

A Note on Some Isomorphic Properties in Projective Tensor Products

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Abstract: A Banach space X is sequentially Right (resp. weak sequentially Right) if every Right subset of X^* is relatively weakly compact (resp. weakly precompact). A Banach space X has the L -limited (resp. the wL -limited) property if every L -limited subset of X^* is relatively weakly compact (resp. weakly precompact). We study Banach spaces with the weak sequentially Right and the wL -limited properties. We investigate whether the projective tensor product of two Banach spaces X and Y has the sequentially Right property when X and Y have the respective property.

Key words: R -sets, L -limited sets, sequentially Right spaces, L -limited property.

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1. INTRODUCTION

A bounded subset A of a Banach space X is called a *Dunford-Pettis* (DP) (resp. *limited*) subset of X if every weakly null (resp. w^* -null) sequence (x_n^*) in X^* tends to 0 uniformly on A ; i.e.,

$$\lim_n \left(\sup \{ |x_n^*(x)| : x \in A \} \right) = 0.$$

A sequence (x_n) is DP (resp. limited) if the set $\{x_n : n \in \mathbb{N}\}$ is DP (resp. limited).

A subset S of X is said to be *weakly precompact* provided that every sequence from S has a weakly Cauchy subsequence. Every DP (resp. limited) set is weakly precompact [37, p. 377], [1] (resp. [4, Proposition]).

An operator $T : X \rightarrow Y$ is called *weakly precompact* (or *almost weakly compact*) if $T(B_X)$ is weakly precompact and *completely continuous* (or *Dunford-Pettis*) if T maps weakly convergent sequences to norm convergent sequences.

In [35] the authors introduced the *Right topology* on a Banach space X . It is the restriction of the Mackey topology $\tau(X^{**}, X)$ to X and it is also the

topology of uniform convergence on absolutely convex $\sigma(X^*, X^{**})$ compact subsets of X^* . Further, $\tau(X^{**}, X)$ can also be viewed as the topology of uniform convergence on relatively $\sigma(X^*, X^{**})$ compact subsets of X^* [26].

A sequence (x_n) in a Banach space X is Right null if and only if it is weakly null and DP (see Proposition 1).

An operator $T : X \rightarrow Y$ is called *pseudo weakly compact* (*pwc*) (or *Dunford-Pettis completely continuous* (*DPcc*)) if it takes Right null sequences in X into norm null sequences in Y ([35], [25]). Every completely continuous operator $T : X \rightarrow Y$ is pseudo weakly compact. If $T : X \rightarrow Y$ is an operator with weakly precompact adjoint, then T is a pseudo weakly compact operator ([18, Corollary 5]).

A subset K of X^* is called a *Right set* (*R-set*) if each Right null sequence (x_n) in X tends to 0 uniformly on K [26]; i.e.,

$$\lim_n \left(\sup\{|x^*(x_n)| : x^* \in K\} \right) = 0.$$

A Banach space X is said to be *sequentially Right* (*SR*) (has *property* (*SR*)) if every pseudo weakly compact operator $T : X \rightarrow Y$ is weakly compact, for any Banach space Y [35]. Banach spaces with property (*V*) are sequentially Right ([35, Corollary 15]).

A subset A of a dual space X^* is called an *L-limited set* if every weakly null limited sequence (x_n) in X converges uniformly on A [39]; i.e.,

$$\lim_n \left(\sup\{|x^*(x_n)| : x^* \in A\} \right) = 0.$$

A Banach space X has the *L-limited property* if every *L-limited* subset of X^* is relatively weakly compact [39]. An operator $T : X \rightarrow Y$ is called *limited completely continuous* (*lcc*) if T maps weakly null limited sequences to norm null sequences [40].

In this paper we introduce the weak sequentially Right (*wSR*) and *wL-limited* properties. A Banach space X is said to have the *weak sequentially Right* (*wSR*) (resp. the *wL-limited*) *property* if every Right (resp. *L-limited*) subset of X^* is weakly precompact. We obtain some characterizations of these properties with respect to some geometric properties of Banach spaces, such as the Gelfand-Phillips property, the Grothendieck property, and properties (*wV*) and (*wL*). We generalize some results from [39]. We also show that property (*SR*) can be lifted from a certain subspace of X to X .

We study whether the projective tensor product $X \otimes_{\pi} Y$ has the (*SR*) (resp. the *L-limited*) property if $L(X, Y^*) = K(X, Y^*)$, and X and Y have the respective property. We prove that in some cases, if $X \otimes_{\pi} Y$ has the (*wSR*) property, then $L(X, Y^*) = K(X, Y^*)$.

2. DEFINITIONS AND NOTATION

Throughout this paper, X , Y , E , and F will denote Banach spaces. The unit ball of X will be denoted by B_X and X^* will denote the continuous linear dual of X . An operator $T : X \rightarrow Y$ will be a continuous and linear function. We will denote the canonical unit vector basis of c_0 by (e_n) and the canonical unit vector basis of ℓ_1 by (e_n^*) . The set of all operators, weakly compact operators, and compact operators from X to Y will be denoted by $L(X, Y)$, $W(X, Y)$, and $K(X, Y)$. The projective tensor product of X and Y will be denoted by $X \otimes_\pi Y$.

A bounded subset A of X^* is called an L -set if each weakly null sequence (x_n) in X tends to 0 uniformly on A ; i.e.,

$$\lim_n \left(\sup \{ |x^*(x_n)| : x^* \in A \} \right) = 0.$$

A Banach space X has the *Dunford-Pettis property* (DPP) if every weakly compact operator $T : X \rightarrow Y$ is completely continuous, for any Banach space Y . Schur spaces, $C(K)$ spaces, and $L_1(\mu)$ spaces have the DPP . The reader can check [8], [9], and [10] for a guide to the extensive classical literature dealing with the DPP .

A Banach space X has the *Dunford-Pettis relatively compact property* ($DPrCP$) if every Dunford-Pettis subset of X is relatively compact [14]. Schur spaces have the $DPrCP$. The space X does not contain a copy of ℓ_1 if and only if X^* has the $DPrCP$ if and only if every L -set in X^* is relatively compact ([14, Theorem 1], [13, Theorem 2]).

The space X has the *Gelfand-Phillips* (GP) property if every limited subset of X is relatively compact. The following spaces have the Gelfand-Phillips property: Schur spaces; spaces with w^* -sequential compact dual unit balls (for example subspaces of weakly compactly generated spaces, separable spaces, spaces whose duals have the Radon-Nikodým property, reflexive spaces, and spaces whose duals do not contain ℓ_1); dual spaces X^* which X not containing ℓ_1 ; Banach spaces with the separable complementation property, i.e., every separable subspace is contained in a complemented separable subspace (for example $L_1(\mu)$ spaces, where μ is a positive measure) [42, p. 31], [4, Proposition], [12, Theorem 3.1 and p. 384], [11, Proposition 5.2], [13, Corollary 5].

A series $\sum x_n$ in X is said to be *weakly unconditionally convergent* (wuc) if for every $x^* \in X^*$, the series $\sum |x^*(x_n)|$ is convergent. An operator $T : X \rightarrow Y$ is called *unconditionally converging* if it maps weakly unconditionally convergent series to unconditionally convergent ones.

A bounded subset A of X^* is called a V -subset of X^* provided that

$$\lim_n \left(\sup \{ |x^*(x_n)| : x^* \in A \} \right) = 0$$

for each wuc series $\sum x_n$ in X .

