = \lim\limits_{\substack{\leftarrow \\ n}} \mathbb{A}_{\mathrm{Spec}\,\mathcal{A}/\mathcal{I}^n} = \lim\limits_{\substack{\leftarrow \\ n}} \mathcal{A}/\mathcal{I}^n = \hat{\mathcal{A}}$. □

Let B be a commutative R-algebra, $J \subset B$ be an ideal and $\hat{\mathcal{B}} = \lim\limits_{\substack{\leftarrow \\ n\in\mathbb{N}}} \mathcal{B}/\mathcal{J}^n$. Consider the ideal $I \otimes B + A \otimes J \subseteq A \otimes B$. Then,

$$\begin{aligned}\operatorname{Spec}\widehat{\mathcal{A}\otimes\mathcal{B}} &= \mathbb{H}om_{\mathcal{R}-alg}(\widehat{\mathcal{A}\otimes\mathcal{B}}, \mathcal{K}) = \lim_{\substack{\rightarrow \\ n}} \mathbb{H}om_{\mathcal{R}-alg}(\mathcal{A}\otimes\mathcal{B}/(\mathcal{I}\otimes\mathcal{B}+\mathcal{A}\otimes\mathcal{J})^n, \mathcal{K}) \\ &= \lim_{\substack{\rightarrow \\ m}} \mathbb{H}om_{\mathcal{R}-alg}(\mathcal{A}/\mathcal{I}^m \otimes \mathcal{B}/\mathcal{J}^m, \mathcal{K}) \\ &= \lim_{\substack{\rightarrow \\ m}} (\mathbb{H}om_{\mathcal{R}-alg}(\mathcal{A}/\mathcal{I}^m, \mathcal{K}) \times \mathbb{H}om_{\mathcal{R}-alg}(\mathcal{B}/\mathcal{J}^m, \mathcal{K})) = \operatorname{Spec}\hat{\mathcal{A}} \times \operatorname{Spec}\hat{\mathcal{B}}\end{aligned}$$

and

$$\hat{\mathcal{A}}\tilde{\otimes}\hat{\mathcal{B}} = (\hat{\mathcal{A}}^*\otimes\hat{\mathcal{B}}^*)^* = (\lim_{\substack{\rightarrow \\ n}}((\mathcal{A}/\mathcal{I}^n)^*\otimes(\mathcal{B}/\mathcal{J}^n)^*))^* \overset{2.21}{=} \lim_{\substack{\leftarrow \\ n}}(\mathcal{A}/\mathcal{I}^n\otimes\mathcal{B}/\mathcal{J}^n) = \widehat{\mathcal{A}\otimes\mathcal{B}}$$

6. Affine functors of monoids

Let $\mathbb{G}$ be a functor of monoids. $\mathcal{R}\mathbb{G}$ is obviously a functor of $\mathcal{R}$-algebras. Given a functor of $\mathcal{R}$-algebras $\mathbb{B}$, it is easy to check the equality

$$\operatorname{Hom}_{mon}(\mathbb{G}, \mathbb{B}) = \operatorname{Hom}_{\mathcal{R}-alg}(\mathcal{R}\mathbb{G}, \mathbb{B}).$$

The closure of dual functors of $\mathcal{R}$-algebras of $\mathbb{G}$ is equal to the closure of dual functors of $\mathcal{R}$-algebras of $\mathcal{R}\mathbb{G}$.

Theorem 6.1. *Let $\mathbb{G}$ be a functor of monoids with a reflexive functor of functions. Then, the closure of dual functors of algebras of $\mathbb{G}$ is $\mathbb{A}_{\mathbb{G}}^*$. That is,*

$$\operatorname{Hom}_{mon}(\mathbb{G}, \mathbb{B}) = \operatorname{Hom}_{\mathcal{R}-alg}(\mathcal{R}\mathbb{G}, \mathbb{B}) = \operatorname{Hom}_{\mathcal{R}-alg}(\mathbb{A}_{\mathbb{G}}^*, \mathbb{B})$$

for every dual functor of $\mathcal{R}$-algebras $\mathbb{B}$.

Proof. $(\mathcal{R}\mathbb{G})^* = \mathbb{A}_{\mathbb{G}}$ is reflexive, so the closure of dual functors of algebras of $\mathbb{G}$ is $\mathbb{A}_{\mathbb{G}}^*$, by Proposition 2.13. □

Theorem 6.2. *Let $\mathbb{G}$ be a functor of monoids with a reflexive functor of functions. The category of quasi-coherent $\mathbb{G}$-modules is equivalent to the category of quasi-coherent $\mathbb{A}_{\mathbb{G}}^*$-modules.*

Likewise, the category of dual functors of $\mathbb{G}$-modules is equivalent to the category of dual functors of $\mathbb{A}_{\mathbb{G}}^*$-modules.

Proof. It a consequence of Proposition 2.13. □

Remark that the structure of functor of algebras of $\mathbb{A}_{\mathbb{G}}^*$ is the only one that makes the morphism $\mathbb{G} \to \mathbb{A}_{\mathbb{G}}^*$ a morphism of functors of monoids.

Theorem 6.3. *Let $\mathbb{G}$ be a functor of monoids with a reflexive functor of functions and let $\mathbb{M}, \mathbb{M}'$ be reflexive functors of $\mathbb{G}$-modules. Assume that $\mathbb{A}_{\mathbb{G}}, \mathbb{M}, \mathbb{M}' \in \mathfrak{F}$ or that $R = K$ is a field. Then, a morphism of $\mathcal{R}$-modules $\mathbb{M} \to \mathbb{M}'$ is a morphism of $\mathbb{G}$-modules if and only if $\mathbb{M}(R) \to \mathbb{M}'(R)$ is a morphism of $\mathbb{A}_{\mathbb{G}}^*(R)$-modules. Let $\mathcal{M}$ be a $\mathbb{G}$-module, then the set of quasi-coherent $\mathbb{G}$-submodules of $\mathcal{M}$ is equal to the set of $\mathbb{A}_{\mathbb{G}}^*(R)$-submodules of M.*

Proof. It a consequence of Theorem 6.2 and Proposition 2.16. □

Example 6.4. *The $\mathbb{C}$-linear representations of $(\mathbb{Z},+)$ are equivalent to the $\mathbb{A}^*_{\mathbb{Z}}$-modules. $\mathbb{A}^*_{\mathbb{Z}}$ is the quasi-coherent algebra associated with $\mathbb{C}[x,1/x]$. The category of $\mathbb{A}^*_{\mathbb{Z}}$-modules is equal to the category of $\mathbb{C}[x,1/x]$-modules. The natural morphism $\mathbb{Z}\to\mathbb{A}^*_{\mathbb{Z}}$ assigns n to x^n. Thus, if V is a finite $\mathbb{C}$-linear representation of $\mathbb{Z}$, then*

$$V=\underset{\alpha,n,m}{\oplus}(\mathbb{C}[x]/(x-\alpha)^n)^m,\qquad (\alpha\neq 0)$$

such that $r\cdot(\overline{p_{\alpha,n,m}(x)})_{\alpha,n,m}=(\overline{x^r\cdot p_{\alpha,n,m}(x)})_{\alpha,n,m}$.

Let $\mathbb{G}$ be a functor of monoids with a reflexive functor of functions. Let $m\colon\mathbb{G}\times\mathbb{G}\to\mathbb{G}$ be the multiplication morphism. Then, the composition morphism of m with the natural morphism $\mathbb{G}\to\mathbb{A}^*_{\mathbb{G}}$ factors through $\mathbb{A}^*_{\mathbb{G}}\otimes\mathbb{A}^*_{\mathbb{G}}$, by 3.12 and 3.13, that is, we have a commutative diagram

$$\begin{array}{ccc}\mathbb{G}\times\mathbb{G} & \xrightarrow{m} & \mathbb{G}\\ \downarrow & & \downarrow\\ \mathbb{A}^*_{\mathbb{G}}\otimes\mathbb{A}^*_{\mathbb{G}} & \xrightarrow{m} & \mathbb{A}^*_{\mathbb{G}}\end{array}$$

Let $e\in\mathbb{G}(R)\subset\mathbb{A}^*_{\mathbb{G}}(R)$ the unit of $\mathbb{G}$. Then, we can define a morphism $e\colon\mathcal{R}\to\mathbb{A}^*_{\mathbb{G}}$. It is easy to check that $\{\mathbb{A}^*_{\mathbb{G}},m,e\}$ is a functor of $\mathcal{R}$-algebras. Moreover, the dual morphisms of the multiplication morphism m and the unit morphism e are the natural morphisms $\mathbb{A}_{\mathbb{G}}\to\mathbb{A}_{\mathbb{G}\times\mathbb{G}}$ and $\mathbb{A}_{\mathbb{G}}\overset{e}{\to}\mathcal{R}$, which are morphisms of $\mathcal{R}$-algebras.

Conversely, let $\mathbb{X}$ be an affine functor and assume that $\mathbb{A}^*_{\mathbb{X}}$ is a functor of $\mathcal{R}$-algebras, such that the dual morphisms m^* and e^*, of the multiplication morphism $m\colon\mathbb{A}^*_{\mathbb{X}}\otimes\mathbb{A}^*_{\mathbb{X}}\to\mathbb{A}^*_{\mathbb{X}}$ and the unit morphism $e\colon\mathcal{R}\to\mathbb{A}^*_{\mathbb{X}}$ are morphisms of $\mathcal{R}$-algebras. Given a point $(x,x')\in\mathbb{X}\times\mathbb{X}\subset\mathbb{H}om_{\mathcal{R}-alg}(\mathbb{A}_{\mathbb{X}\times\mathbb{X}},\mathcal{R})$ then $(x,x')\circ m^*\in\mathbb{H}om_{\mathcal{R}-alg}(\mathbb{A}_{\mathbb{X}},\mathcal{R})=\mathbb{X}$ and we have the commutative diagram

$$\begin{array}{ccc}\mathbb{X}\times\mathbb{X} & \overset{m}{\dashrightarrow} & \mathbb{X}\\ \downarrow & & \downarrow\\ \mathbb{A}^*_{\mathbb{X}}\otimes\mathbb{A}^*_{\mathbb{X}} & \xrightarrow{m} & \mathbb{A}^*_{\mathbb{X}}\end{array}$$

Obviously $e\in\mathbb{H}om_{\mathcal{R}-alg}(\mathbb{A}_{\mathbb{X}},\mathcal{R})=\mathbb{X}$. It is easy to check that $\{\mathbb{X},m,e\}$ is a functor of monoids.

Definition 6.5. *An affine functor $\mathbb{G}=\operatorname{Spec}\mathbb{A}$ is said to be an affine functor of monoids if $\mathbb{G}$ is a functor of monoids.*

Example 6.6. *Affine R-monoid schemes, formal monoids, the completion of an affine monoid scheme along a closed submonoid scheme, $\mathcal{V}$, $\mathbb{E}nd_{\mathcal{R}}\mathcal{V}$ (V being a free R-module) are examples of affine functors of monoids, by 3.7, 5.3 and 9.26.*

Let $\mathbb{G}$ and $\mathbb{G}'$ be affine functors of monoids. Then,

$$\operatorname{Hom}_{mon}(\mathbb{G},\mathbb{G}')=\{f\in\operatorname{Hom}_{\mathcal{R}}(\mathbb{A}_{\mathbb{G}'},\mathbb{A}_{\mathbb{G}})\colon f,\ f^*\text{ are morph. of funct. of }\mathcal{R}\text{-alg.}\}:$$

Let $h\colon\mathbb{G}\to\mathbb{G}'$ be a morphism of functors of monoids. The composition morphism of h with the natural morphism $\mathbb{G}'\to\mathbb{A}^*_{\mathbb{G}'}$ factors through $\mathbb{A}^*_{\mathbb{G}}$, that is, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{h} & \mathbb{G}' \\ \downarrow & & \downarrow \\ \mathbb{A}_{\mathbb{G}}^* & \longrightarrow & \mathbb{A}_{\mathbb{G}'}^* \end{array}$$

The dual morphism $\mathbb{A}_{\mathbb{G}'} \to \mathbb{A}_{\mathbb{G}}$ is the morphism induced by h between the functors of functions. Inversely, let $f\colon \mathbb{A}_{\mathbb{G}'} \to \mathbb{A}_{\mathbb{G}}$ be a morphism of functors of $\mathcal{R}$-algebras, such that f^* is also a morphism of functors of $\mathcal{R}$-algebras. Given $g \in \mathbb{G}$, then $f^*(g) = g \circ f \in \mathbb{H}om_{\mathcal{R}-alg}(\mathbb{A}_{\mathbb{G}'}, \mathcal{R}) = \mathbb{G}'$. Hence, $f^*_{|\mathbb{G}}\colon \mathbb{G} \to \mathbb{G}'$ is a morphism of functors of monoids.

Theorem 6.7. *Let $\mathbb{G} = \operatorname{Spec} \mathbb{A}_{\mathbb{G}}$ be an affine functor. Giving to $\mathbb{G}$ a structure of functor of monoids is equivalent to giving to $\mathbb{A}_{\mathbb{G}}$ a structure of functor of bialgebras. Let $\mathbb{G}$ and $\mathbb{G}'$ be two affine functors of monoids, it holds that*

$$\operatorname{Hom}_{mon}(\mathbb{G}, \mathbb{G}') = \operatorname{Hom}_{bialg}(\mathbb{A}_{\mathbb{G}'}, \mathbb{A}_{\mathbb{G}})$$

Theorem 6.8. *The category of cocommutative bialgebras A is equivalent to the category of formal monoids $\operatorname{Spec} \mathcal{A}^*$ (we assume the R-modules A are projective).*

Theorem 6.9. *Let G be a K-scheme on groups (resp. monoids). Then $\bar{G}$ is a functor of groups (resp. monoids), the natural morphism $\bar{G} \to G$ is a morphism of functors of monoids and*

$$\operatorname{Hom}_{mon}(\operatorname{Spec} \mathcal{C}^*, G) = \operatorname{Hom}_{mon}(\operatorname{Spec} \mathcal{C}^*, \bar{G})$$

for every formal monoid $\operatorname{Spec} \mathcal{C}^*$. If G is commutative, then $\bar{G}$ is commutative.

Proof. Let $\mu\colon G \times G \to G$ the multiplication morphism. By Theorem 4.6, the composition morphism $\overline{G \times G} = \bar{G} \times \bar{G} \to G \times G \to G$ factors through a unique morphism $\mu'\colon \bar{G} \times \bar{G} \to \bar{G}$, that is, we have the commutative diagram:

$$\begin{array}{ccc} \bar{G} \times \bar{G} & \longrightarrow & G \times G \\ \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu} \\ \bar{G} & \longrightarrow & G \end{array}$$

Let $*\colon G \to G$ be the inverse morphism. The composition $\bar{G} \to G \overset{*}{\to} G$ factors through a unique morphism $*'\colon \bar{G} \to \bar{G}$, that is, we have the commutative diagram:

$$\begin{array}{ccc} \bar{G} & \longrightarrow & G \\ \downarrow{\scriptstyle *'} & & \downarrow{\scriptstyle *} \\ \bar{G} & \longrightarrow & G \end{array}$$

Now it is easy to check that $(\bar{G}, \mu', *')$ is a functor of groups and to conclude the proof. □

Proposition 6.10. *Let $\operatorname{Spec} \mathcal{C}^*$ be a formal monoid (resp. group) and let G be a compact K-scheme on monoids. Let $f\colon \operatorname{Spec} \mathcal{C}^* \to G$ be a morphism of functors of monoids and let $f'\colon \operatorname{Spec} C^* \to G$ be the induced morphism.*

Then the (closed) scheme-theoretic image of f', $\operatorname{Im} f'$, is a K-subscheme on monoids (resp. groups) of G. If $\operatorname{Spec} \mathcal{C}^*$ is an abelian formal monoid, then $\operatorname{Im} f'$ is abelian.

Proof. Let $\mu'\colon \operatorname{Spec}\mathcal{C}^* \times \operatorname{Spec}\mathcal{C}^* \to \operatorname{Spec}\mathcal{C}^*$ and $\mu\colon G\times G \to G$ be the operations of the monoids. Consider the commutative diagram:

$$\begin{array}{ccccccc}
\operatorname{Spec}\mathcal{C}^* \times \operatorname{Spec}\mathcal{C}^* & \longrightarrow & \operatorname{Spec}(C\otimes C)^* & \longrightarrow & \operatorname{Spec} C^* \times \operatorname{Spec} C^* & \xrightarrow{f'\times f'} & G\times G \\
\Big\downarrow{\scriptstyle \mu'} & & \Big\downarrow & & & & \Big\downarrow{\scriptstyle \mu} \\
\operatorname{Spec}\mathcal{C}^* & \longrightarrow & \operatorname{Spec} C^* & & \xrightarrow{\qquad f' \qquad} & & G
\end{array}$$

The scheme-theoretic image of $\operatorname{Spec}(C\otimes C)^*$ in $G\times G$ is equal to $\operatorname{Im}(f'\times f')$. As $\operatorname{Im}(f'\times f') = \operatorname{Im} f' \times \operatorname{Im} f'$, we have the commutative diagram:

$$\begin{array}{ccc}
\operatorname{Im} f' \times \operatorname{Im} f' & \hookrightarrow & G\times G \\
\Big\downarrow & & \Big\downarrow \\
\operatorname{Im} f' & \hookrightarrow & G
\end{array}$$

If we denote with $*$ the inverse morphism (with respect to the group law), then we have the commutative square:

$$\begin{array}{ccccc}
\operatorname{Spec}\mathcal{C}^* & \longrightarrow & \operatorname{Spec} C^* & \xrightarrow{f'} & G \\
\Big\downarrow{\scriptstyle *} & & \Big\downarrow & & \Big\downarrow{\scriptstyle *} \\
\operatorname{Spec}\mathcal{C}^* & \longrightarrow & \operatorname{Spec} C^* & \xrightarrow{f'} & G
\end{array}$$

So the inverse of G restricts to $\operatorname{Im} f'$. The rest of the details can be checked in a similar way.

