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# Estimates of Generalized Nevanlinna Counting Function and Applications to Composition Operators

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*Abstract*: Let $\varphi$ be a holomorphic self-map of the unit disc. We study the relationship between the generalized Nevanlinna counting function associated with $\varphi$ and the norms of $\varphi^n$ in the Dirichlet spaces. We give examples of Hilbert-Schmidt composition operators on the Dirichlet spaces.

*Key words*: Generalized Nevanlinna counting function, Dirichlet spaces, composition operators, Hilbert-Schmidt operators.

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## 1. Introduction

Let $\mathbb{D}$ be the unit disk and $\mathbb{T} = \partial\mathbb{D}$ be the unit circle. We denote by $\mathrm{d}A(z) = \mathrm{d}x\,\mathrm{d}y/\pi$ the normalized Lebesgue measure, and for $0 \le \alpha \le 1$, we set

$$\mathrm{d}A_\alpha(z) := (1+\alpha)\big(1-|z|^2\big)^\alpha\,\mathrm{d}A(z)\,.$$

In this paper we are concerned with composition operators on the Dirichlet spaces;

$$\mathcal{D}_\alpha = \left\{ f \in \mathrm{Hol}(\mathbb{D}) \,:\, \|f\|_\alpha^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2\,\mathrm{d}A_\alpha(z) < \infty \right\}.$$

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By [6, p. 14] this norm is comparable to

$$\|f\|_\alpha^2 \asymp \sum_{n=0}^{\infty}(1+n)^{1-\alpha}\big|\widehat{f}(n)\big|^2.$$

Hence $\mathcal{D}_1$ is the usual Hardy space $H^2$, and $\mathcal{D}_0$ is the classical Dirichlet space $\mathcal{D}$.

Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic function. The composition operator on $\mathcal{D}_\alpha$ with symbol $\varphi$ is defined as

$$C_\varphi(f) := f \circ \varphi\,, \qquad f \in \mathcal{D}_\alpha\,.$$

In this paper we study some operator-theoretic properties of $C_\varphi$ such as boundedness, compactness and Hilbert-Schmidt class membership. Recall that $C_\varphi$ is always well–defined on $H^2$, but not on $\mathcal{D}_\alpha$ for $0 \le \alpha < 1$, we refer the reader to the papers [1, 2, 5, 10, 7, 14].

The generalized Nevanlinna counting function associated to $\varphi$, $0 < \alpha \le 1$, is given by

$$N_{\varphi,\alpha}(z) := \sum_{z=\varphi(w),w\in\mathbb{D}} (1-|w|)^\alpha, \qquad z \in \mathbb{D}\,,$$

where each preimage $w$ is counted according to its multiplicity. For $\alpha = 1$, $N_{\varphi,1}$ is comparable with the classical Nevanlinna counting function

$$N_\varphi(z) := N_{\varphi,1}(z) = \sum_{z=\varphi(w),w\in\mathbb{D}} \log(1/|w|)\,, \qquad z \in \mathbb{D}\setminus\{\phi(0)\}.$$

By the Littlewood subordination principle [12, 13], the composition operator $C_\varphi$ is bounded on $H^2$. It is also true that $\sup_{z\in\mathbb{D}} N_\varphi(z)/(1-|z|) < \infty$. In [12] Shapiro gave the following complete characterization of compact composition operators on $H^2$:

$$C_\varphi \text{ is compact on } H^2 \quad \Longleftrightarrow \quad \lim_{|z|\to 1-} \frac{N_\varphi(z)}{1-|z|} = 0\,.$$

The generalized Nevanlinna counting function plays also a key role in the study of composition operators on the weighted spaces $\mathcal{D}_\alpha$, see Theorem 2.3.