A Banach space X has *property (V)* if every V -subset of X^* is relatively weakly compact [33]. A Banach space X has *property (V)* if every unconditionally converging operator T from X to any Banach space Y is weakly compact [33, Proposition 1]. $C(K)$ spaces and reflexive spaces have *property (V)* ([33, Theorem 1, Proposition 7]). A Banach space X has *property (wV)* if every V -subset of X^* is weakly precompact [41].

A Banach space X has the *reciprocal Dunford-Pettis property (RDPP)* if every completely continuous operator T from X to any Banach space Y is weakly compact. The space X has the *RDPP* if and only if every L -set in X^* is relatively weakly compact [28]. Banach spaces with *property (V)* have the *RDPP* [33]. A Banach space X has *property (wL)* if every L -set in X^* is weakly precompact [19].

A topological space S is called *dispersed (or scattered)* if every nonempty closed subset of S has an isolated point. A compact Hausdorff space K is dispersed if and only if $\ell_1 \not\hookrightarrow C(K)$ [34, Main theorem].

The Banach-Mazur distance $d(X, Y)$ between two isomorphic Banach spaces X and Y is defined by $\inf(\|T\|\|T^{-1}\|)$, where the infimum is taken over all isomorphisms T from X onto Y . A Banach space X is called an \mathcal{L}_∞ -space (resp. \mathcal{L}_1 -space) [5, p. 7] if there is a $\lambda \geq 1$ so that every finite dimensional subspace of X is contained in another subspace N with $d(N, \ell_\infty^n) \leq \lambda$ (resp. $d(N, \ell_1^n) \leq \lambda$) for some integer n . Complemented subspaces of $C(K)$ spaces (resp. $L_1(\mu)$ spaces) are \mathcal{L}_∞ -spaces (resp. \mathcal{L}_1 -spaces) ([5, Proposition 1.26]). The dual of an \mathcal{L}_1 -space (resp. \mathcal{L}_∞ -space) is an \mathcal{L}_∞ -space (resp. \mathcal{L}_1 -space) ([5, Proposition 1.27]). The \mathcal{L}_∞ -spaces, \mathcal{L}_1 -spaces, and their duals have the *DPP* ([5, Corollary 1.30]).

3. THE WEAK SEQUENTIALLY RIGHT AND wL -LIMITED PROPERTIES

The following result gives a characterization of Right null sequences.

PROPOSITION 1. *A sequence (x_n) in a Banach space X is Right null if and only if it is weakly null and DP.*

Proof. Suppose that (x_n) is a Right null sequence in X . Then (x_n) is weakly null, since the Right topology is stronger than the weak topology.

Let (x_n^*) be a weakly null sequence in X^* . Since $\{x_n^* : n \in \mathbb{N}\}$ is relatively weakly compact in X^* and (x_n) is Right null, (x_n) converges uniformly on $\{x_n^* : n \in \mathbb{N}\}$. Therefore $\lim_n \sup_i |x_i^*(x_n)| = 0$, and thus $\lim_n |x_n^*(x_n)| = 0$. Hence $\{x_n : n \in \mathbb{N}\}$ is a DP set.

Suppose that (x_n) is a weakly null DP sequence. Let K be a relatively weakly compact subset of X^* . Suppose that (x_n) does not converge uniformly on K . Let $\epsilon > 0$ and let (x_n^*) be a sequence in K so that $|x_n^*(x_n)| > \epsilon$ for all n . Without loss of generality suppose that (x_n^*) converges weakly to x^* , $x^* \in X^*$. Since $(x_n^* - x^*)$ is weakly null in X^* and (x_n) is DP, $\lim_n (x_n^* - x^*)(x_n) = 0$. Thus $\lim_n x_n^*(x_n) = 0$, a contradiction. Hence (x_n) converges uniformly to zero on K , and thus (x_n) is Right null. ■

A Banach space X is sequentially Right if and only if every Right subset of X^* is relatively weakly compact [26, Theorem 3.25]. A Banach space X has the L -limited property if and only if every limited completely continuous operator $T : X \rightarrow Y$ is weakly compact, for every Banach space Y [39, Theorem 2.8]. In the next theorem we give elementary operator theoretic characterizations of weak precompactness, relative weak compactness, and relative norm compactness for Right sets and L -limited sets. The argument contains the theorems in [26] and [39] just cited.

We say that a Banach space X is *weak sequentially Right* (*wSR*) or has the *(wSR) property* (resp. has the *wL-limited property*) if every Right (resp. L -limited) subset of X^* is weakly precompact. If $\ell_1 \not\hookrightarrow X^*$, then X is weak sequentially Right and has the *wL-limited property*, by Rosenthal's theorem ([8, Ch. XI]).

THEOREM 2. *Let X be a Banach space. The following assertions are equivalent:*

1. (i) *For every Banach space Y , every pseudo weakly compact operator $T : X \rightarrow Y$ has a weakly precompact (weakly compact, resp. compact) adjoint.*
- (ii) *Every pseudo weakly compact operator $T : X \rightarrow \ell_\infty$ has a weakly precompact (weakly compact, resp. compact) adjoint.*
- (iii) *Every Right subset of X^* is weakly precompact (relatively weakly compact, resp. relatively compact).*

2. (i) For every Banach space Y , every limited completely continuous operator $T : X \rightarrow Y$ has a weakly precompact (weakly compact, resp. compact) adjoint.
- (ii) Every limited completely continuous operator $T : X \rightarrow \ell_\infty$ has a weakly precompact (weakly compact, resp. compact) adjoint.
- (iii) Every L -limited subset of X^* is weakly precompact (relatively weakly compact, resp. relatively compact).

Proof. We will show that 1.(i) \Rightarrow 1.(ii) \Rightarrow 1.(iii) \Rightarrow 1.(i) in the weakly precompact case as well as 2.(i) \Rightarrow 2.(ii) \Rightarrow 2.(iii) \Rightarrow 2.(i) in the compact case. These two arguments are similar, and the arguments for the remaining implications of the theorem follow the same pattern.

1. (weakly precompact) (i) \Rightarrow (ii) is clear.

(ii) \Rightarrow (iii) Let K be a Right subset of X^* and let (x_n^*) be a sequence in K . Define $T : X \rightarrow \ell_\infty$ by $T(x) = (x_i^*(x))$. Let (x_n) be a Right null sequence in X . Since K is a Right set,

$$\lim_n \|T(x_n)\| = \lim_n \sup_i |x_i^*(x_n)| = 0.$$

Therefore T is pseudo weakly compact, and thus $T^* : \ell_\infty^* \rightarrow X^*$ is weakly precompact. Hence $(T^*(e_n^*)) = (x_n^*)$ has a weakly Cauchy subsequence.

(iii) \Rightarrow (i) Let $T : X \rightarrow Y$ be a pseudo weakly compact operator. Let (x_n) be a Right null sequence in X . If $y^* \in B_{Y^*}$, $\langle T^*(y^*), x_n \rangle \leq \|T(x_n)\| \rightarrow 0$. Then $T^*(B_{Y^*})$ is a Right subset of X^* . Therefore $T^*(B_{Y^*})$ is weakly precompact, and thus T^* is weakly precompact.

2. (compact) (i) \Rightarrow (ii) is clear.