□

Note 6.11. *Let X be a K-scheme and $x\colon \operatorname{Spec} K \to X$ a rational point. Denote by $\hat{X}$ the direct limit of finite subschemes $X_i \subset X$ with support on x. Let $\hat{\mathcal{A}}_X := \lim\limits_{\leftarrow} \mathcal{A}_{X_i}$. as in 4.5, $\hat{X} = \operatorname{Spec}\hat{\mathcal{A}}_X$. If G is a K-scheme on groups, consider as a rational point the identity element.*

Theorems 4.6 and 6.9 remain valid if $\bar{X}$ and $\bar{G}$ are substituted by $\hat{X}$ and $\hat{G}$. If, in addition, C^* is assumed to be local, then the hypothesis of X being compact and separated is no longer necessary in 4.7 and 6.10.

Finally, given an algebraic abelian group G, over an algebraically closed field K, it easy to prove that $\bar{G} = \hat{G}\times G(K)$, where $G(K)$ is the constant functor $G(K)$, that is, consider on $G(K)$ the discrete topology, then $G(K)(S) := \operatorname{Homeo}(\operatorname{Spec} S, G(K))$, for all commutative K-algebra.

Proposition 6.12. *Let $\operatorname{Spec}\mathcal{C}^*$ be a formal monoid and $D_{\mathcal{C}} = \{w \in C^*\colon w(I) = 0$ for some bilateral ideal $I\subset C$ of finite codimension$\} \subset C^*$. Then, $\operatorname{Spec} D_{\mathcal{C}}$ is an affine monoid scheme and*

$$\operatorname{Hom}_{mon}(\operatorname{Spec}\mathcal{C}^*, \operatorname{Spec} A) = \operatorname{Hom}_{mon}(\operatorname{Spec} D_{\mathcal{C}}, \operatorname{Spec} A)$$

for every affine monoid scheme $\operatorname{Spec} A$.

Proof. Observe that $D_{\mathcal{C}} = \lim\limits_{\overrightarrow{I}} (C/I)^*$ and $\mathcal{D}_{\mathcal{C}}^* = \bar{\mathcal{C}}$. Then,

$$\begin{aligned}\mathrm{Hom}_{mon}(\mathrm{Spec}\,\mathcal{C}^*, \mathrm{Spec}\,A) &\overset{6.7}{=} \mathrm{Hom}_{bialg}(\mathcal{A}, \mathcal{C}^*) \overset{2.24}{=} \mathrm{Hom}_{bialg}(\mathcal{C}, \mathcal{A}^*) \\ &\overset{2.27}{=} \mathrm{Hom}_{bialg}(\bar{\mathcal{C}}, \mathcal{A}^*) \overset{2.24}{=} \mathrm{Hom}_{bialg}(\mathcal{A}, \mathcal{D}_{\mathcal{C}}) \overset{6.7}{=} \mathrm{Hom}_{mon}(\mathrm{Spec}\,D_{\mathcal{C}}, \mathrm{Spec}\,A)\end{aligned}$$

□

Note 6.13. *Let X be a K-scheme and A_X the ring of functions of X. The set D_X of distributions of X of finite support is said to be $D_X := \{w \in A_X^* \colon w$ factorizes through a finite quotient algebra of $A_X\}$. Obviously, $\mathcal{D}_X^* = \bar{\mathcal{A}}_X$ and $\mathrm{Spec}\,\mathcal{D}_X^* = \bar{X}$.*

If $\mathrm{Spec}\,\mathcal{C}^*$ is an abelian formal monoid then $G = \mathrm{Spec}\,C$ is an affine abelian monoid scheme and $D_{\mathcal{C}} = D_G$, then

$$\mathrm{Hom}_{mon}(G^\vee, \mathrm{Spec}\,A) = \mathrm{Hom}_{mon}(\mathrm{Spec}\,D_G, \mathrm{Spec}\,A) \tag{3}$$

for every affine monoid scheme $\mathrm{Spec}\,A$.

6.1. **Functorial Cartier Duality.**

Definition 6.14. *Let $\mathbb{G}$ be a functor of abelian monoids. $\mathbb{G}^\vee := \mathbb{H}om_{mon}(\mathbb{G}, \mathcal{R})$ (where we regard $\mathcal{R}$ as a monoid with its multiplication) is said to be the dual monoid of $\mathbb{G}$.*

If $\mathbb{G}$ is a functor of groups, then $\mathbb{G}^\vee = \mathbb{H}om_{grp}(\mathbb{G}, G_m^{\cdot})$.

Theorem 6.15. *Assume that $\mathbb{G}$ is a functor of abelian monoids with a reflexive functor of functions. Then, $\mathbb{G}^\vee = \mathrm{Spec}\,(\mathbb{A}_{\mathbb{G}}^*)$ (in particular, this equality shows that $\mathrm{Spec}\,\mathbb{A}_{\mathbb{G}}^*$ is a functor of abelian monoids).*

Proof. $\mathbb{G}^\vee = \mathbb{H}om_{mon}(\mathbb{G}, \mathcal{R}) \overset{6.1}{=} \mathbb{H}om_{\mathcal{R}-alg}(\mathbb{A}_{\mathbb{G}}^*, \mathcal{R}) = \mathrm{Spec}\,(\mathbb{A}_{\mathbb{G}}^*)$. □

Note 6.16. *Explicitly, $\mathrm{Spec}\,\mathbb{A}_{\mathbb{G}}^* = \mathbb{H}om_{mon}(\mathbb{G}, \mathcal{R})$, $\phi \mapsto \tilde{\phi}$, where $\tilde{\phi}(x) = \phi(x)$, for every $\phi \in \mathrm{Spec}\,\mathbb{A}_{\mathbb{G}}^* = \mathbb{H}om_{\mathcal{R}-alg}(\mathbb{A}_{\mathbb{G}}^*, \mathcal{R})$ and $x \in \mathbb{G} \to \mathbb{A}_{\mathbb{G}}^*$.*

$\mathbb{G}^\vee$ is a functor of abelian monoids ($(f \cdot f')(g) := f(g) \cdot f'(g)$, for every $f, f' \in \mathbb{G}^\vee$ and $g \in \mathbb{G}$), the inclusion $\mathbb{G}^\vee = \mathbb{H}om_{mon}(\mathbb{G}, \mathcal{R}) \subset \mathbb{H}om(\mathbb{G}, \mathcal{R}) = \mathbb{A}_{\mathbb{G}}$ is a morphism of monoids and the diagram

$$\begin{array}{ccc} \mathbb{G}^\vee & \hookrightarrow & \mathbb{H}om(\mathbb{G}, \mathcal{R}) = \mathbb{A}_{\mathbb{G}} \\ \| & & \| \\ \mathrm{Spec}\,\mathbb{A}_{\mathbb{G}}^* & \hookrightarrow & \mathbb{H}om_{\mathcal{R}}(\mathbb{A}_{\mathbb{G}}^*, \mathcal{R}) = \mathbb{A}_{\mathbb{G}}^{**} \end{array}$$

is commutative.

Theorem 6.17. *The category of abelian affine R-monoid schemes $G = \mathrm{Spec}\,A$ is anti-equivalent to the category of abelian formal monoids $\mathrm{Spec}\,\mathcal{A}^*$ (we assume the R-modules A are projective).*

Proof. The functors $\mathrm{Spec}\,A = \mathbb{G} \rightsquigarrow \mathbb{G}^\vee = \mathrm{Spec}\,\mathcal{A}^*$ and $\mathrm{Spec}\,\mathcal{A}^* = \mathbb{G} \rightsquigarrow \mathbb{G}^\vee = \mathrm{Spec}\,A$ establish the anti-equivalence between the category of abelian affine R-monoid schemes and the category of abelian formal monoids:

The morphism $\mathbb{G} \overset{\delta}{\to} \mathbb{G}^{\vee\vee}$, $g \mapsto \delta_g$, where $\delta_g(f) := f(g)$ for every $f \in \mathbb{G}^\vee$, is an isomorphism: It is easy to check that the diagram

$$\begin{array}{ccccc} \operatorname{Spec} \mathbb{A}^{**} & \xrightarrow[6.15]{\sim} & (\operatorname{Spec} \mathbb{A}^*)^\vee & \xleftarrow[6.15]{\sim} & (\operatorname{Spec} \mathbb{A})^{\vee\vee} \\ & \diagdown\!\!\diagdown & & \nearrow_{\delta} & \\ & & \operatorname{Spec} \mathbb{A} & & \end{array}$$

is commutative.

$\operatorname{Hom}_{mon}(\mathbb{G}_1, \mathbb{G}_2) = \operatorname{Hom}_{mon}(\mathbb{G}_2^\vee, \mathbb{G}_1^\vee)$: Every morphism of monoids $\mathbb{G}_1 \to \mathbb{G}_2$, taking $\mathbb{H}om_{mon}(-, \mathcal{R})$, defines a morphism $\mathbb{G}_2^\vee \to \mathbb{G}_1^\vee$. Taking $\mathbb{H}om_{mon}(-, \mathcal{R})$ we get the original morphism $\mathbb{G}_1 \to \mathbb{G}_2$, as it is easy to check.

□

In [7, Ch. I, §2, 14], it is given the Cartier Duality (formal schemes are certain functors over the category of commutative linearly compact algebras over a field).

Assume $G = \operatorname{Spec} A$ and $G' = \operatorname{Spec} B$ are commutative affine monoid schemes, then

$$\begin{aligned} \operatorname{Hom}_{mon}(\operatorname{Spec} D_G, G') &\overset{\text{Eq.3}}{=} \operatorname{Hom}_{mon}(G^\vee, G') = \operatorname{Hom}_{mon}(G'^\vee, G) \\ &\overset{\text{Eq.3}}{=} \operatorname{Hom}_{mon}(\operatorname{Spec} D_{G'}, G) \end{aligned} \tag{4}$$

7. Examples of functors of monoids and Cartier duality

1. **Affine toric varieties.**

Let T be a set with structure of abelian (multiplicative) monoid. Let R be a field. The constant functor $\mathbf{T} = \operatorname{Spec} \prod_T \mathcal{R}$ is an abelian formal monoid. The dual functor is the abelian affine R-monoid scheme $\mathbf{T}^\vee = \operatorname{Spec} \oplus_T \mathcal{R} = \operatorname{Spec} RT$.

An affine group scheme $G = \operatorname{Spec} A$ is linearly reductive if and only if $\mathcal{A}^*$ is linearly reductive. Since $\mathcal{A}^*$ is the inverse limit of its coherent algebra quotients it is easy to check that $\mathcal{A}^*$ is linearly reductive if and only if it is a product of algebras of matrices. Then, $G = \operatorname{Spec} A$ is a linearly reductive commutative group scheme if and only if $\mathcal{A}^* = \prod_T \mathcal{K}$.

We will say that an abelian monoid T is standard if it is finitely generated, its associated group G is torsion-free and the natural morphism $T \to G$ is injective (in the literature, see [3, 6.1], it is called affine monoid). It is easy to prove that T is standard if and only if $RT = \oplus_T R$ is a finitely generated domain over R.

Theorem: The category of abelian monoids (resp. finitely generated monoids, standard monoids) is anti-equivalent to the category of affine semisimple abelian monoid schemes (resp. algebraic affine semisimple abelian monoids, integral algebraic affine semisimple abelian monoids).

If T is standard then $G = \mathbb{Z}^n$ and the morphism $T \to G$ induces a morphism $G_m^n \to \mathbf{T}^\vee$. In particular, G_m^n operates on $\mathbf{T}^\vee$. Furthermore, as RG is the localization of RT by the algebraically closed system T, the morphism $G_m^n \to \mathbf{T}^\vee$ is an open injection. We will say that an integral affine algebraic variety on which the torus operates with a dense orbit is an affine toric variety. It is easy to prove that there exists a one-to-one correspondence between affine toric varieties with a fixed point whose orbit is transitive and dense, and standard monoids.

2. **Group $\mathbb{Z}$.**

Obviously, $\mathbb{Z}^\vee = G_m$ and by Cartier duality $G_m^\vee = \mathbb{Z}$, which is a formal group.

Proposition 7.1. *Let G be an affine K-group scheme. Then,*

$$\{\textit{Rational points of} G\} = \mathrm{Hom}_{mon}(\mathbb{Z}, G) \overset{Eq. 3}{=} \mathrm{Hom}_{mon}(\mathrm{Spec}\, D_{G_m}, G)$$

If G is an affine commutative K-group scheme, by Equation 4,

$$\{\textit{Rational points of} G\} = \mathrm{Hom}_{mon}(\mathrm{Spec}\, D_G, G_m)$$

3. **Functor $\mathrm{rad}\, \mathcal{K}$:**

Let $\mathrm{rad}\, \mathcal{K} \subset G_a$ be the covariant subfunctor of groups defined by $\mathrm{rad}\, \mathcal{K}(S) := \mathrm{rad}\, S = \{s \in S | s^n = 0, \text{ para algún } n \in \mathbb{N}\}$.

Let $K[x]/(x^n) \to K[x]/(x^m)$, $n \geq m$ be the natural quotient morphisms. Consider the projective system of K-álgebras $\{K[x]/(x^n),\, n \in \mathbb{N}\}$ and the inductive system of affine schemes $\{\mathrm{Spec}\, K[x]/(x^n),\, n \in \mathbb{N}\}$.

$\hat{G}_a = \lim\limits_{\substack{\rightarrow \\ n}} \mathrm{Spec}\, \mathcal{K}[x]/(x^n) \overset{4.4}{=} \mathrm{Spec}\, \lim\limits_{\substack{\leftarrow \\ n}} \mathcal{K}[x]/(x^n) = \mathrm{Spec}\, \mathcal{K}[[x]]$.

Proposition 7.2. *It holds that* $\mathrm{rad}\, \mathcal{K} = \hat{G}_a = \mathrm{Spec}\, \mathcal{K}[[x]]$.

Proof. $(\mathrm{Spec}\, K[x]/(x^n))^{\cdot}(S) = \mathrm{Hom}_{K-alg}(K[x]/(x^n), S) = \{s \in S | s^n = 0\}$. Therefore,

$$\hat{G}_a(S) = \lim_{\substack{\rightarrow \\ n}} (\mathrm{Spec}\, K[x]/(x^n))^{\cdot}(S) = \{s \in S | s^n = 0, \text{ para algún } n \in \mathbb{N}\} = \mathrm{rad}\, \mathcal{K}(S)$$

□

Notation 7.3. *Assume* $\mathrm{char}\, K = p$. *We denote* $\alpha_n := \mathrm{Spec}\, K[x]/(x^{p^n}) \subset G_a$. *Given $\mu \in \alpha_1(S) \subset S$ (then $\mu^p = 0$) we denote $e^\mu := \sum_{i=0}^{p-1} \mu^i/i! \in S$. Observe that $e^{\mu+\mu'} = e^\mu \cdot e^{\mu'}$.*

We denote $\mu_n = \mathrm{Spec}\, K[x]/(x^{p^n} - 1) \subset G_m$.

Assume $\mathrm{char}\, K = p$. Given $(\lambda_n) \in \prod_{\mathbb{N}} \alpha_1$ the morphism

$$\mathrm{rad}\, \mathcal{K} \to G_m,\ \mu \mapsto e^{\lambda_0 \mu} \cdot e^{\lambda_1 \mu^p} \cdots e^{\lambda_n \cdot \mu^{p^n}} \cdots$$

is a morphism of functors of groups: observe that the polynomial part of degree less than p^{n+1} of the series $v(x) := e^{\lambda_0 x} \cdot e^{\lambda_1 x^p} \cdots e^{\lambda_n x^{p^n}} \cdots$ is $e^{\lambda_0 x} \cdot e^{\lambda_1 x^p} \cdots e^{\lambda_n x^{p^n}}$. The coefficient of x^{p^n} of $v(x)$ is λ_n, then $\prod_{\mathbb{N}} \alpha_1 \subseteq (\mathrm{rad}\, \mathcal{K})^\vee$.