Generally speaking, it is difficult to give an estimate of $N_{\varphi,\alpha}$. In this work we establish an estimate of the generalized Nevanlinna counting function $N_{\varphi,\alpha}$ in terms of the norms $(\|\varphi^n\|_\alpha)_n$ of the sequence $(\varphi^n)_n$. This allows us to construct some examples of bounded and compact operators on $\mathcal{D}_\alpha$. Precisely, putting

$$\mathcal{D}_\alpha(f) = \int_{\mathbb{D}} |f'(z)|^2 \,\mathrm{d}A_\alpha(z)\,.$$

We shall show that, for $0 < \alpha < 1$,

$$N_{\varphi,\alpha}(1-1/n) \lesssim \mathcal{D}_\alpha\big(\varphi^{n+1}\big)\,, \qquad n \geq 1\,. \tag{1.1}$$

Now, to each $\varphi$ we associate the counting function

$$n_\varphi(z) = \operatorname{card}\{w \,:\, \varphi(w) = z\}, \qquad z \in \mathbb{D}\,.$$

This is the number of roots of the equation $\varphi(w) - z = 0$. We mention also that $N_{\varphi,0} = n_\varphi$.

We shall establish an estimate of $n_\varphi$ in terms of the norm on $\mathcal{D}_0$ of the powers of $\varphi$. More precisely, we show that

$$\inf_{\frac{1}{n+1} \leq 1-|z| \leq \frac{1}{n}} n_\varphi(z) \lesssim \mathcal{D}_0(\varphi^{n+1})\,. \tag{1.2}$$

The paper is organized as follows: In the next section we prove (1.1). In Section 3, we give the proof of (1.2). Section 4 provides some examples of estimates of $N_{\varphi,\alpha}$ and some examples of Hilbert-Schmidt class membership.

Throughout the paper, the notation $A \lesssim B$ means that there is an absolute constant $C$ such that $A \leq CB$. We write $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$.

## 2. The relationship between $N_{\varphi,\alpha}$ and $\mathcal{D}_\alpha(\varphi^n)$

In the sequel we need some basic results. The first lemma gives the change of variable formula in terms of generalizes Nevanlinna counting function, see [13].

Lemma 2.1. *Let $0 \leq \alpha \leq 1$, $\varphi$ be a holomorphic self-map of $\mathbb{D}$ and let $f$ be a measurable function on $\mathbb{D}$. Then*

$$\int_{\mathbb{D}} (f \circ \varphi)(z) |\varphi'(z)|^2 \,\mathrm{d}A_\alpha(z) = (1+\alpha) \int_{\mathbb{D}} f(z) N_{\varphi,\alpha}(z) \,\mathrm{d}A(z)\,.$$

For $\alpha > 0$, the function $N_{\varphi,\alpha}$ satisfies the mean value inequality (see [9]). More precisely, we have:

LEMMA 2.2. *Let $\alpha \in (0,1]$. If $\varphi$ is a holomorphic self-map of $\mathbb{D}$, then*

$$N_{\varphi,\alpha}(z) \le \frac{2}{r^2} \int_{D(z,r)} N_{\varphi,\alpha}(w)\, \mathrm{d}A(w)$$

*for every disk $D(z,r)$ of radius $r$ centered at $z$ with $D(z,r) \subset \mathbb{D}\backslash D(0,1/2)$.*

We need also the following Theorem due to Kellay and Lefèvre [9]

THEOREM 2.3. *If $\varphi$ is a holomorphic self-map of $\mathbb{D}$, then, for $0 < \alpha \le 1$,*

(i) *$C_\varphi$ is bounded on $\mathcal{D}_\alpha$* $\iff$ $N_{\varphi,\alpha} = O\big((1-|z|)^\alpha\big), \quad |z| \to 1-.$

(ii) *$C_\varphi$ is compact on $\mathcal{D}_\alpha$* $\iff$ $N_{\varphi,\alpha} = o\big((1-|z|)^\alpha\big), \quad |z| \to 1-.$

We can now state the main result of this section.