(ii) \Rightarrow (iii) Let K be an L -limited subset of X^* and let (x_n^*) be a sequence in K . Define $T : X \rightarrow \ell_\infty$ as above and note that T is limited completely continuous. Thus $T^* : \ell_\infty^* \rightarrow X^*$ is compact, and $(T^*(e_n^*)) = (x_n^*)$ has a norm convergent subsequence.

(iii) \Rightarrow (i) Let $T : X \rightarrow Y$ be a limited completely continuous operator. Let (x_n) be a weakly null limited sequence in X . If $y^* \in B_{Y^*}$, $\langle T^*(y^*), x_n \rangle \leq \|T(x_n)\| \rightarrow 0$. Then $T^*(B_{Y^*})$ is an L -limited subset of X^* . Therefore $T^*(B_{Y^*})$ is relatively compact, and thus T^* is compact. ■

COROLLARY 3. *If X is weak sequentially Right (has the wL -limited, resp. the L -limited property), then every quotient space of X has the same property.*

Proof. We only prove the result for the weak sequentially Right property. The proofs for the other properties are similar.

Suppose that X is weak sequentially Right. Let Z be a quotient space of X and $Q : X \rightarrow Z$ be a quotient map. Let $T : Z \rightarrow E$ be a pseudo weakly compact operator. Then $TQ : X \rightarrow E$ is pseudo weakly compact, and thus $(TQ)^*$ is weakly precompact by Theorem 2. Since $Q^*T^*(B_E^*)$ is weakly precompact and Q^* is an isomorphism, $T^*(B_E^*)$ is weakly precompact. Apply Theorem 2. ■

COROLLARY 4. *Suppose X is weak sequentially Right and Y is a Banach space. Then an operator $T : X \rightarrow Y$ is pseudo weakly compact if and only if $T^* : Y^* \rightarrow X^*$ is weakly precompact.*

Proof. If $T : X \rightarrow Y$ is pseudo weakly compact, then $T^* : Y^* \rightarrow X^*$ is weakly precompact by Theorem 2, since X is weak sequentially Right.

The converse follows from [18, Corollary 5]. ■

COROLLARY 5. (i) *If X is weak sequentially Right (resp. has the wL -limited property), then every pseudo weakly compact (resp. limited completely continuous) operator $T : X \rightarrow Y$ is weakly precompact.*

(ii) *If X is an infinite dimensional space with the Schur property, then X is not weak sequentially Right (resp. does not have the wL -limited property).*

(iii) *If X is weak sequentially Right (resp. has the wL -limited property), then $\ell_1 \not\hookrightarrow X$.*

Proof. (i) Suppose X is weak sequentially right (resp. has the wL -limited property). Let $T : X \rightarrow Y$ be pseudo weakly compact (resp. limited completely continuous). Then T^* is weakly precompact by Theorem 2. Hence T is weakly precompact, by [2, Corollary 2].

(ii) Since X has the Schur property, the identity operator $i : X \rightarrow X$ is pseudo weakly compact (resp. limited completely continuous). Since X is an infinite dimensional space with the Schur property, i is not weakly precompact. Apply (i).

(iii) Apply Corollary 3 and (ii). ■

COROLLARY 6. *A Banach space X has the L -limited property if every separable subspace of X has the same property.*

Proof. Let $T : X \rightarrow Y$ be a limited completely continuous operator. Then for every closed subspace Z of X , $T|_Z$ is limited completely continuous. Let

(x_n) be a sequence in B_X and let $Z = [x_n : n \in \mathbb{N}]$ be the closed linear span of (x_n) . Since Z is a separable subspace of X , Z has the L -limited property. Since $T|_Z$ is limited completely continuous, it is weakly compact by Theorem 2. Then there is a subsequence (x_{n_k}) of (x_n) so that $(T(x_{n_k}))$ is weakly convergent. Thus T is weakly compact. Apply Theorem 2. ■

EXAMPLE. Corollary 6 cannot be reversed. Indeed, consider ℓ_1 as a subspace of ℓ_∞ . By [39, Theorem 2.11], ℓ_∞ has the L -limited property. However, ℓ_1 does not have the L -limited property, by [39, Corollary 2.9] (or Corollary 5 (ii)).

THEOREM 7. *The Banach space X has the DPP if and only if every Right subset of X^* is an L -set.*

Proof. Suppose X has the DPP. Then every weakly null sequence (x_n) is DP ([9, Theorem 1]). Therefore every Right subset of X^* is an L -set.

Conversely, let $T : X \rightarrow Y$ be a pseudo weakly compact operator. Then $T^*(B_{Y^*})$ is a Right subset of X^* , hence an L -set. Therefore T is completely continuous, and thus X has the DPP by [26, Proposition 3.17], [25, Theorem 1.5], [18, Theorem 10]. ■

COROLLARY 8. *Suppose that X has the DPP. Then the following are equivalent:*

- (i) X does not contain a copy of ℓ_1 .
- (ii) Every L -set in X^* is relatively compact.
- (iii) Every Right subset of X^* is relatively compact.
- (iv) X^* has the Schur property.

Proof. (i) \Leftrightarrow (ii) by [13, Theorem 2]. (ii) \Leftrightarrow (iii) by Theorem 7. (i) \Leftrightarrow (iv) by [9, p. 23]. ■

COROLLARY 9. *X^* has the Schur property if and only if every Right subset of X^* is relatively compact.*

Proof. If X^* has the Schur property, then X has the DPP and X does not contain a copy of ℓ_1 ([9, p. 23]). Hence every Right subset of X^* is relatively compact by Corollary 8.

Conversely, let (x_n^*) be a weakly Cauchy sequence in X^* . Then (x_n^*) is a Right set, by the proof of [26, Corollary 3.26]. Thus (x_n^*) is relatively compact, and X^* has the Schur property. ■

COROLLARY 10. (i) *Suppose X has the DPP and Y has the DPrCP. Then any operator $T : X \rightarrow Y$ is completely continuous.*

(ii) *The space X has the DPP and the DPrCP if and only if X has the Schur property.*

Proof. (i) Let $T : X \rightarrow Y$ be an operator. Since Y has the DPrCP, T is pseudo weakly compact. Then $T^*(B_{Y^*})$ is a Right set, thus an L -set in X^* (by Theorem 7). Hence T is completely continuous.

(ii) Suppose X has the DPP and the DPrCP. Then the identity operator $i : X \rightarrow X$ is completely continuous by (i). Hence X has the Schur property. If X has the Schur property, then X has the DPP and the DPrCP. ■

Corollary 10(i) generalizes [13, Corollary 6] when Y is a dual space E^* with E not containing ℓ_1 (since E^* has the DPrCP [14, Theorem 1]).

A bounded subset A of X^* is called w^* -sequentially compact if every sequence from A has a subsequence which converges to a point in the w^* -topology of X^* .

The following theorem generalizes [39, Theorem 2.2 (b), (c)].