Theorem 7.4. (1) *If* $\mathrm{char}\, K = 0$, $(\mathrm{rad}\, \mathcal{K})^\vee = G_a$.
(2) *If* $\mathrm{char}\, K = p > 0$, $(\mathrm{rad}\, \mathcal{K})^\vee = \prod_{\mathbb{N}} \alpha_1$, *as K-schemes.*

Proof. $\mathbb{H}om(\mathrm{rad}\, \mathcal{K}, \mathcal{K}) = \mathcal{K}[[x]]$. Hence,

$$\mathbb{H}om_{mon}(\mathrm{rad}\, \mathcal{K}, \mathcal{K}) = \{s(x) \in \mathcal{K}[[x]] \colon \begin{array}{l} s(\mu + \mu') = s(\mu) \cdot s(\mu'), \forall\, \mu, \mu' \in \hat{G}_a = \mathrm{rad}\, \mathcal{K} \\ s(0) = 1 \end{array} \}$$

In characteristic zero is easy to prove that $s(x) = e^{\lambda x}$, for some $\lambda \in \mathcal{K}$, then $(\mathrm{rad}\, \mathcal{K})^\vee = G_a$.

Now, in characteristic $p > 0$. If the coefficient of x of $s(x)$ is zero then is easy to prove that $s(x) = t(x^p)$, with $t(x) \in \mathbb{H}om_{mon}(\mathrm{rad}\, \mathcal{K}, \mathcal{K})$. If the coefficient of x of $s(x)$ is λ_0 then the coefficient of x of $(e^{\lambda_0 x})^{-1} \cdot s(x)$ is zero, then $(e^{\lambda_0 x})^{-1} \cdot s(x) =$

$t(x^p)$ and $s(x) = e^{\lambda_0 x} \cdot t(x^p)$. Likewise, $t(x) = e^{\lambda_1 x} \cdot u(x^p)$ and $s(x) = e^{\lambda_0 x} \cdot e^{\lambda_1 x^p} \cdot u(x^{p^2})$, etc. In conclusion, $\{(\lambda_n) \in \prod_{\mathbb{N}} \alpha_1\} = (\operatorname{rad} \mathcal{K})^\vee$

□

Assume $\operatorname{char} K = p$. $(\operatorname{rad} \mathcal{K})^\vee = \prod_{\mathbb{N}} \alpha_1 = \operatorname{Spec} \mathcal{K}[x_0, \ldots, x_n, \ldots]/(x_0^p, \ldots, x_n^p, \ldots)$, as schemes. Observe that

$$e^{\lambda_n \cdot \mu^{p^n}} \cdot e^{\lambda'_n \cdot \mu^{p^n}} = e^{(\lambda_n + \lambda'_n) \cdot \mu^{p^n}} \cdot e^{\sum_{i=1}^{p-1} \frac{\lambda_n^i}{i!} \frac{\lambda'^{p-i}_n}{(p-i)!} \cdot \mu^{p^{n+1}}}.$$

Then, if $*$ denotes the operation of $(\operatorname{rad} \mathcal{K})^\vee = \prod_{\mathbb{N}} \alpha_1$, then

$$(0, \cdots, 0, \lambda_n, 0, \cdots) * ((0, \cdots, 0, \lambda'_n, 0, \cdots) = (0, \cdots, 0, \lambda_n + \lambda'_n, \sum_{i=1}^{p-1} \frac{\lambda_n^i}{i!} \frac{\lambda'^{p-i}_n}{(p-i)!}, 0, \cdots)$$

We have the natural inclusion $(\operatorname{rad} \mathcal{K})^\vee = \operatorname{Spec} \mathcal{K}[[x]]^* \hookrightarrow \mathcal{K}[[x]]$, $(\lambda_n)_{n \in \mathbb{N}} \mapsto e^{\lambda_0 x} \cdots e^{\lambda_n x^{p^n}} \cdots$. We also have the natural inclusion

$$\begin{array}{rcl} \operatorname{rad} \mathcal{K} & \hookrightarrow & \mathbb{H}om((\operatorname{rad} \mathcal{K})^\vee, \mathcal{K}) = \mathcal{K}[x_0, \ldots, x_n, \ldots]/(x_0^p, \ldots, x_n^p, \ldots) \\ \mu & \mapsto & e^{x_0 \mu} \cdots e^{x_n \mu^{p^n}} \cdots \end{array}$$

Note 7.5. *Assume* $\operatorname{char} K = 0$. $(\operatorname{rad} \mathcal{K})^\vee = G_a$ *does not contain any proper subgroup. Then, the quotient groups of* $(\operatorname{rad} \mathcal{K})^\vee$ *are* G_a *and the trivial group.*

Assume $\operatorname{char} K = p$. *The finite subgroups of* G_a *are* $\{\alpha_n\}_{n \in \mathbb{N}}$ *and* $\operatorname{rad} \mathcal{K} = \lim\limits_{\stackrel{\rightarrow}{n}} \alpha_n$. *Then,* $(\operatorname{rad} \mathcal{K})^\vee = \lim\limits_{\stackrel{\leftarrow}{n}} \alpha_n^\vee$. *If* G *is an algebraic group then*

$$\operatorname{Hom}_{grp}((\operatorname{rad} \mathcal{K})^\vee, G) = \operatorname{Hom}_{grp}(\lim_{\stackrel{\leftarrow}{n}} \alpha_n^\vee, G) = \lim_{\stackrel{\rightarrow}{n}} \operatorname{Hom}_{grp}(\alpha_n^\vee, G)$$

That is, every morphism $(\operatorname{rad} \mathcal{K})^\vee \to G$ *is the composition of a projection* $(\operatorname{rad} \mathcal{K})^\vee \to \alpha_n^\vee$ *and an injective morphism* $\alpha_n^\vee \hookrightarrow G$.

Theorem 7.6. *In characteristic* $p > 0$, $(\alpha_n)^\vee = \prod_n \alpha_1$, *as* K*-schemes. Specifically,*

$$(\alpha_n)^\vee = \{(\lambda_n) \in \prod^n \alpha_1\} = \left\{ \begin{array}{rcl} \alpha_n & \to & G_m \\ \mu & \mapsto & e^{\lambda_0 \mu} \cdot e^{\lambda_1 \mu^p} \cdots e^{\lambda_n \cdot \mu^{p^n}} \end{array} \right\}$$

Proof. Proceed as in Theorem 7.4. □

The subgroups of α_n are the obvious subgroups α_r, $r \leq n$, then, by Cartier duality the quotients of $\alpha_n^\vee$ are the obvious quotients $\alpha_r^\vee$, $r \leq n$.

Proposition 7.7. *Assume* $\operatorname{char} K = 0$. *Then,* $\operatorname{rad} \mathcal{K} = \hat{G}_m$.

Proof. The inverse morphism of the morphism $\operatorname{rad} \mathcal{K} \to \hat{G}_m$, $\mu \mapsto e^\mu$ is the morphism $\hat{G}_m \to \operatorname{rad} \mathcal{K}$, $\mu \mapsto \ln(1 + (\mu - 1)) := \sum_{i>0} (-1)^{i+1} (\mu - 1)^i / i!$. □

In characteristic $p > 0$, $\hat{G}_m = \lim\limits_{\stackrel{\rightarrow}{n}} \mu_n$ as functor of groups.

4. **Functor of monoids $G_a^\vee$:**

Let $G_a = \operatorname{Spec} K[x]$ be the additive group.

Theorem 7.8. *Assume* $\operatorname{char} K = 0$. *Then*

$$G_a^\vee = \operatorname{rad} \mathcal{K}$$

Proof. We know that $(\operatorname{rad}\mathcal{K})^\vee = G_a$, by 7.4. By Cartier duality, $G_a^\vee = \operatorname{rad}\mathcal{K}$. □

Explicitly, $\mu \in \operatorname{rad}\mathcal{K}$ defines the morphism $G_a \to G_m$, $\lambda \mapsto e^{\lambda\mu}$. Then, we have the natural inclusion $\operatorname{rad}\mathcal{K} \hookrightarrow \mathcal{K}[x]$, $\mu \mapsto e^{\mu x}$.

Theorem 7.9. *Assume* $\operatorname{char} K = p > 0$. *Then,* $G_a^\vee$ *is isomorphic to* $\oplus_{\mathbb{N}}\alpha_1$ *as functors of sets. Specifically,*

$$G_a^\vee = \{(\lambda_n) \in \oplus_{\mathbb{N}}\alpha_1\} = \left\{ \begin{array}{ccc} G_a & \to & G_m \\ \alpha & \mapsto & e^{\lambda_0\alpha}\cdot e^{\lambda_1\alpha^p}\cdots e^{\lambda_n\cdot\alpha^{p^n}} \end{array} \right.$$

Proof. Proceed as in Theorem 7.4. □

Write $\mathcal{K}[[x_0,\dots,x_n,\dots]]/(x_0^p,\dots,x_n^p,\dots) := \lim\limits_{\overleftarrow{n}} \mathcal{K}[x_0,\dots,x_n]/(x_0^p,\dots,x_n^p)$, then $G_a^\vee = \oplus_{\mathbb{N}}\alpha_1 = \operatorname{Spec}\mathcal{K}[[x_0,\dots,x_n,\dots]]/(x_0^p,\dots,x_n^p,\dots)$, as functors of sets. The dual morphism of natural the inclusion $\operatorname{rad}\mathcal{K} \subset G_a$ is the obvious morphism $G_a^\vee = \oplus_{\mathbb{N}}\alpha_1 \subset \prod_{\mathbb{N}}\alpha_1 = (\operatorname{rad}\mathcal{K})^\vee$.

We have the natural inclusion $G_a^\vee \hookrightarrow \mathcal{K}[x]$, $(\lambda_0,\cdots,\lambda_n) \mapsto e^{\lambda_0 x}\cdots e^{\lambda_n x^{p^n}}$.

Note 7.10. *Let us now assume that* $\operatorname{char} K = 0$. *Let* $\mathbb{Q}$ *be the constant functor of groups* $\mathbb{Q}$ *(with the addition operation), that is, denote* $\mathbb{Q}$ *the discrete topological space, then* $\mathbb{Q}(S) := \operatorname{Homeo}(\operatorname{Spec} S, \mathbb{Q})$*, for all commutative* K*-algebras* S*. Let* $G_M = \mathbb{Q}^\vee$ *be the dual group of* $\mathbb{Q}$*, then* $G_M = \operatorname{Spec}(K[\mathbb{Q}] = K[e^{rx}]_{r\in\mathbb{Q}})$*.* $\mathbb{Q}$ *is the direct limit of its finite* $\mathbb{Z}$*-submodules,* $\mathbb{Q} = \lim\limits_{\overrightarrow{r\in\mathbb{Q}^+}} \mathbb{Z}\cdot r$*, which are isomorphic to* $\mathbb{Z}$*, then* $G_M = \lim\limits_{\overleftarrow{r\in\mathbb{Q}^+}} \operatorname{Spec} K[e^{rx}, e^{-rx}]$ *and* $\operatorname{Spec} K[e^{rx}, e^{-rx}] \simeq G_m$*. Giving a point* $\alpha \in G_M$ *is equal to giving a point* $\alpha \in G_m$*, and to determining* α^r*,for all* $r \in \mathbb{Q}$*. If* G *is an algebraic group, then*

$$\operatorname{Hom}_{mon}(G_M, G) = \lim_{\overrightarrow{r\in\mathbb{Q}^+}} \operatorname{Hom}_{mon}(\operatorname{Spec} K[e^{rx}, e^{-rx}], G)$$

That is, every morphism from G_M *to an algebraic group factors through a projection* $G_M \to G_m$*,* $\alpha \mapsto \alpha^r$*, for some* $r \in \mathbb{Q}$*.*

If $E = \oplus_J\mathbb{Q}$ *is a* $\mathbb{Q}$ *is a vector space, let us also write* E *for the constant functor of monoids* E*, then* $E^\vee := \mathbb{H}om_{mon}(E, G_m) = \prod_J G_M$*. Every morphism of* $E^\vee$ *on an algebraic group factors through* G_m^n*, for some* n*.*

Let us now assume that $\operatorname{char} K = p > 0$*. If* $E = \oplus_J\mathbb{Z}/p\mathbb{Z}$ *is a* $\mathbb{Z}/p\mathbb{Z}$*-vector space then* $E^\vee := \mathbb{H}om_{mon}(E, G_m) = \prod_J \mu_1$*. Every morphism from* $E^\vee$ *to an algebraic group factors through* μ_1^n*, for some* $n \in \mathbb{N}$*.*

Theorem 7.11. *Let* $G_a = \operatorname{Spec} K[x]$ *and let* D_{G_a} *be the set of distributions with finite support of* G_a*. Let* $G_a(K)$ *be the constant functor of groups,* K*. Assume* K *is an algebraically closed field. Then,*

1. $\bar{G}_a = \hat{G}_a \times G_a(K) = \operatorname{rad}\mathcal{K} \times G_a(K)$ *and* $\operatorname{Spec} D_{G_a} = (\operatorname{rad}\mathcal{K})^\vee \times G_a(K)^\vee$.
2. *It holds that*

$$\operatorname{Hom}_{mon}(G_a^\vee, G) \overset{Eq.3}{=} \operatorname{Hom}_{mon}((\operatorname{rad}\mathcal{K})^\vee \times G_a(K)^\vee, G),$$

for all affine group scheme G*.*

(3) *The scheme-theoretic image of every morphism from* $G_a^\vee$ *to an affine algebraic group is isomorphic to*

$$\begin{array}{ll} G_a^\delta \times G_m^n,\ \delta = 0,1, & \text{if } \operatorname{char} K = 0 \\ \alpha_r^\vee \times \mu_1^s, & \text{if } \operatorname{char} K = p > 0 \end{array}$$

8. Exponential map in arbitrary characteristic

8.1. **Vector fields.**

Let $K[\epsilon] = K[x]/(x^2)$. Given a functor of sets $\mathbb{X}$ and the morphism of K-algebras $K[\epsilon] \to K$, $\epsilon \mapsto 0$, let $0^*\colon \mathbb{X}(K[\epsilon]) \to \mathbb{X}(K)$ be the induced morphism. Given $x \in \mathbb{X}(K)$, let

$$T_x\mathbb{X} := \{y \in \mathbb{X}(K[\epsilon]),\ \text{such that } 0^*y = x\}$$

Proposition 8.1. *Let X be a K-scheme and $x \in X$ a rational point. Then,* $T_xX = T_x\bar{X}$.

Proof. Observe that $\bar{X}(K[\epsilon]) = X(K[\epsilon])$ y $\bar{X}(K) = X(K)$. □

More generally, given a commutative K-algebra S and the morphism of S-algebras $S[\epsilon] = S[x]/(x^2) \to S$, $\epsilon \mapsto 0$ let $0^*\colon \mathbb{X}(S[\epsilon]) \to \mathbb{X}(S)$ be the induced morphism. Given $x \in \mathbb{X}(S)$, let

$$T_x\mathbb{X} := \{y \in \mathbb{X}(S[\epsilon]),\ \text{such that } 0^*y = x\}$$

be the vector space of tangent vectors of $\mathbb{X}$ at the point x.

Assume $\mathbb{X} = \operatorname{Spec}\mathbb{A}$. Then $\mathbb{X}(S) = \operatorname{Hom}(\operatorname{Spec} S, \operatorname{Spec}\mathbb{A}) = \operatorname{Hom}_{\mathcal{K}-alg}(\mathbb{A}, \mathcal{S})$ and $\mathbb{X}(S[\epsilon]) = \operatorname{Hom}_{\mathcal{K}-alg}(\mathbb{A}, \mathcal{S}[\epsilon])$. Then, given $x \in \mathbb{X}(S) = \operatorname{Hom}_{\mathcal{K}-alg}(\mathbb{A}, \mathcal{S})$, by the standard arguments

$$T_x \operatorname{Spec}\mathbb{A} = \operatorname{Der}_{\mathcal{K}}(\mathbb{A}, \mathcal{S})$$

Given a morphism of functors of sets $\phi\colon \mathbb{X} \to \mathbb{Y}$, $x \in \mathbb{X}(S)$ and $D_x \in T_x\mathbb{X} \subset \mathbb{X}(S[\epsilon])$ then $\phi(D_x) \in T_{\phi(x)}\mathbb{Y} \subset \mathbb{Y}(S[\epsilon])$.

Definition 8.2. *We will say that* $T\mathbb{X} := \mathbb{H}om(\operatorname{Spec} K[\epsilon], \mathbb{X})$ *is the tangent bundle of* $\mathbb{X}$. *We have a natural morphism* $T\mathbb{X} \overset{0^*}{\to} \mathbb{X}$, $D_x \mapsto 0^*D_x = x$. *We will say that* $\operatorname{Der}\mathbb{X} := \operatorname{Hom}_{\mathbb{X}}(\mathbb{X}, T\mathbb{X})$ *is the set of derivations of* $\mathbb{X}$ *(or the set of vector fields of* $\mathbb{X}$*).*

Giving $D \in \operatorname{Der}\mathbb{X}$ is equivalent to giving for every point $x \in \mathbb{X}(S)$ (and every S) a tangent vector $D_x \in T_x\mathbb{X} \subset \mathbb{X}(S[\epsilon])$, functorially.