THEOREM 2.4. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function and let $\alpha \in (0,1]$. Then there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$ we have*

$$N_{\varphi,\alpha}(z) \le \frac{8\mathrm{e}^4}{1+\alpha} \mathcal{D}_\alpha(\varphi^{n+1}), \qquad \frac{1}{n} \le 1-|z| \le \frac{1}{n-1}.$$

*Proof.* Let $n_1 \in \mathbb{N}$ be large enough so that if $n \ge n_1$, then

$$D\big(1-1/(n-1), 1/2(n+1)\big) \subset \mathbb{D}\backslash D(0,1/2).$$

Let $n \ge n_1$ and suppose that $1/n \le 1-|z| \le 1/n-1$. Then by Lemma 2.2, it follows that

$$\begin{aligned} N_{\varphi,\alpha}(z) &\le 2 \times 4(n+1)^2 \int_{D(z,1/2(n+1))} N_{\varphi,\alpha}(w)\, \mathrm{d}A(w) \\ &= 8(n+1)^2 \int_{D(z,1/2(n+1))} N_{\varphi,\alpha}(w) \frac{|w|^{2n}}{|w|^{2n}}\, \mathrm{d}A(w) \\ &\le 8(n+1)^2 \left[ \sup_{D(z,1/2(n+1))} |w|^{-2n} \right] \int_{D(z,1/2(n+1))} N_{\varphi,\alpha}(w) |w|^{2n}\, \mathrm{d}A(w). \end{aligned}$$

Now, it is easy to see that there exists $n_0 \ge n_1$ large enough so that for each $n \ge n_0$

$$\sup_{D(z,1/2(n+1))} |w|^{-2n} \le \mathrm{e}^4.$$

Therefore,

$$\begin{aligned} N_{\varphi,\alpha}(z) &\le 8\mathrm{e}^4(n+1)^2 \int_{D(z,1/2(n+1))} N_{\varphi,\alpha}(w)|w|^{2n}\,\mathrm{d}A(w) \\ &\le 8\mathrm{e}^4(n+1)^2 \int_{\mathbb{D}} N_{\varphi,\alpha}(w)|w|^{2n}\,\mathrm{d}A(w)\,. \end{aligned}$$

On the other hand, by Lemma 2.1 it follows that

$$\int_{\mathbb{D}} N_{\varphi,\alpha}(w)|w|^{2n}\,\mathrm{d}A(w) = \frac{1}{1+\alpha}\int_{\mathbb{D}} |\varphi'(\eta)|^2|\varphi(\eta)|^{2n}\,\mathrm{d}A_\alpha(\eta)\,.$$

Thus

$$N_{\varphi,\alpha}(z) \le \frac{8\mathrm{e}^4}{1+\alpha}\mathcal{D}_\alpha\big(\varphi^{n+1}\big)\,, \qquad \frac{1}{n} \le 1-|z| \le \frac{1}{n-1}\,.$$

The proof now is complete. ∎

As a consequence of this we obtain

Corollary 2.5. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an holomorphic function and let $\alpha \in (0,1]$, then*

(i) *If $\mathcal{D}_\alpha(\varphi^n) = O(1/n^\alpha)$ then $C_\varphi$ is bounded on $\mathcal{D}_\alpha$.*

(ii) *If $\mathcal{D}_\alpha(\varphi^n) = o(1/n^\alpha)$ then $C_\varphi$ is compact on $\mathcal{D}_\alpha$.*

*Proof.* The proof follows from Theorem 2.4 and Theorem 2.3. ∎

Next, we give another proof which is similar to that given by El-Fallah, Kellay, Shabankhah and Youssfi [5], for the Dirichlet space (i.e., ($\alpha = 0$)), see Corollary 3.4. We consider the test function given by

$$F_\lambda(z) = \frac{\big(1-|\lambda|^2\big)^{1-\frac{\alpha}{2}}}{(1-\overline{\lambda}z)}\,, \qquad \lambda, z \in \mathbb{D}\,,$$

and we recall the following lemma ([5]).