THEOREM 11. *If (x_n^*) is a w^* -Cauchy sequence in X^* , then $\{x_n^* : n \in \mathbb{N}\}$ is an L -limited set.*

Proof. Suppose that (x_n^*) is a w^* -Cauchy sequence in X^* and $\{x_n^* : n \in \mathbb{N}\}$ is not an L -limited set. By passing to a subsequence if necessary, there is an $\epsilon > 0$ and a weakly null limited sequence (x_n) in X such that $|x_n^*(x_n)| > \epsilon$ for all n . Let $k_1 = 1$ and choose $k_2 > k_1$ so that $|x_{k_1}^*(x_{k_2})| < \epsilon/2$. We can do this since (x_n) is weakly null. Continue inductively. Choose $k_n > k_{n-1}$ so that $|x_{k_{n-1}}^*(x_{k_n})| < \epsilon/2$ for all n . Then

$$|(x_{k_n}^* - x_{k_{n-1}}^*)(x_{k_n})| = |x_{k_n}^*(x_{k_n}) - x_{k_{n-1}}^*(x_{k_n})| > \epsilon/2.$$

This is a contradiction, since $(x_{k_n}^* - x_{k_{n-1}}^*)$ is w^* -null in X^* and (x_{k_n}) is limited in X . ■

A Banach space X has the *Grothendieck property* if every w^* -convergent sequence in X^* is weakly convergent [10, p. 179]. A space X is *weakly sequentially complete* if every weakly Cauchy sequence in X is weakly convergent.

COROLLARY 12. (i) *If X has the L -limited property, then X^* is weakly sequentially complete.*

(ii) ([39, Theorem 2.10]) *If X has the L -limited property, then X is a Grothendieck space.*

Proof. (i) Suppose that X has the L -limited property. Let (x_n^*) be a weakly Cauchy sequence in X^* . By Theorem 11, $\{x_n^* : n \in \mathbb{N}\}$ is an L -limited set, and thus relatively weakly compact. Hence (x_n^*) is weakly convergent.

(ii) Let (x_n^*) be a w^* -convergent sequence in X^* . By Theorem 11, (x_n^*) is an L -limited set, thus relatively weakly compact. Hence (x_n^*) is weakly convergent. ■

COROLLARY 13. (i) *A Banach space X with the Gelfand-Phillips property has the wL -limited property if and only if X^* contains no copy of ℓ_1 .*

(ii) *A Banach space X with the $DPrCP$ has the (wSR) property if and only if X^* contains no copy of ℓ_1 .*

(iii) *If X has the wL -limited property, then c_0 is not complemented in X .*

(iv) ([39, Corollary 2.9]) *A Banach space X is reflexive if and only if it has the Gelfand-Phillips property and the L -limited property.*

(v) ([7, Corollary 17]) *A Banach space X is reflexive if and only if it has the $DPrCP$ and the (SR) property.*

Proof. (i) Suppose that X has the Gelfand-Phillips property and the wL -limited property. Then the identity operator $i : X \rightarrow X$ is limited completely continuous (since X has the Gelfand-Phillips property) and $i^* : X^* \rightarrow X^*$ is weakly precompact by Theorem 2. Hence X^* contains no copy of ℓ_1 , by Rosenthal's ℓ_1 theorem. The converse follows by Rosenthal's ℓ_1 theorem.

(ii) The proof is similar to that of (i).

(iii) Suppose that X has the wL -limited property. Since c_0 is separable, it has the Gelfand-Phillips property [4, Proposition]. By (i), c_0 does not have the wL -limited property. Hence c_0 is not complemented in X by Corollary 3.

(iv) If X is reflexive, then it has the Gelfand-Phillips property [4, Proposition] and the L -limited property. Conversely, X^* contains no copy of ℓ_1 by (i) and X^* is weakly sequentially complete by Corollary 12. Then X^* , thus X , is reflexive.

(v) Suppose X is reflexive. Then X has the (SR) property and X^* does not contain a copy of ℓ_1 . Hence X^{**} , thus X , has the $DPrCP$ ([13, Theorem 2]). Conversely, X^* contains no copy of ℓ_1 by (i) and X^* is weakly sequentially complete by [26, Corollary 3.26]. Then X is reflexive. ■

EXAMPLE. The converse of Corollary 12 (i) does not hold. Let X be the first Bourgain-Delbaen space [5, p. 25]. Then X has the Schur property and X^* is weakly sequentially complete. Since X has the Schur property, X does not have the L -limited property (by Corollary 13 (iv)).

COROLLARY 14. (i) *If X has property (wV) , then X is weak sequentially Right.*

(ii) *If X has the L -limited (resp. the wL -limited) property, then X is sequentially Right (resp. weak sequentially Right).*

(iii) *If X is sequentially Right (resp. weak sequentially Right), then it has the $RDPP$ (resp. property (wL)).*

(iv) *If X is an infinite dimensional space with the L -limited property, then X^* does not have the Schur property.*

Proof. (i) Suppose X has property (wV) . Let $T : X \rightarrow Y$ be pseudo weakly compact. Then T is unconditionally converging [35, Proposition 14]. Hence T^* is weakly precompact [19, Theorem 1]. Apply Theorem 2.

(ii) Suppose X has the L -limited (resp. the wL -limited) property. Let (x_n) be a weakly null limited sequence in X . Then (x_n) is a weakly null DP sequence. Hence every Right subset of X^* is L -limited, thus relatively weakly compact (resp. weakly precompact).

(iii) Suppose X is sequentially Right (resp. weak sequentially Right). Every L -set in X^* is a Right set, thus relatively weakly compact (resp. weakly precompact). Hence X has the $RDPP$ [28] (resp. property (wL)).

(iv) Suppose that X has the L -limited property. Then X has the Grothendieck property, by Corollary 12 (ii). By the Josefson-Nissezweig theorem, there is a w^* -null sequence (x_n^*) in X^* of norm one. Then (x_n^*) is weakly null and not norm null, and X^* does not have the Schur property. ■

The fact that a space with property (SR) has the $RDPP$ was obtained in [26, Corollary 3.3].

EXAMPLE. The converse of Corollary 14 (i) is not true. Let Y be the second Bourgain-Delbaen space [5, p. 25]. The space Y is a non-reflexive \mathcal{L}_∞ -space with the DPP that does not contain c_0 or ℓ_1 and such that $Y^* \simeq \ell_1$. The space Y is sequentially Right by Corollary 8. Since Y does not contain c_0 , the identity operator $i : Y \rightarrow Y$ is unconditionally converging ([8, p. 54]) and $i^* : Y^* \rightarrow Y^*$ is not weakly precompact (since $Y^* \simeq \ell_1$). Thus Y does not have property (wV) by [19, Theorem 1].

The converse of Corollary 14 (ii) (strong properties) is not true. The second Bourgain-Delbaen space Y is sequentially Right and does not have the L -limited property (by Corollary 14 (iv)).

The converse of Corollary 14 (iii) (strong properties) is not true. Let J be the original James space [24]. Since J is separable and 1-codimensional in J^{**} , all duals of J are separable and ℓ_1 fails to embed in any of them. Moreover, none of these spaces can be weakly sequentially complete. Thus J and its duals are weak sequentially Right, but none of these spaces are sequentially Right by [26, Corollary 3.26], since their duals are not weakly sequentially complete. Since J does not contain ℓ_1 , every completely continuous operator on J is compact (by a result of Odell [37, p. 377]), and thus weakly compact. Hence J has the *RDPP*.

The following theorem shows that the space E has property (SR) if some subspace of it has property (SR) .

LEMMA 15. ([23, Theorem 2.7]) *Let E be a Banach space, F a reflexive subspace of E (resp. a subspace not containing copies of ℓ_1), and $Q : E \rightarrow E/F$ the quotient map. Let (x_n) be a bounded sequence in E such that $(Q(x_n))$ is weakly convergent (resp. weakly Cauchy). Then (x_n) has a weakly convergent (resp. weakly Cauchy) subsequence.*

Let E be a Banach space and F be a subspace of E^* . Let

$${}^\perp F = \{x \in E : y^*(x) = 0 \text{ for all } y^* \in F\}.$$

THEOREM 16. (i) *Let E be a Banach space and F be a reflexive subspace of E^* . If ${}^\perp F$ has property (SR) (resp. the L -limited property), then E has the same property.*

(ii) *Let E be a Banach space and F be a subspace of E^* not containing copies of ℓ_1 . If ${}^\perp F$ has property (wSR) (resp. the wL -limited property), then E has the same property.*

Proof. We only prove (i) for the (SR) property. The other proofs are similar.