Let $G_a = \mathcal{K}$ be the additive group, then

$$TG_a = \mathbb{H}om(\operatorname{Spec} K[\epsilon], G_a) = \mathbb{H}om_{\mathcal{K}-alg}(\mathcal{K}[x], \mathcal{K}[\epsilon]) = \mathcal{K}[\epsilon]$$

Proposition 8.3. *Let $\mathbb{X}$ be an affine functor, then it holds that*

$$\operatorname{Der}\mathbb{X} = \operatorname{Der}_{\mathcal{K}}(\mathbb{A}_{\mathbb{X}}, \mathbb{A}_{\mathbb{X}})$$

Proof. First, observe that

$$\begin{aligned} \mathbb{A}_{\mathbb{X}\times\operatorname{Spec} K[\epsilon]} &= \mathbb{H}om(\mathbb{X}\times\operatorname{Spec} K[\epsilon], \mathcal{K}) = \mathbb{H}om(\mathbb{X}, \mathbb{H}om(\operatorname{Spec} K[\epsilon], \mathcal{K})) \\ &= \mathbb{H}om(\mathbb{X}, \mathcal{K}[\epsilon]) = \mathbb{A}_{\mathbb{X}}[\epsilon] \end{aligned}$$

Now,

$$\begin{aligned}\operatorname{Der}\mathbb{X} &= \operatorname{Hom}_{\mathbb{X}}(\mathbb{X}, T\mathbb{X}) = \operatorname{Hom}_{\mathbb{X}}(\mathbb{X}, \mathbb{H}om(\operatorname{Spec} K[\epsilon], \mathbb{X}))\\ &= \{f \in \operatorname{Hom}(\mathbb{X}\times \operatorname{Spec} K[\epsilon], \mathbb{X})\colon f(x,0)=x,\, \forall x\in\mathbb{X}\}\\ &= \{h\in \operatorname{Hom}_{\mathcal{K}-alg}(\mathbb{A}_{\mathbb{X}}, \mathbb{A}_{\mathbb{X}}[\epsilon])\colon h = \operatorname{Id} \mod (\epsilon)\} = \operatorname{Der}_{\mathcal{K}}(\mathbb{A}_{\mathbb{X}}, \mathbb{A}_{\mathbb{X}})\end{aligned}$$

□

Given a K-scheme X, it is well known that

$$\operatorname{Der} X = T_{\operatorname{Id}}\mathbb{E}nd\, X \subset End_{K[\epsilon]-sch}(X\times_K \operatorname{Spec} K[\epsilon]), \quad D\mapsto e^{\epsilon D},$$

(topologically $e^{\epsilon D}$ is the identity morphism and over the ring of functions $e^{\epsilon D}(a+b\epsilon) := a + \epsilon\cdot D(a) + b\epsilon$, for all $a+b\epsilon\in\mathcal{O}_X[\epsilon]$).

Theorem 8.4. *Let $\mathbb{X}$ be a functor of sets. It holds that*

$$T_{\operatorname{Id}}\mathbb{E}nd\,\mathbb{X} = \operatorname{Der}\mathbb{X}$$

Proof. It is a consequence of the equalities

$$\begin{aligned}\operatorname{Hom}(\mathbb{X}_{|K[\epsilon]}, \mathbb{X}_{|K[\epsilon]}) &= \operatorname{Hom}_{\operatorname{Spec} K[\epsilon]}(\mathbb{X}\times\operatorname{Spec} K[\epsilon], \mathbb{X}\times \operatorname{Spec} K[\epsilon])\\ &= \operatorname{Hom}(\mathbb{X}\times\operatorname{Spec} K[\epsilon], \mathbb{X}) = \operatorname{Hom}(\mathbb{X}, \mathbb{H}om(\operatorname{Spec} K[\epsilon], \mathbb{X}))\end{aligned}$$

□

Giving $D\in\operatorname{Der}\mathbb{X}$, we denote $\mathbb{X}\to\mathbb{H}om(\operatorname{Spec} K[\epsilon],\mathbb{X})$, $x\mapsto D_x$ the corresponding morphism and we denote $e^{\epsilon D}\colon \mathbb{X}_{|K[\epsilon]}\to\mathbb{X}_{|K[\epsilon]}$ the corresponding endomorphism. Given $x\in\mathbb{X}(S)\subset\mathbb{X}(S[\epsilon])$, it is easy to check that $e^{\epsilon D}(x) = D_x$.

Definition 8.5. *Let $\mathbb{G}$ be a functor of groups. We say that a vector field $D\in\operatorname{Der}\mathbb{G}$ (for each point $g\in\mathbb{G}(S)$ we have a tangent vector D_g at g) is $\mathbb{G}$-invariant if $g'\cdot D_g = D_{g'\cdot g}$ for all pair of points g, g' of $\mathbb{G}$. We will say that* $\operatorname{Derinv}\mathbb{G}$ *is the set of all invariant vector fields of $\mathbb{G}$.*

Proposition 8.6. *Let $\mathbb{G}$ be a functor of groups and let $e\in\mathbb{G}(K)$ be the identity element. It holds that* $\operatorname{Derinv}\mathbb{G} = T_e\mathbb{G}$.

Proof. The morphisms $\operatorname{Derinv}\mathbb{G}\to T_e\mathbb{G}$, $D\mapsto D_e$, $T_e\mathbb{G}\to\operatorname{Derinv}\mathbb{G}$, $D_e\mapsto D$, where $D_g := g\cdot D_e$ for all $g\in\mathbb{G}$, are mutually inverses. □

Definition 8.7. *Given $D_e\in T_e\mathbb{G}$, we will say that $D\in\operatorname{Derinv}\mathbb{G}$, such that $D_g := g\cdot D_e$, for all $g\in\mathbb{G}$, is the invariant field associated with D_e.*

Proposition 8.8. *Let $f\colon\mathbb{G}_1\to\mathbb{G}_2$ be a morphism of functors of groups, $D_e\in T_e\mathbb{G}_1$ and $D'_e = f(D_e)$. If D y D' are the invariant fields associated with D_e and D'_e respectively, then $f(D_g) = D'_{f(g)}$, for all $g\in\mathbb{G}_1$.*

Proof.

$$f(D_g) = f(g\cdot D_e) = f(g)\cdot f(D_e) = f(g)\cdot(D'_e) = D'_{f(g)}$$

□

8.2. **Analytic one-parameter group.**

Theorem 8.9. *Let $\mathbb{X}$ be an affine functor of sets. It holds that*

$$\mathrm{Der}(\mathbb{X}) = \mathrm{Der}_{\mathcal{K}}(\mathbb{A}_{\mathbb{X}}, \mathbb{A}_{\mathbb{X}}) = \mathrm{Hom}_{mon}(G_a^\vee, \mathbb{E}nd\,\mathbb{X})$$

Proof. Let $\mathbb{M}$ be a dual functor of $\mathcal{K}$-modules. By Theorem 6.1, giving a $\mathcal{K}[x]$-module structure on $\mathbb{M}$ (equivalently, giving a linear endomorphism on $\mathbb{M}$) is equivalent to giving a $G_a^\vee$-module structure on $\mathbb{M}$. If $\mathbb{M}$ is a $\mathcal{K}[x]$-module then through the inclusion $G_a^\vee \hookrightarrow K[x]$, $G_a^\vee$ operates on $\mathbb{M}$. Let us follow the notations

$$\mathbb{E}nd_K\mathbb{M} = \mathrm{Hom}_{K-alg}(\mathcal{K}[x], \mathbb{E}nd_K\mathbb{M}) = \mathrm{Hom}_{mon}(G_a^\vee, \mathbb{E}nd_{\mathcal{K}}\mathbb{M}), \quad T \mapsto e^{xT}$$

where $e^{xT}\colon G_a^\vee \to \mathbb{E}nd_{\mathcal{K}}\mathbb{M}$ is defined by $e^{xT}(\mu) = e^{\mu T}$ (if $\mathrm{char}\, K = 0$, $\mu \in \mathrm{rad}\,\mathcal{K} = G_a^\vee$ and $e^{\mu T} := \sum_i (\mu \cdot T)^i / i!$; if $\mathrm{char}\, K = p > 0$, $\mu = (\mu_0, \dots, \mu_r) \in \oplus_{\mathbb{N}} \alpha_1 = G_a^\vee$ and $e^{\mu T} := e^{\mu_0 T} \cdot e^{\mu_1 T^p} \cdots e^{\mu_r T^{p^r}}$).

Likewise, giving a $\mathcal{K}[x_1, \dots, x_n]$-structure on $\mathbb{M}$ is equivalent to giving a $G_a^\vee \times \overset{n}{\cdots} \times G_a^\vee$-module structure on $\mathbb{M}$.

Let us follow the notations

$$(\mathbb{E}nd_K\mathbb{M})^n \supseteq \mathrm{Hom}_{K-alg}(K[x_1, \dots, x_n], \mathbb{E}nd_K\mathbb{M}) = \mathrm{Hom}_{mon}((G_a^\vee)^n, \mathbb{E}nd_K\mathbb{M}),$$
$$(T_1, \dots, T_n) \overset{Not.}{\mapsto} e^{x_1 T_1} \cdots e^{x_n T_n}$$

If $T_1, T_2 \in \mathbb{E}nd_K\mathbb{M}$ commute then $e^{x(T_1+T_2)} = e^{xT_1} \cdot e^{xT_2}$. Let $D \in End_K\mathbb{A}_{\mathbb{X}}$, that is, $\mathbb{A}_{\mathbb{X}}$ is a $\mathcal{K}[x]$-module (hence it is a $G_a^\vee$-module: given $\mu \in G_a^\vee$, $\mu \cdot a = e^{\mu D} a$). Consider $D \otimes 1 + 1 \otimes D \in \mathbb{E}nd_K(\mathbb{A}_{\mathbb{X}} \otimes \mathbb{A}_{\mathbb{X}})$, then $\mathbb{A}_{\mathbb{X}} \otimes \mathbb{A}_{\mathbb{X}}$ is a $\mathcal{K}[x]$-module (hence it is a $G_a^\vee$-module: $\mu \cdot (a \otimes b) = e^{\mu(D \otimes 1 + 1 \otimes D)}(a \otimes b) = e^{\mu D} a \otimes e^{\mu D} b$). Now , D is a derivation if and only if the morphism $\mathbb{A}_{\mathbb{X}} \otimes \mathbb{A}_{\mathbb{X}} \to \mathbb{A}_{\mathbb{X}}$, $a \otimes b \mapsto ab$ is a morphism of $\mathcal{K}[x]$-modules, which is equivalent to say that the morphism is a morphism of $G_a^\vee$-módulos, that is, $G_a^\vee$ operates on $\mathbb{A}_{\mathbb{X}}$ by morphisms of K-algebras. That is, the diagram

$$\begin{array}{ccc} \mathbb{E}nd_K\mathbb{A}_{\mathbb{X}} & = & \mathrm{Hom}_{mon}(G_a^\vee, \mathbb{E}nd_K\mathbb{A}_{\mathbb{X}}) \\ \uparrow & & \uparrow \\ \mathrm{Der}_K\,\mathbb{A}_{\mathbb{X}} & = & \mathrm{Hom}_{mon}(G_a^\vee, \mathbb{E}nd_{K-alg}\mathbb{A}_{\mathbb{X}}) \end{array}$$

is commutative. □

Theorem 8.10. *Let X be a K-scheme. It holds that*

$$\mathrm{Hom}_{mon}(G_a^\vee, \mathbb{E}nd\,X) = \mathrm{Der}\,X$$

Proof. Given a point $\mu \in G_a^\vee(S)$ there exists a finite, local and rational K-subalgebra $C \subset S$, such that $\mu \in G_a^\vee(C)$. Moreover, if $C \to K$ is the quotient morphism then $G_a^\vee(C) \to G_a^\vee(K) = \{0\}$, $\mu \mapsto 0$. Therefore, given a morphism $f\colon G_a^\vee \to \mathbb{E}ndX$, then $f(\mu) \in \mathbb{E}ndX(C)$, that is, $f(\mu)$ is an endomorphism $X_C \to X_C$. Moreover, by base change $C \to K$, $f(\mu) = \mathrm{Id}$. Topologically $X_C = X$, then $f(\mu)$ is topologically equal to the identity morphism, hence it is affine. Let $\{U_i\}$ be a covering of X by open affine sets. Giving a morphism $G_a^\vee \to \mathbb{E}ndX$ is equivalent to giving morphisms $G_a^\vee \to \mathbb{E}nd\,U_i$ which coincide on $U_i \cap U_j$. That is, giving a morphism $G_a^\vee \to \mathbb{E}ndX$ is equivalent to giving derivations on each U_i which coincide on $U_i \cap U_j$. That is,

$$\mathrm{Hom}_{mon}(G_a^\vee, \mathbb{E}ndX) = \mathrm{Der}\,X$$

□

Let D be a vector field on an affine functor of sets $\mathbb{X}$. Consider the morphism

$$exp_D\colon G_a^\vee \to \mathbb{E}nd\,\mathbb{X},\quad \mu\mapsto e^{\mu D}$$

If X is a K-scheme, then $e^{\mu D}$ is an affine morphism because it is topologically the identity morphism. $e^{\mu D}\colon \mathbb{X}_{|C}\to\mathbb{X}_{|C}$ ($\mu\in G_a^\vee(C)$) induces the morphism $e^{\mu D}\colon (\mathbb{A}_{\mathbb{X}})_{|C}\to(\mathbb{A}_{\mathbb{X}})_{|C}$ defined by

$$e^{\mu D}(a):=\left\{\begin{array}{ll}\sum_n \frac{\mu^n D^n(a)}{n!}, & \text{if } \operatorname{char} K=0\\ e^{\mu_0 D}(a)\cdots e^{\mu_n D^{p^n}}(a), & \text{if} \operatorname{char} K=p>0\end{array}\right.$$

If X is a projective K-variety, then there exists a K-scheme whose functor of points is $\mathbb{E}nd\,X^{\cdot}$ (see [9]), and we will also denote it $\mathbb{E}nd\,X$. Thus, $exp_D\colon G_a^\vee\to\mathbb{E}nd\,X$ factors through a unique morphism $exp_D\colon \operatorname{Spec} K[x]^*\to\mathbb{E}nd\,X$.

Definition 8.11. *The morphism $exp_D\colon G_a^\vee\to\mathbb{E}nd\,\mathbb{X}$ is said to be the analytic one-parameter group associated with D.*

Let $y\in\mathbb{X}$ be a rational point. We will say that $exp_{D,y}\colon G_a^\vee\to\mathbb{X}$, $exp_{D,y}(\mu):=exp_D(\mu)(y)$ is the analytic integral curve of D passing through y.

The induced morphism between the rings of functions by $exp_{D,y}$ is $\mathbb{A}_{\mathbb{X}}\to\mathcal{K}[x]^*$,

$$a\mapsto\left\{\begin{array}{ll}\sum_i D^i(a)(y)/i!\cdot x^i\in\mathcal{K}[[x]], & \text{if } \operatorname{char} K=0\\ (\sum_{i=0}^{p-1} D^i(a)(y)/i!\cdot x_0^i)\cdots(\sum_{i=0}^{p-1} D^{ip^n}(a)(y)/i!\cdot x_n^i)\cdots, & \text{if } \operatorname{char} K=p>0\end{array}\right.$$

If X is a K-scheme, the morphism $exp_{D,y}\colon G_a^\vee\to X$ factors through $\operatorname{Spec} K[x]^*$ and we also write it $exp_{D,y}\colon \operatorname{Spec} K[x]^*\to X$. If $U=\operatorname{Spec} A$ is an affine open set containing y, then the morphism induced by $exp_{D,y}$ between the rings of functions is written as it has been written above.

Theorem 8.12. *Let $\mathbb{G}$ be an affine functor of groups or a K-group scheme. It holds that*

$$\operatorname{Hom}_{grp}(G_a^\vee,\mathbb{G})=\operatorname{Derinv}(\mathbb{G})=T_e\mathbb{G}$$

Proof. 1. Given $g\in\mathbb{G}$, let $L_g\colon\mathbb{G}\to\mathbb{G}$ be the morphism defined by $L_g(g')=gg'$. $\mathbb{G}$ operates on $\mathbb{E}nd\,\mathbb{G}$ by $g*f:=L_g\circ f\circ L_{g^{-1}}$, for all $g\in\mathbb{G}$ and $f\in\mathbb{E}nd\,\mathbb{G}$. The morphism f is $\mathbb{G}$-invariant if and only if $f(g)=g\cdot f(e)$, for all $g\in\mathbb{G}$. Hence, $(\mathbb{E}nd\,\mathbb{G})^{\mathbb{G}}=\mathbb{G}$.

2. $\mathbb{G}$ operates on $\operatorname{Der}\mathbb{G}$ by $(g*D)_h:=g\cdot D_{g^{-1}h}$ for all $g,h\in\mathbb{G}$ and $D\in\operatorname{Der}\mathbb{G}$. Obviously, $(\operatorname{Der}\mathbb{G})^{\mathbb{G}}=\operatorname{Derinv}\mathbb{G}$.

3. $\mathbb{G}$ operates on $\operatorname{Hom}_{mon}(G_a^\vee,\mathbb{E}nd\,\mathbb{G})$ by $(g*F)(\mu)=g*(F(\mu))$, for all $g\in\mathbb{G}$, $F\in\operatorname{Hom}_{mon}(G_a^\vee,\mathbb{E}nd\,\mathbb{G})$ and $\mu\in G_a^\vee$.