Lemma 2.6. *Let $\varphi \in \mathcal{D}_\alpha$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $0 < \alpha \le 1$. Thus*

(i) *$C_\varphi$ is bounded on $\mathcal{D}_\alpha$* $\iff \sup_{\lambda\in\mathbb{D}} \|F_\lambda \circ \varphi\|_\alpha < \infty\,.$

(ii) *$C_\varphi$ is compact on $\mathcal{D}_\alpha$* $\iff \lim_{|\lambda|\to 1^-} \|F_\lambda \circ \varphi\|_\alpha = 0\,.$

*Second proof of Corollary* 2.5. We assume that $\varphi(0)=0$. If $\mathcal{D}_\alpha(\varphi^n)=o(1/n^\alpha)$ then

$$\begin{aligned}
\mathcal{D}_\alpha(C_\varphi(F_\lambda)) &= \int_{\mathbb{D}} |\,(F_\lambda(\varphi(w))'\,|^2 \,\mathrm{d}A_\alpha(w)\\
&\le c_1\big(1-|\lambda|^2\big)^{2-\alpha}\int_{\mathbb{D}}\frac{|\varphi'(w)|^2}{(1-|\lambda\varphi(w)|^2)^4}\,\mathrm{d}A_\alpha(w)\\
&\le c_2\big(1-|\lambda|^2\big)^{2-\alpha}\sum_{n\ge 0}(1+n)^3|\lambda|^{2n}\int_{\mathbb{D}}|\varphi'(w)|^2|\varphi(w)|^{2n}\,\mathrm{d}A_\alpha(w)\\
&\le c_3\big(1-|\lambda|^2\big)^{2-\alpha}\sum_{n\ge 0}(1+n)|\lambda|^{2n}\mathcal{D}_\alpha(\varphi^{n+1})\\
&\le c_4\big(1-|\lambda|^2\big)^{2-\alpha}\Bigg[\sum_{0\le n\le N}(1+n)^{1-\alpha}|\lambda|^{2n}+o\Big(\sum_{n\ge N}(1+n)^{1-\alpha}|\lambda|^{2n}\Big)\Bigg]\\
&\asymp o(1)\,, \qquad |\lambda|\to 1-\,.
\end{aligned}$$

where $c_1,c_2,c_3$ and $c_4$ are positives constants. Thus $C_\varphi$ is compact. A similar proof can be given for the boundedness.

## 3. The relationship between $n_\varphi$ and $\mathcal{D}(\varphi^n)$

We need the following lemma

Lemma 3.1. *Let $\varphi:\mathbb{D}\to\mathbb{D}$ be a holomorphic function. Then*

$$\int_{1-\frac{1}{m}\le|z|\le 1} n_\varphi(z)\,\mathrm{d}A(z)\le\frac{\mathrm{e}^4}{(1+m)^2}\mathcal{D}_0\big(\varphi^{m+1}\big)\,,\qquad m\ge 2\,.$$

*Proof.* Since $N_{\varphi,0}=n_\varphi$, by Lemma 2.1 we have

$$\begin{aligned}
\mathcal{D}_0(\varphi^{m+1}) &= (m+1)^2\int_{\mathbb{D}}|\varphi'(z)|^2|\varphi^m(z)|^2\,\mathrm{d}A(z)\\
&= (m+1)^2\int_{\mathbb{D}} n_\varphi(w)|w|^{2m}\,\mathrm{d}A(w)\\
&\ge (m+1)^2\int_{1-\frac{1}{m}\le|w|\le 1} n_\varphi(w)|w|^{2m}\,\mathrm{d}A(w)
\end{aligned}$$

$$\geq (m+1)^2\big(1-\frac{1}{m}\big)^{2m}\int_{1-\frac{1}{m}\leq|z|\leq 1} n_\varphi(w)\,\mathrm{d}A(w)$$
$$\geq \mathrm{e}^{-4}(m+1)^2\int_{1-\frac{1}{m}\leq|z|\leq 1} n_\varphi(w)\,\mathrm{d}A(w)\,,\qquad m\geq 2\,,$$

and this completes the proof. ∎

We obtain the following result which is the main theorem in this section and gives a relationship between the mean behavior of $n_\varphi$ and the norm of $\varphi^m$.