Suppose that ${}^\perp F$ has property (SR) . Let $Q : E^* \rightarrow E^*/F$ be the quotient map and $i : E^*/F \rightarrow ({}^\perp F)^*$ be the natural surjective isomorphism ([31, Theorem 1.10.16]). It is known that $iQ : E^* \rightarrow ({}^\perp F)^*$ is $w^* - w^*$ continuous, since $iQ(x^*)$ is the restriction of x^* to ${}^\perp F$ ([31, Theorem 1.10.16]). Then there is an operator $S : {}^\perp F \rightarrow E$ such that $iQ = S^*$.

Let $T : E \rightarrow G$ be a pseudo weakly compact operator. Then $TS : {}^\perp F \rightarrow G$ is pseudo weakly compact. Since ${}^\perp F$ has property (SR) , TS has a weakly compact adjoint, by Theorem 2. Since $S^*T^* = iQT^*$ is weakly compact and i is a surjective isomorphism, QT^* is weakly compact. Let (x_n^*) be a sequence in B_{G^*} . By passing to a subsequence, we can assume that $(QT^*(x_n^*))$ is weakly convergent. Hence $(T^*(x_n^*))$ has a weakly convergent subsequence by Lemma 15. Thus E has property (SR) . ■

The $w^* - w$ continuous operators from X^* to Y will be denoted by $L_{w^*}(X^*, Y)$.

THEOREM 17. *Let X be a Banach space and A be a bounded subset of X^* . The following are equivalent:*

- (i) A is an L -limited set.
- (ii) Every operator $T \in L_{w^*}(X^*, c_0)$ that is w^* -norm sequentially continuous maps A into a relatively compact set.

Proof. (i) \Rightarrow (ii) Let $T \in L_{w^*}(X^*, c_0)$ be an operator so that T is w^* -norm sequentially continuous. Note that $T^* \in L_{w^*}(\ell_1, X)$, $(x_n) = (T^*(e_n^*))$ is a weakly null sequence in X , and $T(x^*) = (x^*(x_i))_i$. If (x_n^*) is a w^* -null sequence in X^* and $y \in B_{\ell_1}$, then

$$|\langle x_n^*, T^*(y) \rangle| \leq \|T(x_n^*)\| \rightarrow 0.$$

Hence $T^*(B_{\ell_1})$, thus (x_n) , is limited. Since A is an L -limited set, $\sup_{x^* \in A} |x^*(x_n)| \rightarrow 0$. Therefore $T(A)$ is relatively compact in c_0 , by the characterization of relatively compact subsets of c_0 .

(ii) \Rightarrow (i) Let (x_n) be a weakly null limited sequence in X . Define $T : X^* \rightarrow c_0$ by $T(x^*) = (x^*(x_n))_n$. Note that $T^*(b) = \sum b_n x_n$, $b = (b_n) \in \ell_1$, $T^*(\ell_1) \subseteq X$, and $T \in L_{w^*}(X^*, c_0)$. If (x_n^*) is a w^* -null sequence in X^* , then

$$\|T(x_n^*)\| = \sup_i |x_n^*(x_i)| \rightarrow 0,$$

since (x_i) is limited. Hence T is w^* -norm sequentially continuous operator, and $T(A)$ is relatively compact in c_0 . By the characterization of relatively compact subsets of c_0 , $\sup_{x^* \in A} |x^*(x_n)| \rightarrow 0$, and thus A is an L -limited subset of X^* . ■

An operator $T : X \rightarrow Y$ is called *limited* if $T(B_X)$ is a limited subset of Y ([4]). The operator T is limited if and only if $T^* : Y^* \rightarrow X^*$ is w^* -norm sequentially continuous.

COROLLARY 18. *Let X be a Banach space and A be a bounded subset of X^* . The following are equivalent:*

- (i) *A is an L -limited set.*
- (ii) *For every limited operator $S \in L_{w^*}(\ell_1, X)$, $S^*(A)$ is relatively compact.*

Proof. (i) \Rightarrow (ii) Let $S \in L_{w^*}(\ell_1, X)$ be a limited operator. Then $S^* \in L_{w^*}(X^*, c_0)$ and S^* is w^* -norm sequentially continuous. By Theorem 17, $S^*(A)$ is relatively compact.

(ii) \Rightarrow (i) Let $T \in L_{w^*}(X^*, c_0)$ be a w^* -norm sequentially continuous operator and let $S = T^*$. Then $S \in L_{w^*}(\ell_1, X)$, S is limited, and $S^*(A)$ is relatively compact. By Theorem 17, A is an L -limited set. ■

COROLLARY 19. *Suppose that A is a bounded subset of X^* such that for every $\epsilon > 0$, there is an L -limited subset A_ϵ of X^* such that $A \subseteq A_\epsilon + \epsilon B_{X^*}$. Then A is an L -limited set.*

Proof. Suppose that A satisfies the hypothesis. Let $\epsilon > 0$ and A_ϵ as in the hypothesis. Let $T \in L_{w^*}(X^*, c_0)$ be an operator such that T is w^* -norm sequentially continuous and $\|T\| \leq 1$. Then $T(A) \subseteq T(A_\epsilon) + \epsilon B_{c_0}$, and $T(A_\epsilon)$ is relatively compact by Theorem 17. Then $T(A)$ is relatively compact [8, p. 5], and thus A is an L -limited set by Theorem 17. ■

4. THE (wSR) AND wL -LIMITED PROPERTIES IN PROJECTIVE TENSOR PRODUCTS

In this section we consider the (SR) and L -limited properties in the projective tensor product $X \otimes_\pi Y$. We begin by noting that there are examples of Banach spaces X and Y such that $X \otimes_\pi Y$ has the (SR) and L -limited properties. If $1 < q' < p < \infty$, then $L(\ell_p, \ell_{q'}) = K(\ell_p, \ell_{q'})$ ([36], [10, p. 247]). If q is the conjugate of q' , then $\ell_p \otimes_\pi \ell_q$ is reflexive (by [38, Theorem 4.19], [10, p. 248]), and thus has the (SR) and L -limited properties. Then the spaces $X = \ell_p$ and $Y = \ell_q$ are as desired.

If $H \subseteq L(X, Y)$, $x \in X$ and $y^* \in Y^*$, let $H(x) = \{T(x) : T \in H\}$ and $H^*(y^*) = \{T^*(y^*) : T \in H\}$.

In the proofs of Theorems 23 and 25 we will need the following results.

THEOREM 20. ([20, Theorem 1]) *Let H be a subset of $K(X, Y)$ such that*

- (i) *$H(x)$ is weakly precompact compact for all $x \in X$.*
- (ii) *$H^*(y^*)$ is relatively weakly compact for all $y^* \in Y^*$.*

Then H is weakly precompact.

THEOREM 21. ([20, Theorem 3]) *Suppose that $L(X, Y) = K(X, Y)$ and H is a subset of $K(X, Y)$ such that:*

- (i) *$H(x)$ is relatively weakly compact for all $x \in X$.*
- (ii) *$H^*(y^*)$ is relatively weakly compact for all $y^* \in Y^*$.*

Then H is relatively weakly compact.