4. Taking $\mathbb{G}$-invariants on $\operatorname{Der}(\mathbb{G})=\operatorname{Hom}_{mon}(G_a^\vee,\mathbb{E}nd\,\mathbb{G})$, by 1.,2. and 3. we obtain

$$\operatorname{Derinv}(\mathbb{G})=\operatorname{Hom}_{grp}(G_a^\vee,\mathbb{G})$$

□

8.3. **Existence and uniqueness of "analytic" solutions of an algebraic differential equation in arbitrary characteristic.**

Proposition 8.13. *If $\mu \in G_a^\vee(S)$, then*

$$T_\mu G_a^\vee = \begin{cases} S, & \textit{if } \operatorname{char} K = 0 \\ \oplus_{\mathbb{N}} S, & \textit{if } \operatorname{char} K = p > 0 \end{cases}$$

Proof. $G_a^\vee(S[\epsilon]) = \{\mu + \lambda \cdot \epsilon$, with $\mu \in G_a^\vee(S)$ and $\lambda \in S$ if $\operatorname{char} K = 0$, $\lambda \in \oplus_{\mathbb{N}} S$ if $\operatorname{char} K = p > 0\}$. □

In characteristic zero, let us denote $\mu + \lambda \cdot \epsilon = \lambda(\partial_{x_0})_\mu$. In positive characteristic $p > 0$, $G_a^\vee = \oplus_{\mathbb{N}} \alpha_1 = \operatorname{Spec} K[[x_0, \ldots, x_n, \ldots]]/(x_0^p, \ldots, x_n^p, \ldots)$. Let us denote $\mu + \lambda \cdot \epsilon = \sum_i \lambda_i (\partial_{x_i})_{\mu_i}$. The morphism $G_a^\vee \to \operatorname{Spec} K[x]^*$ induces a morphism $T_\mu G_a^\vee \to T_\mu \operatorname{Spec} K[x]^*$, that maps $\mu + \lambda \cdot \epsilon$ to $\sum_i \lambda_i (\partial_{x_i})_{\mu_i}$.

The linear map induced between the tangent spaces by exp_D is:

$$T_0 G_a^\vee \overset{exp_D}{\to} T_{Id} \mathbb{A}ut\mathbb{X} = \operatorname{Der} \mathbb{X}, \quad \epsilon = (\partial_{x_0})_0 \overset{exp_D}{\mapsto} e^{\epsilon D} = D$$

Then, by Theorem 8.10 we have the following theorem.

Theorem 8.14 (existence and uniqueness)**.** *Let $\mathbb{X}$ be an affine functor of sets or a K-scheme. If D is a vector field on $\mathbb{X}$, then exp_D is the only morphism of functors of groups $f \colon G_a^\vee \to \mathbb{A}ut\,\mathbb{X}$ such that $f((\partial_{x_0})_0) = D$.*

Let $x \in \mathbb{X}(K)$ be a rational point. The morphism $\chi_x \colon \mathbb{E}nd\,\mathbb{X} \to \mathbb{X}$, $\chi_x(\tau) = \tau(x)$, maps $e^{\epsilon D} = D$ to $e^{\epsilon D}(x) = D_x$. Then,

$$exp_{D,x}((\partial_{x_0})_0) = \chi_x(exp_D((\partial_{x_0})_0)) = \chi_x(D) = D_x$$

Notation 8.15. *Let $\delta \in \operatorname{Derinv} G_a^\vee$ be the invariant field associated with $(\partial_{x_0})_0$. That is, $\delta_\mu = \mu * (\partial_{x_0})_0$, where $\mu \in G_a^\vee$ and $*$ is the operation of $G_a^\vee$. Specifically, $\delta = \partial_{x_0} + x_0^{p-1}/(p-1)!\partial_{x_1} + x_0^{p-1}/(p-1)!x_1^{p-1}/(p-1)!\partial_{x_2} + \cdots$.*

It can be checked that: $\delta^{p^n} = \partial_{x_n} + x_n^{p-1}/(p-1)!\partial_{x_{n+1}} + x_n^{p-1}/(p-1)!x_{n+1}^{p-1}/(p-1)!\partial_{x_{n+2}} + \cdots$. Therefore: $(\delta^{p^n})_0 = (\partial_{x_n})_0$ and:

$$\operatorname{Derinv} G_a^\vee = \langle \delta, \ldots, \delta^{p^n}, \ldots \rangle$$

Let $\mu \in G_a^\vee(S)$, $x \in \mathbb{X}(S)$ and denote $*$ the operation of $G_a^\vee$. We have the commutative diagram

$$\begin{array}{ccc} G_a^\vee & \xrightarrow{exp_{D,x}} & \mathbb{X} \\ \downarrow{\scriptstyle \mu *} & & \downarrow{\scriptstyle exp_D(\mu)} \\ G_a^\vee & \xrightarrow{exp_{D,x}} & \mathbb{X} \end{array} \qquad \begin{array}{ccc} \delta_0 & \longmapsto & D_x \\ \downarrow & & \downarrow \\ \delta_\mu & \longmapsto & exp_{D,x}(\delta_\mu) = exp_D(\mu)(D_x) \end{array}$$

Let $\epsilon = \delta_0 \in T_0 G_a^\vee$. From the commutative diagram

$$\begin{array}{ccc} (G_a^\vee)_{|K[\epsilon]} & \xrightarrow{exp_{D,x}} & \mathbb{X}_{|K[\epsilon]} \\ \downarrow{\scriptstyle \epsilon *} & & \downarrow{\scriptstyle exp_D(\epsilon) = e^{\epsilon D}} \\ (G_a^\vee)_{|K[\epsilon]} & \xrightarrow{exp_{D,x}} & \mathbb{X}_{|K[\epsilon]} \end{array} \qquad \begin{array}{ccc} \mu & \longmapsto & exp_{D,x}(\mu) \\ \downarrow & & \downarrow \\ \delta_\mu & \longmapsto & exp_{D,x}(\delta_\mu) = D_{\exp_{D,x}(\mu)} \end{array}$$

we obtain that $exp_D(\mu)(D_x) = exp_{D,x}(\delta_\mu) = D_{exp_{D,x}(\mu)} = D_{exp_D(\mu)(x)}$.

Lemma 8.16. *Let* $a \in \mathbb{A}_{G_a^\vee} = \mathcal{K}[x]^*$. *Then,* $a \in \mathcal{K} \iff \delta(a) = 0 \iff \delta(a)(\mu) = 0$ *for all* $\mu \in G_a^\vee$.

Proof. Obviously $a \in \mathbb{H}om(\mathbb{X}, \mathcal{K}) = \mathbb{A}_{\mathbb{X}}$ is zero if and only if $a(x) = 0$ for all $x \in \mathbb{X}$.

Let us only prove that $a \in \mathcal{K}$ if $\delta(a) = 0$, and $\operatorname{char} K = p$: Obviously $\delta^{p^n}(a) = 0$ for all $n \in \mathbb{N}$. As

$$\mathcal{K}[x]^* = \mathcal{K}[[x_0, \ldots, x_n, \ldots]]/(x_0^p, \ldots, x_n^p, \ldots) = \{ \sum_{\alpha \in \oplus_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}} \lambda_\alpha x^\alpha, \, \lambda_\alpha \in \mathcal{K} \}$$

and $T_\mu G_a^\vee = \langle \delta_\mu, \ldots, (\delta^{p^n})_\mu, \ldots \rangle = \langle (\partial_{x_i})_\mu \rangle$, then $\partial_{x_i} a = 0$, for all $i \in \mathbb{N}$, and $a \in \mathcal{K}$.

□

Theorem 8.17 (existence and uniqueness)**.** *Let* $\mathbb{X}$ *be an affine functor of sets or a* K*-scheme, let* $x \in \mathbb{X}(K)$ *be a rational point and let* D *be a vector field on* $\mathbb{X}$*. Then,* $exp_{D,x}$ *is the only morphism* $f \colon G_a^\vee \to \mathbb{X}$ *such that* $f(0) = x$ *and* $f(\delta_\mu) = D_{f(\mu)}$, *for every point* $\mu \in G_a^\vee$.

Proof. We already know that $exp_{D,x}(0) = e^{0D}(x) = \operatorname{Id}(x) = x$ and $exp_{D,x}(\delta_\mu) = D_{exp_{D,x}(\mu)}$.

We still have to prove the uniqueness of f.
Let $\tilde{D}$ be the only invariant field on $\mathbb{A}utX$ such that $\tilde{D}_{\mathrm{Id}} = e^{\epsilon D}$. Recall that the morphism $\chi_x \colon \mathbb{A}utX \to X$, $\chi_x(\phi) = \phi(x)$ maps $\tilde{D}_{\mathrm{Id}} = e^{\epsilon D}$ to $e^{\epsilon D}(x) = D_x$. We have the commutative diagram

$$\begin{array}{ccc} \mathbb{A}ut\,\mathbb{X} & \xrightarrow{\chi_x} & \mathbb{X} \\ \downarrow{\scriptstyle \phi\cdot} & & \downarrow{\scriptstyle \phi} \\ \mathbb{A}ut\,\mathbb{X} & \xrightarrow{\chi_x} & \mathbb{X} \end{array} \qquad \begin{array}{ccc} \tilde{D}_{\mathrm{Id}} & \overset{\chi_x}{\longmapsto} & D_x \\ \downarrow{\scriptstyle \phi\cdot} & & \downarrow{\scriptstyle \phi} \\ \tilde{D}_\phi & \overset{\chi_x}{\longmapsto} & \chi_x(\tilde{D}_\phi) = \phi(D_x) \end{array}$$

The composition:

$$G_a^\vee \overset{exp_{-D} \times f}{\longrightarrow} \mathbb{A}ut\,\mathbb{X} \times \mathbb{X} \overset{\circ}{\to} \mathbb{X}$$

maps δ_μ to zero, for all $\mu \in G_a^\vee$:

$$\begin{array}{ccccc} T_\mu G_a^\vee & \to & T_{exp_{-D}(\mu)} \mathbb{A}ut\mathbb{X} \times T_{f(\mu)}\mathbb{X} & \to & T_{exp_{-D}(\mu) f(\mu)}\mathbb{X} \\ \delta_\mu & \mapsto & (-\tilde{D}_{exp_{-D}(\mu)}, D_{f(\mu)}) & \mapsto & -exp_{-D}(\mu) D_{f(\mu)} + exp_{-D}(\mu) D_{f(\mu)} \end{array}$$

Since $\mathcal{K} = \{a \in \mathbb{A}_{G_a^\vee} \colon \delta_\mu(a) = 0, \text{ for all } \mu \in G_a^\vee\}$, the composition $G_a^\vee \overset{exp_{-D} \times f}{\longrightarrow} \mathbb{A}ut\mathbb{X} \times \mathbb{X} \to \mathbb{X}$, is constant, that is, $exp_{-D}(\mu) f(\mu) = x$, and $f(\mu) = exp_D(\mu) x = exp_{D,x}(\mu)$.

□

Theorem 8.18 (existence and uniqueness)**.** *Let* X *be a* K*-scheme, let* $y \in X$ *be a rational point and let* D *be a vector field on* X*. Then,* $exp_{D,y}$ *is the only morphism* $f \colon \operatorname{Spec} K[x]^* \to X$ *such that* $f(0) = y$ *and* $f(\delta_\mu) = D_{f(\mu)}$, *for every point* $\mu \in \operatorname{Spec} K[x]^*(S)$.

Proof. By 4.7 and 8.17, we only have to prove that given a morphism $f \colon \operatorname{Spec} K[x]^* \to X$ such that $f(0) = y$ and $f(\delta_\mu) = D_{f(\mu)}$, for every point $\mu \in G_a^\vee(S) \subset (\operatorname{Spec} K[x]^*)(S)$, then $f(\delta_\mu) = D_{f(\mu)}$, for every $\mu \in \operatorname{Spec} K[x]^*(S)$. Consider the diagram

$$\begin{array}{ccc} A_{X,y} & \xrightarrow{f^*} & K[x]^* \\ \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle \delta} \\ A_{X,y} & \xrightarrow{f^*} & K[x]^* \end{array}$$

By the hypothesis $(f^* \circ D - \delta \circ f^*)(a)(\mu) = 0$ for all $a \in A_{X,y}$ and $\mu \in G_a^\vee(S)$. Then, $(f^* \circ D - \delta \circ f^*)(a) = 0$ and $f^* \circ D = \delta \circ f^*$. Hence, $f(\delta_\mu) = D_{f(\mu)}$, for every $\mu \in \operatorname{Spec} K[x]^*(S)$. □

8.4. **Algebraic group associated with a vector field.**

Let X be a K-scheme, let D be a vector field on X and let $\tilde{D}$ be the invariant field of $\mathbb{E}nd\mathbb{X}$ associated with the tangent vector D at $\mathrm{Id} \in \mathbb{E}nd\,\mathbb{X}$.

Proposition 8.19. *The analytic integral curve of $\tilde{D}$ passing through* Id *is exp_D, that is, $exp_{\tilde{D}_{\mathrm{Id}}} = exp_D$.*

Proof. exp_D is a morphism of groups, so $exp_D(\delta_\mu) = \tilde{D}_{exp_D(\mu)}$. It follows from Theorem 8.17 that $exp_D = exp_{\tilde{D}_{\mathrm{Id}}}$.

□

Proposition 8.20. *The (closed) scheme-theoretic image of $exp_{D,x}$, $\operatorname{Im} exp_{D,x}$, is the minimal subvariety of X tangent to D at x.*

Proof. Let C be the minimal subvariety of X tangent to D at x. Let

$$exp^C_{D,x} \colon \operatorname{Spec} K[x]^* \to C$$

be the analytic integral curve associated with $D_{|C}$ (passing through x). Obviously, the composition of $exp^C_{D,x}$ with the inclusion $C \hookrightarrow X$ is $exp_{D,x}$, so $\operatorname{Im} exp_{D,x} \subseteq C$. It is easy to check that $\operatorname{Im} exp_{D,x}$ is a subvariety tangent to D at x, so it is the minimal subvariety of X tangent to D at x. □

The ideal of functions vanishing at $\operatorname{Im} exp_{D,x}$ is the greatest ideal whose germ at x is the ideal of all functions f such that $D^n f(x) = 0$ for every $n \geq 0$, if $\operatorname{char} K = 0$, or such that $D^{ip^n} f(x) = 0$ for some $n \geq 0$ and every $0 \leq i < p$, if $\operatorname{char} K = p > 0$.

Corollary 8.21. *Let X be a projective variety. The (closed) scheme-theoretic image of exp_D is the minimal closed subvariety of $\mathbb{A}utX$ tangent to $\tilde{D}$ at* Id. *$\operatorname{Im} exp_D$ is a commutative group and the closure of $(\operatorname{Im} exp_D) \cdot x$ coincides with $\operatorname{Im} exp_{D,x}$.*

Proof. The scheme-theoretic image of exp_D is the minimal closed subvariety of $\mathbb{A}utX$ tangent to $\tilde{D}$ at Id, by 8.19 and 8.20. $\operatorname{Im} exp_D$ is a commutative group by 6.10. By definition, the composition

$$\begin{array}{ccccc} \operatorname{Spec} K[x]^* & \xrightarrow{exp_D} & \mathbb{A}ut\,X & \longrightarrow & X \\ & & f & \longmapsto & f \cdot x = f(x) \end{array}$$

is $exp_{D,x}$. Therefore, $\operatorname{Im} exp_{D,x}$ is the scheme-theoretic image of $\operatorname{Im} exp_D$, that is, the closure of $\operatorname{Im} exp_D \cdot x$ coincides with $\operatorname{Im} exp_{D,x}$.

□

Definition 8.22. *Let X be a projective variety. We will say that* $\operatorname{Im} exp_D$ *is the algebraic group associated with D.*

The requirement of projectiveness is to assure the existence of the scheme $\mathbb{A}ut\, X$. It could be possible to define the algebraic group $\operatorname{Im} exp_D$ whenever exp_D factors through a group scheme G included in $\mathbb{A}ut\, X$ (it is enough, for this, that $D \in T_{\mathrm{Id}} G$) because in that case we can also define $\operatorname{Im} exp_D$.

Theorem 8.23. *Assume* $\operatorname{char} K = 0$. *Let* $X = \operatorname{Spec} A$ *be an affine K-scheme. Then,* $\operatorname{Hom}_{mon}(G_a, \mathbb{E}nd\, X) = \{D \in \operatorname{Der} X \colon$ *for each $a \in A$ there exists an $n \in \mathbb{N}$, such that* $D^n(a) = 0\}$.