THEOREM 3.2. *Let $\varphi:\mathbb{D}\to\mathbb{D}$ be a holomorphic function. Then*

$$\inf_{1-\frac{1}{m}\leq|z|\leq 1-\frac{1}{m+1}} n_\varphi(z)\leq \frac{\mathrm{e}^4}{\pi}\mathcal{D}_0\big(\varphi^{m+1}\big)\,,\qquad m\geq 2\,.$$

*Proof.* This follows from Lemma 3.1 and the following inequality

$$\int_{1-\frac{1}{m}\leq|z|\leq 1} n_\varphi(z)\,\mathrm{d}A(z)\geq\frac{\pi}{(m+1)^2}\inf_{1-\frac{1}{m}\leq|z|\leq 1-\frac{1}{m+1}} n_\varphi(z)\,.$$

∎

The Carleson window is defined as

$$W(\zeta,\delta)=\big\{z\in\mathbb{D}\,:\,|z|>1-\delta\,,\ |\arg(\overline{\zeta}z)|<\delta\big\},\qquad \zeta\in\mathbb{T}\,.$$

For $\zeta\in\mathbb{T}$ and $\delta\in(0,1)$, set

$$\mathcal{N}(\zeta,\delta):=\int_{W(\zeta,\delta)} n_\varphi(w)\,\mathrm{d}A(w)\,.$$

We shall make use of the following lemma due to Zorboska [14, 9]

LEMMA 3.3. *Let $\varphi$ be a holomorphic self-map of $\mathbb{D}$. Then*

(i) $C_\varphi$ *is bounded on* $\mathcal{D}$ $\iff$ $\sup_{\zeta\in\mathbb{T}}\mathcal{N}(\zeta,\delta)=O(\delta^2)\quad\delta\to 0\,.$

(ii) $C_\varphi$ *is compact on* $\mathcal{D}$ $\iff$ $\sup_{\zeta\in\mathbb{T}}\mathcal{N}(\zeta,\delta)=o(\delta^2)\quad\delta\to 0\,.$

From Lemma 3.2 and Lemma 3.3 we obtain the following result which was first proved by El-Fallah-Kellay-Shabankah-Youssfi [5].

Corollary 3.4. *Let $\varphi$ be a holomorphic self-map of $\mathbb{D}$. Then*

(i) *If $\mathcal{D}_0(\varphi^n) = O(1)$ then $C_\varphi$ is bounded on $\mathcal{D}$.*

(ii) *If $\mathcal{D}_0(\varphi^n) = o(1)$ then $C_\varphi$ is compact on $\mathcal{D}$.*

*Proof.* Suppose that (i) holds. Let $\delta > 0$ and let $m \geq 1$ such that $1/(m+1) \leq \delta \leq 1/m$. By Lemma 3.2 we have

$$\sup_{\zeta\in\mathbb{T}} \mathcal{N}(\zeta,\delta) \leq \int_{1-\frac{1}{m}\leq|z|\leq 1} n_\varphi(z)\,\mathrm{d}A(z) = O\big(1/(1+m)^2\big) = O\big(\delta^2\big),$$

and Lemma 3.3 gives the result. A similar proof can be given for the compactness. ∎

Li–Queffélec-Rodríguez–Piazza have shown that this result is essentially optimal [11].

## 4. Examples

Recall that for $f \in H^2$, the radial limit $f^*$ of $f$ is given by

$$f^*\big(\mathrm{e}^{it}\big) := \lim_{r\to 1-} f\big(r\mathrm{e}^{it}\big).$$

By Fatou's Theorem, the radial limit $f^*$ exists almost everywhere on $\mathbb{T}$. Note that $\log|f^*| \in L^1(\mathbb{T})$. The function $f$ is said to be outer if

$$\log|f(0)| = \int_{\mathbb{T}} \log|f^*(\zeta)| \frac{|\,\mathrm{d}\zeta|}{2\pi}\,.$$