LEMMA 22. *Suppose $L(X, Y^*) = K(X, Y^*)$. If (x_n) is a weakly null DP sequence in X and (y_n) is a DP sequence in Y , then $(x_n \otimes y_n)$ is a weakly null DP sequence in $X \otimes_\pi Y$.*

Proof. Suppose that (x_n) is weakly null DP in X and $\|y_n\| \leq M$ for all $n \in \mathbb{N}$. Let $T \in L(X, Y^*) \simeq (X \otimes_\pi Y)^*$ ([10, p. 230]). Since T is completely continuous,

$$\langle T, x_n \otimes y_n \rangle \leq M \|T(x_n)\| \rightarrow 0.$$

Thus $(x_n \otimes y_n)$ is weakly null in $X \otimes_\pi Y$.

Let (A_n) be a weakly null sequence in $(X \otimes_\pi Y)^* \simeq L(X, Y^*)$ and let $x^{**} \in X^{**}$. Since the map $\gamma_{x^{**}} : L(X, Y^*) = K(X, Y^*) \rightarrow Y^*$, $\gamma_{x^{**}}(T) = T^{**}(x^{**})$ is linear and bounded, $(A_n^{**}(x^{**}))$ is weakly null in Y^* . Therefore

$$\langle x^{**}, A_n^*(y_n) \rangle = \langle A_n^{**}(x^{**}), y_n \rangle \rightarrow 0,$$

since (y_n) is DP in Y . Hence $(A_n^*(y_n))$ is weakly null in X^* . Then

$$\langle A_n, x_n \otimes y_n \rangle = \langle A_n^*(y_n), x_n \rangle \rightarrow 0,$$

since (x_n) is DP in X . Thus $(x_n \otimes y_n)$ is DP in $X \otimes_\pi Y$. ■

THEOREM 23. ([7, Theorem 18]) *Suppose that $L(X, Y^*) = K(X, Y^*)$. If X and Y are sequentially Right, then $X \otimes_\pi Y$ is sequentially right.*

Proof. Let H be a Right subset of $(X \otimes_{\pi} Y)^* \simeq L(X, Y^*) = K(X, Y^*)$. We will use Theorem 20. We will verify the conditions (i) and (ii) of this theorem. Let (T_n) be a sequence in H and let $x \in X$. We prove that $\{T_n(x) : n \in \mathbb{N}\}$ is a Right subset of Y^* . Let (y_n) be a Right null sequence in Y . Thus (y_n) is weakly null and DP. For each n ,

$$\langle T_n(x), y_n \rangle = \langle T_n, x \otimes y_n \rangle.$$

We show that $(x \otimes y_n)$ is Right null in $X \otimes_{\pi} Y$. If $T \in (X \otimes_{\pi} Y)^* \simeq L(X, Y^*)$ ([10, p. 230]), then

$$|\langle T, x \otimes y_n \rangle| = |\langle T(x), y_n \rangle| \rightarrow 0,$$

since (y_n) is weakly null. Thus $(x \otimes y_n)$ is weakly null. Let (A_n) be a weakly null sequence in $(X \otimes_{\pi} Y)^* \simeq L(X, Y^*)$. Since the map $\phi_x : L(X, Y^*) \rightarrow Y^*$, $\phi_x(T) = T(x)$ is linear and bounded, $(A_n(x))$ is weakly null in Y^* . Therefore

$$|\langle A_n, x \otimes y_n \rangle| = |\langle A_n(x), y_n \rangle| \rightarrow 0,$$

since (y_n) is DP in Y . Thus $(x \otimes y_n)$ is DP and $(x \otimes y_n)$ is Right null. Since (T_n) is a Right set,

$$|\langle T_n, x \otimes y_n \rangle| = |\langle T_n(x), y_n \rangle| \rightarrow 0.$$

Thus $\{T_n(x) : n \in \mathbb{N}\}$ is a Right subset of Y^* , hence relatively weakly compact (by Theorem 2). We thus verified (i) of Theorem 20.

Let $y^{**} \in Y^{**}$. We show that $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ is a Right subset of X^* . Let (x_n) be a Right null sequence in X . Thus (x_n) is weakly null and DP. For each n ,

$$\langle T_n^*(y^{**}), x_n \rangle = \langle y^{**}, T_n(x_n) \rangle.$$

It is enough to show that $(T_n(x_n))$ is weakly null in Y^* . Let (y_n) be a Right null sequence in Y . By Lemma 22 and Proposition 1, $(x_n \otimes y_n)$ is Right null in $X \otimes_{\pi} Y$. Since (T_n) is a Right set,

$$|\langle T_n, x_n \otimes y_n \rangle| = |\langle T_n(x_n), y_n \rangle| \rightarrow 0.$$

Therefore $(T_n(x_n))$ is a Right subset of Y^* , thus relatively weakly compact (by Theorem 2). By passing to a subsequence, we can assume that $(T_n(x_n))$ is weakly convergent. Let $y \in Y$. An argument similar to the one above shows that $(x_n \otimes y)$ is Right null in $X \otimes_{\pi} Y$. Then

$$|\langle T_n, x_n \otimes y \rangle| = |\langle T_n(x_n), y \rangle| \rightarrow 0,$$

since (T_n) is a Right set. Hence $(T_n(x_n))$ is w^* -null. Since $(T_n(x_n))$ is also weakly convergent, $(T_n(x_n))$ is weakly null. Then $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ is a Right subset of X^* . Hence $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ is relatively weakly compact (by Theorem 2). By Theorem 20, H is weakly precompact. We can assume without loss of generality that (T_n) is weakly Cauchy. Since X and Y are sequentially Right, X^* and Y^* are both weakly sequentially complete [26, Corollary 3.26], and thus $L(X, Y^*) = K(X, Y^*)$ is weakly sequentially complete, by [22, Theorem 3.10]. Then (T_n) is weakly convergent. ■

Remark. Theorem 23 can also be proved as follows. Let H be a Right subset of $(X \otimes_\pi Y)^* \simeq L(X, Y^*) = K(X, Y^*)$ and let (T_n) be a sequence in H . By the proof of Theorem 23, $\{T_n(x) : n \in \mathbb{N}\}$ and $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ are relatively weakly compact for all $x \in X$ and $y^{**} \in Y^{**}$. By Theorem 21, H is relatively weakly compact.

LEMMA 24. *Suppose $L(X, Y^*) = K(X, Y^*)$. If (x_n) is a weakly null limited sequence in X and (y_n) is a limited sequence in Y , then $(x_n \otimes y_n)$ is a weakly null limited sequence in $X \otimes_\pi Y$.*

Proof. By Lemma 22, $(x_n \otimes y_n)$ is a weakly null. Let (A_n) be a w^* -null sequence in $(X \otimes_\pi Y)^* \simeq L(X, Y^*)$. Then $(A_n^*(x))$ is a w^* -null sequence in Y^* . If $x \in X$, then $\langle A_n(x), y_n \rangle = \langle A_n^*(y_n), x \rangle \rightarrow 0$, since (y_n) is limited in Y . Hence $(A_n^*(y_n))$ is w^* -null in X^* . Since (x_n) is limited,

$$\langle A_n, x_n \otimes y_n \rangle = \langle A_n^*(y_n), x_n \rangle \rightarrow 0.$$

Thus $(x_n \otimes y_n)$ is limited in $X \otimes_\pi Y$. ■

THEOREM 25. ([7, Theorem 25]) *Suppose that $L(X, Y^*) = K(X, Y^*)$. If X and Y have the L -limited property, then $X \otimes_\pi Y$ has the L -limited property.*

Proof. The proof is similar to the proof of Theorem 23 and uses Lemma 24. ■

Remark. Theorem 25 can also be proved with a method similar to the one in the previous remark.