Proof. Let V be a vector space. Endowing V with a G_a-module structure is equivalent to endowing $\mathcal{V}$ with a $\mathcal{K}[[x]]$-module structure. Endowing $\mathcal{V}$ with a $\mathcal{K}[[x]]$-module structure of is equivalent to defining an endomorphism $T \colon V \to V$ $(T = x\cdot)$ such that for each $v \in V$ there exists an $n \in \mathbb{N}$ so that $T^n(v) = 0$. Therefore,

$$\left\{ \begin{array}{l} T \in End_K V \colon \text{ for each } v \in V, \text{ there exists an } n \in \mathbb{N} \\ \text{such that } T^n(v) = 0 \end{array} \right\} = \operatorname{Hom}(G_a, \mathbb{E}nd_K V)$$
$$T \mapsto e^{xT}$$

Arguing as in 8.9, we can prove this theorem. □

Theorem 8.24. *Let* $X = \operatorname{Spec} K[\xi_1, \dots, \xi_n]$ *be an affine variety and assume* $\operatorname{char} K = 0$. *Let D be a vector field of X. Then, the morphism* $exp_D \colon G_a^\vee \to \mathbb{A}utX$ *factors through an unipotent group (that is G_a) if and only if there exists $m >> 0$ such that $D^m(\xi_i) = 0$, for all i.*

Theorem 8.25. *Assume* $\operatorname{char} K = p > 0$ *and let X be a K-scheme. It holds that*

$$\operatorname{Hom}_{grp.}(\alpha_n^\vee, \mathbb{E}ndX) = \{D \in \operatorname{Der} X \colon D^{p^n} = 0\}$$

Proof. 1. Let V be a K-vector space. Endowing V with a $\alpha_n^\vee$-module structure is equivalent to endowing V of a $K[x]/(x^{p^n})$-module structure. Then,

$$\{T \in \mathbb{E}nd_K V \colon T^{p^n} = 0\} = \operatorname{Hom}_{funct}(\alpha_n^\vee, \mathbb{E}nd_K V),\ T \mapsto e^{xT}$$

2. Suppose, now, that V is a K-algebra. Then, as in 8.9,

$$\{D \in \operatorname{Der}_K(V, V) \colon D^{p^n} = 0\} = \operatorname{Hom}_{grp}(\alpha_n^\vee, \mathbb{E}nd_{K-alg} V)$$

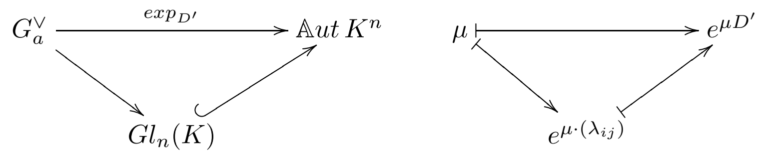
3. Now, we can prove the theorem as in 8.10. □

Theorem 8.26. *Let X be a complete K-variety and* $\operatorname{char} K = p > 0$. *Let D be a vector field of X. Then, the morphism* $exp_D \colon G_a^\vee \to \mathbb{A}utX$ *factors through a unipotent group, that is, $\alpha_n^\vee$ if and only if $D^{p^n} = 0$.*

8.5. Algebraic groups associated with the vector fields of $\mathbb{P}^n$.

Let K be an algebraically closed field and consider the natural morphism $\pi \colon K^n \backslash 0 \to \mathbb{P}^{n-1}$. We are going to compute the algebraic group associated with a field on $\mathbb{P}^{n-1}(K)$.

As it is well known, if D is a vector field on $\mathbb{P}^{n-1}$, then $D = \pi D'$ for some vector field $D' = \sum_{ij} \lambda_{ij} x_i \partial_{x_j}$ on $K^n \backslash 0$. We have the commutative diagram:



The morphism $\operatorname{Spec} K[x]^* \to \operatorname{Spec} K[x_{ij}, \det(x_{ij})^{-1}]$, $\mu \mapsto e^{\mu\cdot(\lambda_{ij})}$ is induced by $K[x_{ij}, \det(x_{ij})^{-1}] \to K[x]^*$, $x_{rs} \mapsto x_{rs}(e^{x\cdot(\lambda_{ij})})$ (if $\operatorname{char} K = 0$ then $K[x]^* = K[[x]]$, if $\operatorname{char} K = p > 0$ then $K[x]^* = (K[[x_i]]/(x_i^p))_{i\in\mathbb{N}}$).

Therefore, the algebraic group associated with D' is:

$$G = \operatorname{Spec} K[x_{rs}(e^{x\cdot(\lambda_{ij})}), e^{-x\cdot\operatorname{tr}((\lambda_{ij}))}]$$

Changing the base of K^n, we can suppose that the matrix (λ_{ij}) is in its Jordan form, with eigenvalues $\lambda_1, \dots, \lambda_s$.

Now there are two cases, depending on the characteristic of K.

Theorem 8.27. *Let K be an algebraically closed field of characteristic zero. Let $D = \pi(\sum_{ij} \lambda_{ij} x_i \partial_{x_j})$ be a vector field on $\mathbb{P}^{n-1}$ and let G be its associated algebraic group. Then,*

$$G \simeq G_m^r \times G_a^\delta$$

where r is the dimension of the $\mathbb{Q}$-affine space generated by the eigenvalues of the matrix (λ_{ij}) in $\mathbb{C}$, $\delta = 0$ in case the matrix is diagonalizable and $\delta = 1$ otherwise.

Proof. As the matrix (λ_{ij}) is in its Jordan form, it is easy to check that:

$$G = \operatorname{Spec} K[e^{x\lambda_1}, \dots, e^{x\lambda_s}, e^{-x(\lambda_1+\dots+\lambda_s)}, \delta\cdot x]$$

where $\delta = 0$ in case the Jordan matrix is diagonal and $\delta = 1$ otherwise. Moreover, if, reordering, $\lambda_1, \dots, \lambda_r$ is a base of the $\mathbb{Z}$-module generated by $\lambda_1, \dots, \lambda_s$ in K (or, equivalently, $\lambda_1, \dots, \lambda_r$ is a base of the $\mathbb{Q}$-vector space generated by $\lambda_1, \dots, \lambda_s$ in K) then, as $x, e^{x\lambda_1}, \dots, e^{x\lambda_r}$ are algebraically independent, we have that:

$$\begin{aligned} G &= \operatorname{Spec} K[e^{x\lambda_1}, \dots, e^{x\lambda_s}, e^{-x(\lambda_1+\dots+\lambda_s)}, \delta\cdot x] \\ &= \operatorname{Spec} K[e^{x\lambda_1}, e^{-x\lambda_1}] \otimes \dots \otimes K[e^{x\lambda_r}, e^{-x\lambda_r}] \otimes K[\delta x] = G_m^r \times G_a^\delta \end{aligned}$$

But this group G is also the algebraic group associated with D. To see this, recall that two vector fields $D' = \sum_{ij} \lambda_{ij} x_i \partial_{x_j}$ and $D' = \sum_{ij} \mu_{ij} x_i \partial_{x_j}$ on $K^n - \{0\}$ projet to the same vector field D on $\mathbb{P}^{n-1}$ if and only if they differ on $\lambda\cdot(\sum_i x_i \partial_{x_i})$, that is, if the matrix (μ_{ij}) differs from (λ_{ij}) in $\lambda\cdot\operatorname{Id}$. We can then assume that the matrix $M = (\lambda_{ij})$ has the eigenvalue zero, and in this case the affine space generated by the eigenvalues is the vector space generated by the eigenvalues.

Let $v \in K^n$ be an eigenvector such that $M\cdot v = 0$. If H_v stands for the closed subgroup of $Gl_n(K)$ of the automorphisms $h \in Gl_n(K)$ such that $h\cdot v = v$, then it is clear that the morphism $exp_{D'} \colon \hat{G}_a \to Gl_n(K)$, $\mu \mapsto e^{\mu\cdot M}$ factors through H_v. Analogously, let H'_v be the closed subgroup of the projectivities that let $\bar{v} \in \mathbb{P}^{n-1}$ fixed.

At the Lie algebras, the natural morphism $Gl_n(K) \to PGl_n(K) = \mathbb{A}ut\,\mathbb{P}^{n-1}$ maps D' to D. Therefore, exp_D coincides with the composition $\hat{G}_a \overset{exp_{D'}}{\to} Gl_n(K) \to$

$\mathbb{A}ut\, \mathbb{P}^{n-1}$. As a consequence, we have:

$$\begin{array}{ccccc} \hat{G}_a & \xrightarrow{exp_{D'}} & Gl_n(K) & \longrightarrow & PGl_n(K) \\ & \searrow & \uparrow & & \uparrow \\ & & H_v & = & H'_v \end{array}$$

that shows that the algebraic group associated with D' is the same as that associated with D, so we are done. $\square$

Theorem 8.28. *Let K be an algebraically closed field of characteristic $p > 0$. Let $D = \pi(\sum_{ij} \lambda_{ij} x_i \partial_{x_j})$ be a vector field on $\mathbb{P}^{n-1}(K)$, and let G be its associated algebraic group. Then,*

$$G \simeq \mu_1^r \times \alpha_{m+1}^\vee$$

where r is the dimension of the $\mathbb{Z}/p\mathbb{Z}$-affine space generated by the eigenvalues of the matrix (λ_{ij}) in K, and m is such that, if s is the greatest of the orders of the Jordan boxes, then $p^m \leq s - 1 < p^{m+1}$ (if $s = 1$ we say that $m = -1$).

Proof. Everything is analogous to the previous theorem, except the calculation of the algebraic group G associated with $D' = \sum_{ij} \lambda_{ij} x_i \partial_{x_j}$ on $K^n - \{0\}$.

Reordering, let $\lambda_1, \dots, \lambda_r$ be a base of the $\mathbb{Z}/p\mathbb{Z}$-vector space generated by $\lambda_1, \dots, \lambda_s$ in K. Let s be the greatest of the orders of the Jordan boxes and $m \in \mathbb{N}$ such that $p^m \leq s - 1 < p^{m+1}$ (if $s = 1$ we say that $m = -1$). A similar computation to that used in the characteristic cero case, shows that:

$$\begin{aligned} G &= \operatorname{Spec} K[e^{x\lambda_1}, \dots, e^{x\lambda_s}, e^{-x(\lambda_1 + \cdots + \lambda_s)}, x_0, \dots, x_m]/(x_0^p, \dots, x_n^p, \dots) \\ &= \mu_1^r \times \alpha_{m+1}^\vee \end{aligned}$$

$\square$

9. Appendix

9.1. **Tannakian Categories.**

In this subsection we use Theorem 6.2 to derive the so called Tannaka's theorem (see [2] and references therein for the standard treatment).

Let K be a field.

Definition 9.1. *A neutralized K-linear category (C, ω) is an abelian category C together with a "fibre" functor $\omega \colon \mathsf{C} \rightsquigarrow \mathrm{Vect}_K$ into the category of finite dimensional K-vector spaces such that ω is exact, additive and for every $X, X' \in \mathrm{Ob}(\mathsf{C})$,*

$$\mathrm{Hom}_{\mathsf{C}}(X, X') \subset \mathrm{Hom}_K(\omega(X), \omega(X'))$$

is a K-linear vector subspace.

A *K-linear morphism* between neutralized K-linear categories $F \colon (\mathsf{C}, \omega) \to (\bar{\mathsf{C}}, \bar{\omega})$ is an additive functor $F \colon \mathsf{C} \rightsquigarrow \bar{\mathsf{C}}$ such that $\bar{\omega} \circ F = \omega$.

Example 9.2. *Let A be finite a K-algebra. The category Mod_A of finitely generated modules over A together with the forgetful functor is a neutralized K-linear category.*

Recall also that morphisms of K-algebras $A \to B$ correspond to K-linear morphisms $\mathrm{Mod}_B \to \mathrm{Mod}_A$.

If (C, ω) is a neutralized K-linear category and $X \in \operatorname{Ob} \mathsf{C}$ is an object, we will denote by $\langle X \rangle$ the full subcategory of C whose objects are (isomorphic to) quotients of subobjects of finite direct sums $X \oplus \ldots \oplus X$.

By standard arguments, it can be proved the following:

Theorem 9.3 (Main Theorem)**.** *Let $\langle X \rangle$ be a neutralized K-linear category generated by an object X. There exists a (weak) equivalence of neutralized K-linear categories $\langle X \rangle \simeq \operatorname{Mod}_{A^X}$, where A^X is a finite K-algebra unique up to isomorphisms.*

Moreover, every K-linear morphism $F\colon \langle X \rangle \rightsquigarrow \langle \bar{X} \rangle$ induces a unique morphism of K-algebras $f\colon A^{\bar{X}} \to A^X$.

A neutralized K-linear category (C, ω) is said to *admit a set of generators* if there exists a filtering set I of objects in C such that: $\mathsf{C} = \lim\limits_{\substack{\rightarrow \\ X \in I}} \langle X \rangle$.

In this case, a standard argument passing to the limit allows to prove:

$$\mathsf{C} = \lim_{\substack{\rightarrow \\ X \in I}} \langle X \rangle \simeq \lim_{\rightarrow} \operatorname{Mod}_{A^X} = \operatorname{Mod}_{\mathcal{C}^*}$$

where $\mathcal{C}^* := \lim\limits_{\leftarrow} \mathcal{A}^X$, that is a $\mathcal{K}$-algebra scheme and $\operatorname{Mod}_{\mathcal{C}^*}$ is the category of $\mathcal{K}$-coherent $\mathcal{C}^*$-modules.

Let (C, ω) and $(\bar{\mathsf{C}}, \bar{\omega})$ be two neutralized K-linear categories that admit a set of generators. Every K-linear morphism $F\colon (\mathsf{C}, \omega) \rightsquigarrow (\bar{\mathsf{C}}, \bar{\omega})$ induces a unique morphism of $\mathcal{K}$-algebra schemes $f\colon \bar{\mathcal{C}}^* \to \mathcal{C}^*$.

Definition 9.4. *A tensor product on a neutralized K-linear category (C, ω) is a bilinear functor $\otimes\colon \mathsf{C} \times \mathsf{C} \rightsquigarrow \mathsf{C}$ that fits into the square:*

$$\begin{CD}
\mathsf{C} \times \mathsf{C} @>{\otimes}>> \mathsf{C} \\
@V{\omega \times \omega}VV @VV{\omega}V \\
\operatorname{Vect}_K \times \operatorname{Vect}_K @>{\otimes_K}>> \operatorname{Vect}_K
\end{CD}$$

(where the symbol $\otimes_K$ denotes the standard tensor product on vector spaces) and satisfies:

a) Associativity and commutativity.

b) Unity. There exists an object K together with functorial isomorphisms for every object X:

$$X \otimes K \simeq X \simeq K \otimes X$$

that through ω become the natural identifications $\omega(X) \otimes_K K = \omega(X) = K \otimes \omega(X)$.

c) Duals. There exists a covariant additive functor $^\vee\colon \mathsf{C} \to \mathsf{C}^\circ$, satisfying:

$$\begin{CD}
\mathsf{C} @>{\vee}>> \mathsf{C}^\circ \\
@V{\omega}VV @VV{\omega^*}V \\
\operatorname{Vect}_K @>{*}>> \operatorname{Vect}_K^\circ
\end{CD}$$

where $\omega^*(X) := \omega(X)^*$. There also exists functorial isomorphisms $(X^\vee)^\vee = X$ and a morphism $\mathcal{K} \to X \otimes X^\vee$ such that via ω is the natural morphism $K \to \omega(X) \otimes_K \omega(X)^*$.

Definition 9.5. *A Tannakian category neutralized over K is a triple $(\mathsf{C},\omega,\otimes)$ where (C,ω) is a neutralized K-linear category that admits a set of generators and $\otimes$ is a tensor product on (C,ω).*

Now it is not difficult to check that the existence of a tensor product in a neutralized K-linear category $\mathsf{C}\simeq \mathrm{Mod}_{\mathcal{C}^*}$ amounts to the existence of a comultiplication on the algebra scheme $\mathcal{C}^*$. As a consequence:

Theorem 9.6. *Let $(\mathsf{C},\omega,\otimes)$ be a Tannakian category neutralized over K. There exists a unique (up to isomorphism) cocommutative Hopf algebra $\mathcal{K}$-scheme $\mathcal{C}^*$ such that $(\mathsf{C},\omega,\otimes)$ is equivalent to the category $\mathrm{Mod}_{\mathcal{C}^*}$.*

Corollary 9.7 (Tannaka's Theorem)**.** *If $(\mathsf{C},\omega,\otimes)$ is a Tannakian category neutralized over K, then there exists a unique (up to isomorphism) affine K-group scheme G such that $(\mathsf{C},\omega,\otimes)$ is equivalent to the category of finite linear representations of G.*

Proof. By the previous theorem, there exists a scheme of Hopf algebras $\mathcal{C}^*$ such that $\mathsf{C}\simeq\mathrm{Mod}_{\mathcal{C}^*}$. If we define the affine group scheme $G:=\mathrm{Spec}\, C$, then the statement follows from Theorem 6.2. □

9.2. **Lie Algebras and Infinitesimal Formal Monoids in Characteristic Zero.**

Notation 9.8. *In this subsection all algebra schemes are assumed to be commutative and $R=K$ is assumed to be a field of characteristic zero.*

Let $f\colon N\to M$ be a morphism of R-modules and let $f^*\colon\mathcal{M}^*\to\mathcal{N}^*$ be the dual morphism. $\mathrm{Ker} f^*=\mathcal{M}_1^*$ and $\mathrm{Coker}\, f^*=\mathcal{N}_1^*$, where $M_1=\mathrm{Coker}\, f$ and $N_1=\mathrm{Ker}\, f$.