In this case the function has the following integral representation

$$f(z) = \exp\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \log|f^*(\zeta)| \frac{|\,\mathrm{d}\zeta|}{2\pi}\,, \qquad z \in \mathbb{D}\,.$$

Let $K$ be a closed set of $\mathbb{T}$, and let $\Omega \in \mathcal{C}^1([0.2\pi])$, such that $\Omega(0) = 0$ and

$$\int_{\mathbb{T}} \Omega(\,\mathrm{d}(\zeta,K))|\,\mathrm{d}\zeta| < \infty\,.$$

The distance function corresponding to $\Omega, K$ is the outer function $\varphi_{\Omega,K}$ satisfying

$$|\varphi_{\Omega,K}(\zeta)| = \mathrm{e}^{-\Omega(\,\mathrm{d}(\zeta,K))} \qquad \text{a.e. on } \mathbb{T}. \tag{4.1}$$

Hence

$$\varphi_{\Omega,K}(z) = \exp \int_{\mathbb{D}} \frac{z+\zeta}{z-\zeta} \Omega(\,\mathrm{d}(\zeta, K)) \frac{|\,\mathrm{d}\zeta|}{2\pi}\,, \qquad z \in \mathbb{D}\,.$$

If $\Omega$ satisfies Dini's condition

$$\int_0^\pi \frac{\Omega(t)}{t}\,\mathrm{d}t < \infty\,,$$

then the function $\varphi_{\Omega,K}$ belongs to the Disk algebra [8, pp. 105 − 106]. In this case we have $|\phi(z)| \leq 1$ and the set of the contact points of $\varphi_{\Omega,K}$ on the circle coincides with $K$, this means that

$$|\varphi_{\Omega,K}| = 1 \quad \text{ on } K.$$

We recall here the construction of the generalized Cantor set on $\mathbb{T}$. Let $K_0 = \mathbb{T}$ and $\ell_0 = 2\pi$. Let $(a_n)_{n\geq 1}$ be a positive decreasing sequence with $a_1 < 1/2$. We remove an interval of length $a_1$ from the middle of $K_0$. Denote the union of two remaining intervals by $K_1$, and denote the length of each interval in $K_1$ by $\ell_1$. Then we remove two intervals, each of length $a_2$, from the middle of intervals in $K_1$. Let $K_2$ denote the union of the resulting four pairwise disjoint intervals of equal length $\ell_2$. After $n$ steps, we obtain a compact set $K_n$ which is union of $2^n$ closed intervals of length $\ell_n$. Note that $2\ell_n + a_n = \ell_{n-1}$. The compact $K = \bigcap_{n\geq 1} K_n$ is called the generalized Cantor set. It is easy to see that $K$ has Lebesgue measure zero if and only if $\sum_{n=1}^{\infty} 2^{n-1} a_n = 2\pi$. The classical Cantor set corresponds to $\ell_n = (1/3)^n$.

Let $\varepsilon > 0$. For a closed subset $K$ of $\mathbb{T}$, the $\varepsilon$–neighborhood of $K$ is given by

$$K_\varepsilon = \{\zeta \in \mathbb{T} \,:\, \mathrm{d}(\zeta, K) \leq \varepsilon\}\,.$$

Let $K$ be the generalized Cantor set associated to a sequence $(a_n)_n$. If

$$\lambda_K := \sup_{n\geq 1} \frac{a_{n+1}}{a_n} < \frac{1}{2}\,, \tag{4.2}$$

then

$$|K_\varepsilon| = O(\varepsilon^{\mu_K}) \qquad \varepsilon \to 0\,, \tag{4.3}$$

where $\mu_K = 1 - \log 2/|\log \lambda_K|$ (see [4]). The classical Cantor set $K$ corresponds to $\mu_K = 1 - \log 2/\log 3$.

We have the following formula which allows us to calculate explicitly the norm for the outer function $\varphi_{\Omega,K}$.