The fact that the (SR) and L -limited properties are inherited by quotients, immediately implies the following result.

COROLLARY 26. (i) Suppose that $L(X^*, Y^*) = K(X^*, Y^*)$, and X^* and Y are sequentially Right. Then the space $N_1(X, Y)$ of all nuclear operators from X to Y is sequentially Right.

(ii) Suppose that $L(X^*, Y^*) = K(X^*, Y^*)$, and X^* and Y have the L -limited property. Then the space $N_1(X, Y)$ of all nuclear operators from X to Y has the L -limited property.

Proof. It is known that $N_1(X, Y)$ is a quotient of $X^* \otimes_\pi Y$ ([38, p. 41]).

(i) Apply Theorem 23. (ii) Apply Theorem 25. ■

Observation 1. If $T : Y \rightarrow X^*$ be an operator such that $T^*|_X$ is (weakly) compact, then T is (weakly) compact. To see this, let $T : Y \rightarrow X^*$ be an operator such that $T^*|_X$ is (weakly) compact. Let $S = T^*|_X$. Suppose $x^{**} \in B_{X^{**}}$ and choose a net (x_α) in B_X which is w^* -convergent to x^{**} . Then $(T^*(x_\alpha)) \xrightarrow{w^*} T^*(x^{**})$. Now, $(T^*(x_\alpha)) \subseteq S(B_X)$, which is a relatively (weakly) compact set. Then $(T^*(x_\alpha)) \rightarrow T^*(x^{**})$ (resp. $(T^*(x_\alpha)) \xrightarrow{w} T^*(x^{**})$). Hence $T^*(B_{X^{**}}) \subseteq \overline{S(B_X)}$, which is relatively (weakly) compact. Therefore $T^*(B_{X^{**}})$ is relatively (weakly) compact, and thus T is (weakly) compact.

It follows that if $L(X, Y^*) = K(X, Y^*)$, then $L(Y, X^*) = K(Y, X^*)$.

The following result improves Corollaries 19 and 21 of [7].

COROLLARY 27. If X is sequentially Right and Y^* has the Schur property (or Y is sequentially Right and X^* has the Schur property), then $X \otimes_\pi Y$ is sequentially Right.

Proof. Since Y^* has the Schur property, every Right set in Y^* is relatively compact (by Corollary 9). Let $T : X \rightarrow Y^*$ be an operator. Then T is pseudo weakly compact (since Y^* has the Schur property), hence compact (by Theorem 2). Apply Theorem 23. ■

THEOREM 28. Suppose that $L(X, Y^*) = K(X, Y^*)$. The following statements are equivalent:

1. (i) X and Y are sequentially Right and at least one of them does not contain ℓ_1 .
(ii) $X \otimes_\pi Y$ is sequentially Right.
2. (i) X and Y have the L -limited property and at least one of them does not contain ℓ_1 .
(ii) $X \otimes_\pi Y$ has the L -limited property.

Proof. We only prove 1. The other proof is similar.

(i) \Rightarrow (ii) by Theorem 23.

(ii) \Rightarrow (i) Suppose that $X \otimes_{\pi} Y$ is sequentially Right. Then X and Y are sequentially Right, since the sequentially Right property is inherited by quotients [26, Proposition 3.8]. We will show that $\ell_1 \not\hookrightarrow X$ or $\ell_1 \not\hookrightarrow Y$. Suppose that $\ell_1 \hookrightarrow X$ and $\ell_1 \hookrightarrow Y$. Hence $L_1 \hookrightarrow X^*$ ([32, Theorem 3.4], [8, p. 212]). Also, the Rademacher functions span ℓ_2 inside of L_1 , and thus $\ell_2 \hookrightarrow X^*$. Similarly $\ell_2 \hookrightarrow Y^*$. Then $c_0 \hookrightarrow K(X, Y^*)$ ([15, Theorem 3], [21, Corollary 21]). Thus $\ell_1 \xrightarrow{c} X \otimes_{\pi} Y$ ([3, Theorem 4], [8, Theorem 10, p. 48]), a contradiction with Corollary 5 (iii). ■

Observation 2. If $\ell_1 \hookrightarrow X$ and $\ell_1 \hookrightarrow Y$, then $\ell_2 \hookrightarrow X^*$ and $\ell_2 \hookrightarrow Y^*$, and $c_0 \hookrightarrow K(X, Y^*)$ ([15, Theorem 3], [21, Corollary 21]). More generally, if $\ell_1 \hookrightarrow X$ and $\ell_p \hookrightarrow Y^*$, $p \geq 2$, then $c_0 \hookrightarrow K(X, Y^*)$ ([15], [21]). Thus $\ell_1 \xrightarrow{c} X \otimes_{\pi} Y$ ([3, Theorem 4], [8, Theorem 10, p. 48]). Hence $X \otimes_{\pi} Y$ is not weak sequentially Right (and does not have the wL -limited property), by Corollary 5 (iii).

COROLLARY 29. *Suppose that $L(X, Y^*) = K(X, Y^*)$.*

1. *If $X \otimes_{\pi} Y$ is weak sequentially Right, then X and Y are weak sequentially Right and at least one of them does not contain ℓ_1 .*
2. *If $X \otimes_{\pi} Y$ has the wL -limited property, then X and Y have the wL -limited property and at least one of them does not contain ℓ_1 .*

Proof. We only prove 1. The other proof is similar. If $X \otimes_{\pi} Y$ is weak sequentially Right, then X and Y are weak sequentially Right, since the weak sequentially Right property is inherited by quotients (by Corollary 3). Apply Observation 2. ■

COROLLARY 30. ([7, Theorem 22]) *Suppose that X and Y have the DPP. The following statements are equivalent:*

- (i) *X and Y are sequentially Right and at least one of them does not contain ℓ_1 .*
- (ii) *$X \otimes_{\pi} Y$ is sequentially Right.*

Proof. (i) \Rightarrow (ii) Suppose that X and Y have the *DPP*. Without loss of generality suppose that $\ell_1 \not\hookrightarrow X$. Then X^* has the Schur property [9]. Apply Corollary 27.

(ii) \Rightarrow (i) by Observation 2. ■

By Corollary 30, the space $C(K_1) \otimes_{\pi} C(K_2)$ is sequentially Right if and only if either K_1 or K_2 is dispersed.

Next we present some results about the necessity of the condition $L(X, Y^*) = K(X, Y^*)$. It is implicit in [6] that a Banach space X has all bilinear forms weakly sequentially continuous if and only if every operator $S : X \rightarrow X^*$ transforms weakly null sequences into L -sets. Emmanuelle shows in [13] that a Banach space X does not contain ℓ_1 if and only if every L -set in X^* is relatively compact. Then, it is easy to see that if X and Y are not containing ℓ_1 , then $L(X, Y^*) = K(X, Y^*)$ if and only if every operator $T : X \rightarrow Y^*$ transforms weakly null sequences into L -sets (for more details see [6]).