Let $\mathcal{C}^*$ be a commutative algebra scheme. If $f^*\colon\mathcal{M}^*\to\mathcal{N}^*$ is a morphism of $\mathcal{C}^*$-modules then $\mathrm{Coker}\, f^*$ and $\mathrm{Ker}\, f^*$, are $\mathcal{C}^*$-modules.

Let $\mathcal{I}_j^*\hookrightarrow\mathcal{C}^*$ ideal schemes and $m\colon\mathcal{I}_1^*\otimes\cdots\otimes\mathcal{I}_n^*\to\mathcal{C}^*$ the obvious multiplication morphism. We denote by $\mathcal{I}_1^*\cdots\mathcal{I}_n^*=\mathcal{J}^*$ the module scheme closure of $\mathrm{Im}\, m$ in $\mathcal{C}^*$, which is an ideal scheme of $\mathcal{A}^*$: the dual morphism of m, $c\colon C\to I_1\otimes\cdots\otimes I_n$, is a morphism of $\mathcal{C}^*$-modules and $J=\mathrm{Im}\, c$.

Given a functor of $\mathcal{K}$-modules $\mathbb{M}$ we will denote its $\mathcal{K}$-module scheme closure $\bar{\mathbb{M}}$. Observe that $\mathbb{M}^*(K)=\bar{\mathbb{M}}^*(K)$. Hence, $\bar{\mathbb{M}}=\mathcal{N}^*$, where $N=\mathbb{M}^*(K)$ (see [1, 2.7]). We say that a morphism of functors of $\mathcal{K}$-modules $\mathbb{M}\to\mathbb{N}$ is dense if $\bar{\mathbb{M}}\to\bar{\mathbb{N}}$ is surjective, that is, if $\mathbb{N}^*(K)\to\mathbb{M}^*(K)$ is injective.

We have

$$\mathcal{I}_1^*\otimes\cdots\otimes\mathcal{I}_n^*\overset{\text{dense}}{\to}\mathcal{I}_1^*\cdots\mathcal{I}_n^*\hookrightarrow\mathcal{C}^*$$

$\mathcal{I}_1^*\cdots\mathcal{I}_n^*=\mathcal{I}_1^*\cdot(\mathcal{I}_2^*\cdots\mathcal{I}_n^*)$: observe the diagram

$$\mathcal{I}_1^*\otimes(\mathcal{I}_2^*\otimes\cdots\otimes\mathcal{I}_n^*)\overset{\text{dense}}{\to}\mathcal{I}_1^*\otimes\mathcal{I}_2^*\cdots\mathcal{I}_n^*\overset{\text{dense}}{\to}\mathcal{I}_1^*\cdot(\mathcal{I}_2^*\cdots\mathcal{I}_n^*)\hookrightarrow\mathcal{C}^*$$

$\mathcal{I}_1^*\cdot(\mathcal{I}_2^*\cdot\mathcal{I}_3^*)=\mathcal{I}_1^*\cdot\mathcal{I}_2^*\cdot\mathcal{I}_3^*=(\mathcal{I}_1^*\cdot\mathcal{I}_2^*)\cdot\mathcal{I}_3^*$.

Notation 9.9. *Let $\mathbb{M}$ be an $\mathcal{R}$-module. We denote $S^n\mathbb{M}$ the functor of $\mathcal{R}$-modules defined by $(S^n\mathbb{M})(S):=S^n(\mathbb{M}(S))$ the n-th symmetric power of the S-module $\mathbb{M}(S)$. Let $S_n\mathbb{M}$ be the functor of modules defined by $(S_n\mathbb{M})(S):=(\mathbb{M}(S)\otimes_S\overset{n}{\cdots}\otimes_S\mathbb{M}(S))^{S_n}$. The natural morphism $S_n\mathbb{M}\to S^n\mathbb{M}$ is an isomorphism when R is a $\mathbb{Q}$-algebra.*

Denote $\mathcal{I}^{*n}=\mathcal{I}^*\overset{n}{\cdots}\mathcal{I}^*$. The composition $\mathcal{I}^*\otimes\overset{n}{\cdots}\otimes\mathcal{I}^*\to\mathcal{I}^{*n}\to\mathcal{I}^{*n}/\mathcal{I}^{*n+1}$ is dense and factors through $S^n(\mathcal{I}^*/\mathcal{I}^{*2})$. Then, the morphism

$$\overline{S^n(\mathcal{I}^*/\mathcal{I}^{*2})}\overset{m}{\to}\mathcal{I}^{*n}/\mathcal{I}^{*n+1}$$

is surjective.

Observe that $S^n\mathcal{M}$ is the quasi-coherent module associated to the K-module S^nM and $\overline{S^n\mathcal{M}^*}=(S^n\mathcal{M})^*$, because

$$\begin{aligned}\mathrm{Hom}_{\mathcal{K}}(\overline{S^n\mathcal{M}^*},\mathcal{K})&=\mathrm{Hom}_{\mathcal{K}}(S^n\mathcal{M}^*,\mathcal{K})=\mathrm{Hom}_{\mathcal{K}}(\mathcal{M}^*\otimes\overset{n}{\cdots}\otimes\mathcal{M}^*,\mathcal{K})^{S_n}\\&=(M\otimes\overset{n}{\cdots}\otimes M)^{S_n}=S^nM\end{aligned}$$

Definition 9.10. *Let $\mathcal{A}^*$ be a commutative bialgebra scheme, $e\colon\mathcal{A}^*\to\mathcal{K}$ its counit and $\mathcal{I}^*=\mathrm{Ker}\,e$. We will say that $\mathbb{G}:=\mathrm{Spec}\,\mathcal{A}^*$ is an infinitesimal formal monoid if $\mathcal{A}^*=\varprojlim_i\mathcal{A}^*/\mathcal{I}^{*i}$.*

Theorem 9.11. *Let $\mathbb{G}=\mathrm{Spec}\,\mathcal{A}^*$ be an infinitesimal formal monoid, $e\colon\mathcal{A}^*\to\mathcal{K}$ the unit of $\mathbb{G}$ and $\mathcal{I}^*=\mathrm{Ker}\,e$. The natural morphism*

$$\overline{S^n(\mathcal{I}^*/\mathcal{I}^{*2})}\overset{m}{\to}\mathcal{I}^{*n}/\mathcal{I}^{*n+1}$$

is an isomorphism.

Proof. Let us construct the inverse morphism $\mathcal{I}^{*n}/\mathcal{I}^{*n+1}\to\overline{S^n(\mathcal{I}^*/\mathcal{I}^{*2})}$ of m: consider the multiplication morphism $\mathbb{G}\times\overset{n}{\cdots}\times\mathbb{G}\to\mathbb{G}$, $(g_1,\dots,g_n)\mapsto g_1\cdots g_n$, which corresponds to the comultiplication morphism $c\colon\mathcal{A}^*\to\mathcal{A}^*\tilde{\otimes}\cdots\tilde{\otimes}\mathcal{A}^*$.

For any $f\in\mathcal{I}^*$ we have that

$$c(f)=\sum_{j=1}^n 1\otimes\cdots\otimes\overset{j}{f}\otimes\cdots\otimes 1 \bmod \sum_{r\neq s}^n\mathcal{A}^*\tilde{\otimes}\cdots\tilde{\otimes}\underset{r}{\mathcal{I}^*}\tilde{\otimes}\cdots\tilde{\otimes}\underset{s}{\mathcal{I}^*}\tilde{\otimes}\cdots\tilde{\otimes}\mathcal{A}^*$$

because the classes of $c(f)$ and $\tilde{f}:=\sum_{j=1}^n 1\otimes\cdots\otimes\overset{j}{f}\otimes\cdots\otimes 1$ in $\mathcal{A}^*/\mathcal{I}^*\otimes\cdots\otimes\overset{s}{\mathcal{A}^*}\otimes\cdots\otimes\mathcal{A}^*/\mathcal{I}^*=\mathcal{A}^*$ are equal to f, for every s, so

$$c(f)-\tilde{f}\in\cap_{s=1}^n(\sum_{r\neq s}^n\mathcal{A}^*\tilde{\otimes}\cdots\tilde{\otimes}\underset{r}{\mathcal{I}^*}\tilde{\otimes}\cdots\tilde{\otimes}\mathcal{A})=\sum_{r\neq s}^n\mathcal{A}^*\tilde{\otimes}\cdots\tilde{\otimes}\underset{r}{\mathcal{I}^*}\tilde{\otimes}\cdots\tilde{\otimes}\underset{s}{\mathcal{I}^*}\tilde{\otimes}\cdots\tilde{\otimes}\mathcal{A}^*$$

(for the latter equation recall $\mathcal{A}^*=\mathcal{K}\oplus\mathcal{I}^*$). Therefore, we obtain the morphism

$$\begin{array}{ccl}\mathcal{I}^{*n}/\mathcal{I}^{*n+1} & \overset{\bar{c}}{\longrightarrow} & \mathcal{I}^*/\mathcal{I}^{*2}\tilde{\otimes}\overset{n}{\cdots}\tilde{\otimes}\mathcal{I}^*/\mathcal{I}^{*2}\subset\mathcal{A}^*/\mathcal{I}^{*2}\tilde{\otimes}\overset{n}{\cdots}\tilde{\otimes}\mathcal{A}^*/\mathcal{I}^{*2}\\ f_1\cdots f_n & \mapsto & \overline{c(f_1\cdots f_n)}=\overline{c(f_1)\cdots c(f_n)}=\overline{\sum_{\sigma\in S_n}f_{\sigma(1)}\otimes\cdots\otimes f_{\sigma(n)}}\end{array}$$

for every $f_1,\dots,f_n\in\mathcal{I}^*$, that defines a morphism $\bar{c}\colon\mathcal{I}^{*n}/\mathcal{I}^{*n+1}\to\overline{S^n(\mathcal{I}^*/\mathcal{I}^{*2})}$. Now it can be checked that $\bar{c}\circ m\colon\overline{S^n(\mathcal{I}^*/\mathcal{I}^{*2})}\to\overline{S^n(\mathcal{I}^*/\mathcal{I}^{*2})}$ is equal to the homothety with scale factor $n!$.

□

Definition 9.12. *If A is a bialgebra, we say that an element is primitive if $c(a)=a\otimes 1+1\otimes a$, where c is the comultiplication of A.*

It can be checked that $a\in A$ is a primitive element if and only if $a\in T_e\mathbb{G}:=\mathrm{Der}_{\mathcal{K}}(\mathcal{A}^*,\mathcal{K})=\mathrm{Hom}_{\mathcal{K}}(\mathcal{I}^*/\mathcal{I}^{*2},\mathcal{K})$.

The inclusion $T_e\mathbb{G}\hookrightarrow A$ is a morphism of Lie algebras that extends to a morphism of algebras $U(T_e\mathbb{G})\to A$, where $U(T_e\mathbb{G})$ is the universal algebra of $T_e\mathbb{G}$.

Let L be a Lie algebra. $U(L)$ is a quotient of the tensorial algebra of L, $T^{\cdot}L$. It is easy to see, ([12, I.III.4.]) that $S^{\cdot}L$ has a surjective morphism onto the graduated algebra by the filtration of $U(L)$, $\{U(L)_n := \left[\oplus_{i\leq n} T^i L\right]\}$.

Let $\mathbb{G} = \operatorname{Spec} \mathcal{A}^*$ be an infinitesimal formal group. Let us denote $\mathcal{A}_n = (\mathcal{A}^*/\mathcal{I}^{*n+1})^*$. The equality $\mathcal{A}^* = \varprojlim_i \mathcal{A}^*/\mathcal{I}^{*i}$ is equivalent to the equality $\mathcal{A} = \varinjlim_i \mathcal{A}_i$. Observe that $A_i \cdot A_j \subseteq A_{i+j}$: let $c\colon \mathcal{A}^* \to \mathcal{A}^* \tilde{\otimes} \mathcal{A}^*$ be the comultiplication. Then, $c(\mathcal{I}^*) \subset \mathcal{I}^* \tilde{\otimes} \mathcal{A}^* + \mathcal{A}^* \tilde{\otimes} \mathcal{I}^*$, so that $c(\mathcal{I}^{*i+j+1}) \subseteq \mathcal{I}^{*i+1} \tilde{\otimes} \mathcal{A}^* + \mathcal{A}^* \tilde{\otimes} \mathcal{I}^{*j+1}$. The dual morphism of

$$\mathcal{A}^*/\mathcal{I}^{*i+j+1} \overset{c}{\to} \mathcal{A}^*/\mathcal{I}^{*i+1} \tilde{\otimes} \mathcal{A}^*/\mathcal{I}^{*j+1}$$

is the multiplication morphism $A_i \otimes A_j \to A_{i+j}$. The morphism $U(L) \to A$ maps $U(L)_1$ into A_1, so $U(L)_n$ maps into A_n. Lastly, it is easy to check that $\mathcal{A}_n/\mathcal{A}_{n-1} = (\mathcal{I}^{*n}/\mathcal{I}^{*n+1})^* =: \mathcal{L}_n$.

Theorem 9.13. *Let $\mathbb{G} = \operatorname{Spec} \mathcal{A}^*$ an infinitesimal formal group, and write $L := T_e\mathbb{G}$. Then,*

(1) *$U(L) \to A$ is an isomorphism of bialgebras.*
(2) *The morphism $U(L)_n/U(L)_{n-1} \to A_n/A_{n-1}$ is an isomorphism.*
(3) *L is the module of primitive elements of $U(L)$ and $U(L)_n/U(L)_{n-1} = S^n L$.*

Proof. From the commutative diagram

$$\begin{array}{ccc} S^n L & \overset{9.11}{=\!=\!=\!=} & L_n \\ & \underset{\text{surj}}{\searrow} & \uparrow \\ & & U(L)_n/U(L)_{n-1} \end{array}$$

it easily follows (2). By induction on n, it is easy to see that $U(L)_n \to A_n$ is an isomorphism, then $U(L) \to A$ is an isomorphism.

Moreover, $U(L) \to A$ is a morphism of coalgebras because it maps L, that are primitive elements of $U(L)$, into primitive elements of A and $U(L)$ is generated algebraically by L. Finally, the module of primitive elements of A is L, so the module of primitive elements of $U(L)$ is precisely L.

□

Corollary 9.14. *Let $\mathbb{G} = \operatorname{Spec} \mathcal{A}^*$, $\mathbb{G}' = \operatorname{Spec} \mathcal{B}^*$ be infinitesimal formal groups. Then,*

$$\operatorname{Hom}_{grp}(\mathbb{G}, \mathbb{G}') = \operatorname{Hom}_{Lie}(T_e\mathbb{G}, T_e\mathbb{G}')$$

Proof. It follows from:

$$\begin{aligned} \operatorname{Hom}_{grp}(\mathbb{G}, \mathbb{G}') &= \operatorname{Hom}_{bialg}(\mathcal{B}^*, \mathcal{A}^*) = \operatorname{Hom}_{bialg}(A, B) \\ &= \operatorname{Hom}_{bialg}(U(T_e\mathbb{G}), U(T_e\mathbb{G}')) = \operatorname{Hom}_{Lie}(T_e\mathbb{G}, T_e\mathbb{G}') \end{aligned}$$

□

Note 9.15. *If L is a Lie algebra, consider $\mathbb{G} = \operatorname{Spec} \mathcal{U}(\mathcal{L})^*$. Let $\bar{L} = T_e\mathbb{G}$, that is, the primitive elements of $U(L)$. We have a natural morphism $L \to \bar{L}$. With* *adequate basis in* L *and* $\bar{L}$ *we have the commutative diagram:*

$$\begin{array}{ccc} S^{\cdot}L & \longrightarrow & S^{\cdot}\bar{L} \\ \downarrow{\scriptstyle surj} & & \|{\scriptstyle 9.13(2)} \\ U(L) & \overset{9.13(1)}{=\!=\!=} & U(\bar{L}) \end{array}$$

that allows to prove that the morphism $L \to \bar{L}$ *is surjective.*

Let us also outline very briefly that the morphism $L \to \bar{L}$*,* $D \mapsto \bar{D}$ *is injective (see* [12, 5.4]*). We only have to prove that there exists a faithful linear representation of* L*, since* $U(L) = U(\bar{L})$*. If* L *is commutative, then* $S^{\cdot}L = U(L)$ *and the morphism* $L \to U(L)$ *is injective. Let* Z *be the kernel of the surjection* $L \to \bar{L}$ *(notice that* $[L, Z] = 0$*). Let* $\mathbb{G}_Z$ *and* $\mathbb{G}_{\bar{L}}$ *be the formal groups associated to* Z *and* $\bar{L}$*. It is enough to see that there exists a morphism of Lie algebras* $L \to \mathrm{Der}_K(\mathbb{G}_Z \times \mathbb{G}_{\bar{L}})$ *injective. To do that, it is enough to prove that there exists a section of Lie algebras* $w\colon \bar{L}\otimes_K U(\bar{L})^* \to L\otimes_K U(\bar{L})^*$ *of the natural surjection* $L\otimes_K U(\bar{L})^* \to \bar{L}\otimes_K U(\bar{L})^*$*. Let* $s\colon \bar{L} \to L$ *be any* K*-linear section. It can be checked that the 2-form of* $\mathbb{G}_{\bar{L}}$ *with values in* Z*,* $w_2\colon \bar{L} \times \bar{L} \to Z$*,* $w_2(\bar{D}, \bar{D}') = s([\bar{D}, \bar{D}']) - [D, D']$ *is closed. By Poincare Lemma, there exists a 1-form of* $\mathbb{G}_{\bar{L}}$ *with values in* Z*,* $w'\colon \bar{L} \otimes_K U(\bar{L})^* \to Z \otimes_K U(\bar{L})^*$*, such that* $dw' = w_2$*. The section of Lie algebras that we were looking for is* $w = s + w'$*.*

Theorem 9.16. *The category of infinitesimal formal groups is equivalent to the category of Lie algebras.*

Proof. The functors giving the equivalence assign to each infinitesimal formal group $\mathbb{G}$ its tangent space at the identity $T_e\mathbb{G}$ and to each Lie algebra L, the group $\mathrm{Spec}\,\mathcal{U}(L)^*$. □

Corollary 9.17. *The category of linear representations of an infinitesimal formal group* $\mathbb{G}$ *is equivalent to the category of linear representations of its Lie algebra* $T_e\mathbb{G}$*.*

Proof. The category of linear representations of the formal group $\mathbb{G} = \mathrm{Spec}\,\mathcal{A}^*$ is equivalent to the category of A-modules, that is equivalent to the category of linear representations of the Lie algebra $T_e\mathbb{G}$, because A is the universal algebra associated to $T_e\mathbb{G}$. □

Let $G = \mathrm{Spec}\,A$ be an affine K-group scheme and I_e the ideal of functions that vanish at the identity element of G. Let J be the set of ideals of finite codimension of A that are included in I_e and let us denote $\mathrm{Dist}(G) := \lim\limits_{\substack{\to \\ I\in J}} (A/I)^*$.

Corollary 9.18. *Let* $G = \mathrm{Spec}\,A$ *be an affine* K*-group scheme. There exists a canonical isomorphism of bialgebras:*

$$U(T_eG) = \mathrm{Dist}\,G$$

Therefore, $U(T_eG)^* = \hat{A}$ *and the infinitesimal formal group associated to* T_eG *is* $\hat{G}$*.*

Proof. Let $\hat{\mathcal{A}} := \lim\limits_{\substack{\leftarrow \\ I\in J}} \mathcal{A}/\mathcal{I}$ and $\hat{G} = \operatorname{Spec} \hat{\mathcal{A}}$. Observe that $\operatorname{Hom}_{\mathcal{K}}(\hat{\mathcal{A}}, \mathcal{K}) = \operatorname{Dist} G$. Moreover,

$$T_e G = \operatorname{Hom}_{\operatorname{Spec} K}(\operatorname{Spec} K[x]/(x^2), G) = \operatorname{Hom}_{\operatorname{Spec} \mathcal{K}}(\operatorname{Spec} \mathcal{K}[x]/(x^2), \hat{G}) = T_e \hat{G}$$

Therefore, by Theorem 9.13, $\operatorname{Dist} G = U(T_e\hat{G}) = U(T_e G)$. □

(See [4, III.6.1], where G is algebraic).

Corollary 9.19. *If $G = \operatorname{Spec} A$ is a commutative unipotent K-group, then it is isomorphic to $\mathcal{V}^*$, where $V = T_e G^\vee$.*

Proof. G is a commutative unipotent K-group if and only if $G^\vee$ is a commutative infinitesimal formal group. By Theorem 9.13, $G^\vee = \operatorname{Spec}(U(T_e G^\vee))^*$. As $T_e G^\vee \subset A$ is a trivial Lie algebra, $G = \operatorname{Spec} U(T_e G^\vee) = \operatorname{Spec} S^\cdot(T_e G^\vee) = \mathcal{V}^*$. □

9.3. **Another examples of affine functors.**

Definition 9.20. *Let $\mathbb{M}$ be a functor of $\mathcal{R}$-modules and let $\mathbb{A}_{\mathbb{M}}$ be the functor of functions of $\mathbb{X} = \mathbb{M}$. We define*

$$[\mathbb{A}_{\mathbb{M}}]_n := \{F \in \mathbb{A}_{\mathbb{M}} \colon F(\lambda \cdot m) = \lambda^n \cdot F(m), \text{ for all } \lambda \in \mathcal{R} \text{ and } m \in \mathbb{M}\}.$$

Proposition 9.21. *Let $\mathbb{M}$ be a functor of $\mathcal{R}$-modules. Then,*

$$\oplus_n [\mathbb{A}_{\mathbb{M}}]_n \subseteq \mathbb{A}_{\mathbb{M}} \subseteq \prod_n [\mathbb{A}_{\mathbb{M}}]_n,$$

which are inclusions of $\prod_{\mathbb{N}} \mathcal{R}$-modules. Specifically,

$$\mathbb{A}_{\mathbb{M}} = \{\sum_n F_n \in \prod_n [\mathbb{A}_{\mathbb{M}}]_n : \text{for each } m \in \mathbb{M} \text{ there exists } r \in \mathbb{N} \text{ such that } F_n(m) = 0, \text{ for all } n > r\}$$

Proof. Given $m \in \mathbb{M}$, let $G^m \colon \mathcal{R} \to \mathcal{R}$ be defined by $G^m(\lambda) = F(\lambda m)$. Then, $G^m(x) = \sum_n r_n^m x^n \in \mathbb{A}_{\mathcal{R}} = \mathcal{R}[x]$ ($r_n^m = 0$, for all $n >> 0$) and $F(\lambda m) = \sum_n r_n^m \lambda^n$. Let $F_n \colon \mathbb{M} \to \mathcal{R}$ be defined by $F_n(m) = r_n^m$ (given m, $F_n(m) = 0$, for all $n >> 0$). Observe that $F(\lambda(\mu m)) = \sum_n r_n^m (\lambda\mu)^n = \sum_n (r_n^m \mu^n) \cdot \lambda^n$, then $F_n(\mu m) = r_n^m \mu^n = \mu^n F_n(m)$ and $F_n \in [\mathbb{A}_{\mathbb{M}}]_n$. Moreover, $F(m) = \sum_n r_n^m = \sum_n F_n(m)$.

Finally, let $(w_n) \in \prod_n [\mathbb{A}_{\mathbb{M}}]_n$ such that for each $m \in \mathbb{M}$ there exists $r \in \mathbb{N}$ so that $w_n(m) = 0$, for all $n > r$. If $F := \sum_n w_n = 0$ then $w_n = 0$, for all n: $0 = (\sum_n w_n)(\lambda m) = \sum_n w_n(m)\lambda^n$, for all λ, then $w_n(m) = 0$ for all n, and $w_n = 0$ for all n.

□

Proposition 9.22. *Let R be a commutative $\mathbb{Q}$-algebra. Let $\mathbb{M}$ be a functor of $\mathcal{R}$-modules. Then,*

$$[\mathbb{A}_{\mathbb{M}}]_n = (S_n \mathbb{M})^*.$$

Proof. Let $i \colon \mathbb{M} \to S_n\mathbb{M}$ be the morphism of functors of sets defined by $i(m) = m \otimes \overset{n}{\cdots} \otimes m$ and $i^* \colon \mathbb{A}_{S_n\mathbb{M}} \to \mathbb{A}_{\mathbb{M}}$ the morphism induced over the functors of functions. If $w \in (S_n\mathbb{M})^*$ then $i^*(w) = w \circ i \in [\mathbb{A}_{\mathbb{M}}]_n$. If $i^*(w) = w \circ i = 0$ then $w = 0$: By hypothesis, $w(m \otimes \overset{n}{\cdots} \otimes m) = 0$, for all $m \in \mathbb{M}$. Given $m_1, \dots, m_n \in \mathbb{M}$

and $m=\sum_i \lambda_i m_i$ (λ_i being variables), then $0=w(m\otimes\overset{n}{\cdots}\otimes m)=\sum_{|\alpha|=n}\lambda^\alpha w(m^\alpha)$. Hence, $w(m^\alpha)=0$, and $w=0$.

Now, let $F\in[\mathbb{A}_{\mathbb{M}}]_n$. Given $m_1,\dots,m_r\in\mathbb{M}$ let $G\colon\mathcal{R}^r\to\mathcal{R}$ be defined by $G((\lambda_i))=F(\sum_i\lambda_i m_i)$. Then, $G=\sum_{|\alpha|=n}r_\alpha x^\alpha\in[\mathbb{A}_{\mathcal{R}^r}]_n=[\mathcal{R}[x_1,\dots,x_r]]_n$. That is, $F(\sum_i\lambda_i m_i)=\sum_{|\alpha|=n}r_\alpha\lambda^\alpha$. Let

$$F_n\colon\mathbb{M}\times\overset{n}{\cdots}\times\mathbb{M}\to\mathcal{R},\ F_n(m_1,\dots,m_n):=r_{(1,\overset{n}{\dots},1)}$$

Let us check that F_n is a symmetric n-multilinear mapping of $\mathbb{M}^n$, that is, $F_n\in(S^n\mathbb{M})^*$. Obviously F_n is symmetric. $F(\lambda_1m_1+\cdots+\lambda_n\mu m_n)=\sum_{|\alpha|=n}r_\alpha\mu^{\alpha_n}\lambda^\alpha$, then $F_n(m_1,\cdots,\mu m_n)=\mu\cdot F_n(m_1,\dots,m_n)$. Let us write

$$\begin{aligned}F(\lambda_1m_1+\cdots+\lambda_{n+1}m_{n+1})&=\sum_{|\beta|=n}a_\beta\lambda^\beta\\&=\lambda_1\cdots\lambda_{n-1}\cdot(a_{(1,\dots,1,1,0)}\lambda_n+a_{(1,\dots,1,0,1)}\lambda_{n+1})\\&+\sum_{|\beta|=n,(\beta_1,\dots,\beta_{n-1})\neq(1,\dots,1)}a_\beta\lambda^\beta\end{aligned}$$

Considering $\lambda_{n+1}=0$ we obtain $F_n(m_1,\dots,m_{n-1},m_n)=a_{(1,\dots,1,1,0)}$. Considering $\lambda_n=0$ we obtain $F_n(m_1,\dots,m_{n-1},m_{n+1})=a_{(1,\dots,1,0,1)}$. Considering $\lambda_n=\lambda_{n+1}$ we obtain $F_n(m_1,\dots,m_n+m_{n+1})=a_{(1,\dots,1,1,0)}+a_{(1,\dots,1,0,1)}$. Hence F_n is linear.

Let $\tilde F_n\colon\mathbb{M}\otimes\overset{n}{\cdots}\otimes\mathbb{M}\to\mathcal{R}$ be the morphism defined by F_n. Let us prove that $n!\cdot F$ is the composite morphism

$$\mathbb{M}\overset{i}{\to}S_n\mathbb{M}\subset\mathbb{M}\otimes\overset{n}{\cdots}\otimes\mathbb{M}\overset{\tilde F_n}{\to}\mathcal{R}$$

Given $m\in\mathbb{M}$, write $a_n:=F(m)$. Then, $F(\lambda\cdot m)=\lambda^n a_n$. $F(\lambda_1m+\cdots+\lambda_nm)=F((\lambda_1+\cdots+\lambda_n)m)=(\lambda_1+\cdots+\lambda_n)^na_n=n!(\lambda_1\cdots\lambda_n)a_n+\cdots$, hence, $(\tilde F_n\circ i)(m)=F_n(m,\overset{n}{\dots},m)=n!a_n=n!\cdot F(m)$.

□

Proposition 9.23. *Let M be a flat R-module. Then,*

$$[\mathbb{A}_{\mathcal{M}}]_n=(S_n\mathcal{M})^*,$$

which is a reflexive functor of modules.

Proof. By Govorov-Lazard Theorem ([8, A6.6]), M is a direct limit of free modules of finite type, $M=\varinjlim_i V_i$. Then,

$$[\mathbb{A}_{\mathcal{M}}]_n=\varprojlim_i[\mathbb{A}_{\mathcal{V}_i}]_n=\varprojlim_i((S_n\mathcal{V}_i)^*)=(\varinjlim_i S_n\mathcal{V}_i)^*=(S_n\mathcal{M})^*$$

Observe that $S_n\mathcal{M}$ is a quasi-coherent module because it is a direct limit of quasi-coherent modules. Hence, $(S_n\mathcal{M})^*$ is reflexive.

□

Proposition 9.24. *Let R be a commutative $\mathbb{Q}$-algebra. Let $\mathcal{M}$ be a quasi-coherent $\mathcal{R}$-module, then $\mathbb{X}=\mathcal{M}$ is an affine functor.*

Proof. By [10, 5.1] and Lemma 9.22, $\mathbb{A}_{\mathcal{M}}$ is reflexive.

A morphism $\phi\colon\mathbb{A}_{\mathcal{M}}\to\mathcal{R}$ is determined by the restriction of ϕ on $\oplus_n(S_n\mathcal{M})^*$ (see [10, 5.1]). Observe that $(S^\cdot\mathcal{M}^*)^{**}=(\prod_n S_n\mathcal{M})^*=\oplus_n(S_n\mathcal{M})^*$. Then,

$$\begin{aligned}\operatorname{Spec}\mathbb{A}_{\mathcal{M}} &\subseteq \mathbb{H}om_{\mathcal{R}-alg}(\oplus_n(S_n\mathcal{M})^*,\mathcal{R}) \overset{2.13}{=} \mathbb{H}om_{\mathcal{R}-alg}(S^{\cdot}\mathcal{M}^*,\mathcal{R}) = \mathbb{H}om_{\mathcal{R}}(\mathcal{M}^*,\mathcal{R})\\ &= \mathcal{M}\end{aligned}$$

The composition of the natural morphism $\mathcal{M} \to \operatorname{Spec}\mathbb{A}_{\mathcal{M}}$ with the inclusion $\operatorname{Spec}\mathbb{A}_{\mathcal{M}} \subset \mathcal{M}$ is the identity morphism. Therefore, $\operatorname{Spec}\mathbb{A}_{\mathcal{M}} = \mathcal{M}$. □

Proposition 9.25. *Let M be a flat R-module. Then, $\mathcal{M}$ is an affine functor.*

Proof. Proceed as in the proof of Proposition 9.24. □

Theorem 9.26. *Let $\mathbb{M} \in \mathfrak{F}$. Then, $\mathbb{M}$ is an affine functor.*

Proof. There exist inclusions $\oplus_I\mathcal{R} \subseteq \mathbb{M} \subseteq \prod_I \mathcal{R}$, which are morphisms of $\prod_I \mathcal{R}$-modules.

Let us prove that $\mathbb{A}_{\mathbb{M}} \subseteq \mathbb{A}_{\oplus_I\mathcal{R}}$: Let $f \in \mathbb{A}_{\mathbb{M}}$ such that $f_{|\oplus_I\mathcal{R}} = 0$. Given $m = (r_i)_{i\in I} \in \mathbb{M} \subseteq \prod_I \mathcal{R}$ let $F\colon \prod_I \mathcal{R} \to \mathcal{R}$ be defined by $F((\lambda_i)) = f((\lambda_i r_i))$. Then, $F \in \mathbb{A}_{\prod_I \mathcal{R}} = S^{\cdot}(\oplus_I\mathcal{R}) = \mathcal{R}[x_i]_{i\in I}$. Hence, there exists a finite subset $J \subseteq I$ such that $F \in \mathcal{R}[x_j]_{j\in J}$. Therefore $F((\lambda_i r_i)_{i\in I}) = F((\lambda_j r_j)_{j\in J})$ and $f((r_i)_{i\in I}) = f((r_j)_{j\in J}) = 0$, that is, $f = 0$.

In conclusion we have

$$\underset{n\in\mathbb{N}}{\oplus} S^n(\oplus_I\mathcal{R}) = \mathbb{A}_{\prod_I \mathcal{R}} \subseteq \mathbb{A}_{\mathbb{M}} \subseteq \mathbb{A}_{\oplus_I\mathcal{R}} \overset{9.23}{\subseteq} \prod_{n\in\mathbb{N}} (S_n(\oplus_I\mathcal{R}))^*$$

Then, $\mathbb{A}_{\mathbb{M}} \in \mathfrak{F}$ and it is reflexive.

Any morphism of $\mathcal{R}$-modules $\phi\colon \mathbb{A}_{\mathbb{M}} \to \mathcal{R}$ is determined by its restriction to $S^{\cdot}(\oplus_I\mathcal{R})$, by [10, 5.1]. Then, any morphism of $\mathcal{R}$-algebras $\varphi\colon \mathbb{A}_{\mathbb{M}} \to \mathcal{R}$ is determined by its restriction to $\oplus_I\mathcal{R}$. Since $\oplus_I\mathcal{R} = (\prod_I \mathcal{R})^* \subseteq \mathbb{M}^*$, φ is determined by its restriction to $\mathbb{M}^*$. Hence, $\operatorname{Spec}\mathbb{A}_{\mathbb{M}} \subseteq \mathbb{M}^{**} = \mathbb{M}$. Then, $\operatorname{Spec}\mathbb{A}_{\mathbb{M}} = \mathbb{M}$.

□

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