A Banach space X has the *approximation property* if for each norm compact subset M of X and $\epsilon > 0$, there is a finite rank operator $T : X \rightarrow X$ such that $\|Tx - x\| < \epsilon$ for all $x \in M$. If in addition T can be found with $\|T\| \leq 1$, then X is said to have the *metric approximation property*. $C(K)$ spaces, c_0 , ℓ_p , $1 \leq p < \infty$, $L_p(\mu)$ (μ any measure), $1 \leq p < \infty$, and their duals have the metric approximation property [10, p. 238].

A separable Banach space X has an *unconditional compact expansion of the identity (u.c.e.i)* if there is a sequence (A_n) of compact operators from X to X such that $\sum A_n(x)$ converges unconditionally to x for all $x \in X$ [17]. In this case, (A_n) is called an (u.c.e.i.) of X .

A sequence (X_n) of closed subspaces of a Banach space X is called an *unconditional Schauder decomposition* of X if every $x \in X$ has a unique representation of the form $x = \sum x_n$, with $x_n \in X_n$, for every n , and the series converges unconditionally [30, p. 48].

The space X has (Rademacher) *cotype* q for some $2 \leq q \leq \infty$ if there is a constant C such that for every n and every x_1, x_2, \dots, x_n in X ,

$$\left(\sum_{i=1}^n \|x_i\|^q \right)^{1/q} \leq C \left(\int_0^1 \|r_i(t)x_i\|^q dt \right)^{1/q},$$

where (r_n) are the Radamacher functions. A Hilbert space has cotype 2 [8, p. 118]. \mathcal{L}_p -spaces have cotype 2, if $1 \leq p \leq 2$ [8, p. 118].

THEOREM 31. *Assume one of the following holds:*

- (i) *If $T : X \rightarrow Y^*$ is an operator which is not compact, then there is a sequence (T_n) in $K(X, Y^*)$ such that for each $x \in X$, the series $\sum T_n(x)$ converges unconditionally to $T(x)$.*
- (ii) *X is an \mathcal{L}_∞ -space and Y^* is a subspace of an \mathcal{L}_1 -space.*
- (iii) *$X = C(K)$, K a compact Hausdorff space, and Y^* is a space with cotype 2.*
- (iv) *Either X or Y^* has an (u.c.e.i.).*
- (v) *X has the DPP and $\ell_1 \hookrightarrow Y$.*
- (vi) *X and Y have the DPP.*

If $X \otimes_\pi Y$ is weak sequentially Right, then $L(X, Y^) = K(X, Y^*)$.*

Proof. Suppose that $X \otimes_\pi Y$ is weak sequentially Right. Then X and Y are weak sequentially Right.

(i) Let $T : X \rightarrow Y^*$ be a noncompact operator. Let (T_n) be a sequence as in the hypothesis. By the Uniform Boundedness Principle, $\{\sum_{n \in A} T_n : A \subseteq \mathbb{N}, A \text{ finite}\}$ is bounded in $K(X, Y^*)$. Then $\sum T_n$ is wuc and not unconditionally convergent (since T is noncompact). Hence $c_0 \hookrightarrow K(X, Y^*)$ ([3, Theorem 5]), $\ell_1 \xhookrightarrow{c} X \otimes_\pi Y$ ([3, Theorem 4]), and we have a contradiction with Corollary 5 (iii).

Suppose (ii) or (iii) holds. It is known that any operator $T : X \rightarrow Y^*$ is 2-absolutely summing ([8, p. 189]), hence it factorizes through a Hilbert space. If $L(X, Y^*) \neq K(X, Y^*)$, then $c_0 \hookrightarrow K(X, Y^*)$ (by [16, Remark 3]), a contradiction.

(iv) If $L(X, Y^*) \neq K(X, Y^*)$, then $c_0 \hookrightarrow K(X, Y^*)$ (by [27, Theorem 6]), a contradiction.

(v) Suppose that X has the DPP and $\ell_1 \hookrightarrow Y$. By Observation 1, $\ell_1 \not\hookrightarrow X$. Then X^* has the Schur property ([9, Theorem 3]). Let $T : Y \rightarrow X^*$ be an operator. Then T is pseudo weakly compact (since X^* has the Schur property), and thus weakly precompact (by Corollary 5 (i)). Then $L(Y, X^*) = K(Y, X^*)$. Hence $L(X, Y^*) = K(X, Y^*)$, by Observation 1.

(vi) Suppose that X and Y have the DPP. Then $L(X, Y^*) = K(X, Y^*)$, either by (v) if $\ell_1 \hookrightarrow Y$, or since Y^* has the Schur property ([9, Theorem 3]) if $\ell_1 \not\hookrightarrow Y$ (by an argument similar to the one in (v)). ■

Assumption (i) of the previous theorem is satisfied, for instance, if X^* (or Y^*) has an (u.c.e.i.).

EXAMPLES. By Theorem 31, the space $\ell_p \otimes \ell_q$, where $1 < p \leq q' < \infty$ and q and q' are conjugate, is not weak sequentially Right, since the natural inclusion map $i : \ell_p \rightarrow \ell_{q'}$ is not compact.

The space $C(K) \otimes_{\pi} \ell_p$, with K not dispersed and $1 < p \leq 2$, is not weak sequentially Right (by Observation 2, since $\ell_1 \hookrightarrow C(K)$ and $\ell_2 \hookrightarrow \ell_p^*$).

For $1 < p_1, p_2 < \infty$, $L_{p_1}[0, 1] \otimes_{\pi} L_{p_2}[0, 1]$ is not weak sequentially Right by Corollary 5 (iii), since $\ell_1 \xhookrightarrow{c} L_{p_1}[0, 1] \otimes_{\pi} L_{p_2}[0, 1]$ ([38, Corollary 2.26]).

THEOREM 32. (i) Suppose Y^* is complemented in a Banach space Z which has an unconditional Schauder decomposition (Z_n) , and $L(X, Z_n) = K(X, Z_n)$ for all n . If $X \otimes_{\pi} Y$ is weak sequentially Right, then $L(X, Y^*) = K(X, Y^*)$.

(ii) Suppose either X^* or Y^* has the metric approximation property. If $X \otimes_{\pi} Y$ is sequentially Right, then $W(X, Y^*) = K(X, Y^*)$.

Proof. (i) Let $T : X \rightarrow Y^*$ be a noncompact operator, $P_n : Z \rightarrow Z_n$, $P_n(\sum z_i) = z_n$, and let P be the projection of Z onto Y^* . Define $T_n : X \rightarrow Y^*$ by $T_n(x) = PP_nT(x)$, $x \in X$, $n \in \mathbb{N}$. Note that P_nT is compact since $L(X, Z_n) = K(X, Z_n)$. Then T_n is compact for each n . For each $z \in Z$, $\sum P_n(z)$ converges unconditionally to z ; thus $\sum T_n(x)$ converges unconditionally to $T(x)$ for each $x \in X$. Then $\sum T_n$ is wuc and not unconditionally converging. Hence $c_0 \hookrightarrow K(X, Y^*)$ ([3, Theorem 5]), and we obtain a contradiction.

(ii) Since $X \otimes_{\pi} Y$ is sequentially Right, $(X \otimes_{\pi} Y)^* \simeq L(X, Y^*)$ is weakly sequentially complete ([26, Corollary 3.26]). Under assumption (ii), [29, Corollary 2.4] implies $W(X, Y^*) = K(X, Y^*)$.

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