LEMMA 4.1. *Let $\alpha \in [0,1]$. Let $K$ be a generalized Cantor set associated to a sequence $(a_n)_n$ satisfying (4.2), and let $\Omega : [0,2\pi] \to \mathbb{R}^+$ be an increasing function such that $t \to \Omega(t^\gamma)$ is concave for some $\gamma > 2/(1-\alpha)$.Then*

$$\mathcal{D}_\alpha(\varphi_{\Omega,K}) \leq c \int_0^{2\pi} \Omega'(t)^2 \mathrm{e}^{-2\Omega(t)} t^\alpha |K_t| \,\mathrm{d}t\,,$$

*where c is a positive constant.*

*Proof.* For the proof we refer to [4, Theorem 3.2] and [3, Theorem 4.1]. ∎

4.1. EXAMPLES OF ESTIMATES OF GENERALIZED NEVANLINNA COUNTING FUNCTION. Here we gives some estimate of generalized Nevanlinna counting function associated to distance function given by (4.1). This allows to give some examples of bounded and compact composition operators on the Dirichlet spaces by Corollary 2.5.

We begin with the case of the Hardy space ($\alpha = 1$).

LEMMA 4.2. *Let $K$ be a closet set of $\mathbb{T}$ and let $\Omega : [0,2\pi] \to \mathbb{R}^+$ be an increasing function such that $\Omega(0) = 0$. Let $\varphi = \varphi_{\Omega,K}$, then*

$$N_\varphi(z) \lesssim \inf_{\varepsilon>0} \Big\{ |K_\varepsilon| + \mathrm{e}^{-2\frac{\Omega(\varepsilon)}{1-|z|}} \Big\}, \qquad |z| < 1\,.$$

*Proof.* Let $\varepsilon > 0$. By Lemma 2.4 and for $1/n \leq 1-|z| \leq 1/(n-1)$, $n \geq 2$, we have

$$\begin{aligned} N_\varphi(z) &\lesssim \int_{\mathbb{T}} \mathrm{e}^{-2(n+1)\Omega(\,\mathrm{d}(\zeta,K))} \frac{|\,\mathrm{d}\zeta|}{2\pi} \\ &= \int_{\zeta\in K_\varepsilon} \mathrm{e}^{-2(n+1)\Omega(\,\mathrm{d}(\zeta,K))} \frac{|\,\mathrm{d}\zeta|}{2\pi} + \int_{\zeta\in \mathbb{T}\setminus K_\varepsilon} \mathrm{e}^{-2(n+1)\Omega(\,\mathrm{d}(\zeta,K))} \frac{|\,\mathrm{d}\zeta|}{2\pi} \\ &\lesssim |K_\varepsilon| + \mathrm{e}^{-2(n+1)\Omega(\varepsilon)}. \end{aligned}$$

∎

THEOREM 4.3. *Let $K$ be a generalized Cantor set associated to a sequence $(a_n)_n$ satisfying (4.2) and let $\Omega(t) = t^\beta$ such that $\beta > \mu_K$, then*

$$N_\varphi(z) = O\Big((1-|z|)^{\mu_K/\beta} (\log 1/(1-|z|))^{\mu_K/\beta}\Big), \qquad |z| \to 1-\,.$$

*Proof.* By (4.3) and Lemma 4.2, we get

$$N_\varphi(z) \lesssim \inf_{\varepsilon>0} \Big\{ \varepsilon^{\mu_K} + \mathrm{e}^{-\frac{2\varepsilon^\beta}{1-|z|}} \Big\}, \qquad |z| < 1\,.$$

It suffice to choose $\varepsilon^\beta = (1-|z|)\big(\log 1/(1-|z|)^{\mu_K/\beta}\big)$. ∎

Now we consider the Dirichlet space $\mathcal{D}_\alpha$ where $0 < \alpha < 1$.

Theorem 4.4. *Let $0 < \alpha < 1$. Let $K$ be a generalized Cantor set associated to a sequence $(a_n)_n$ satisfying (4.2) such that $\alpha + \mu_K \geq 1$. Let $\Omega(t) = t^\beta$ such that $\beta < \min\{(1-\alpha)/2, \alpha + \mu_K - 1\}$. Let $\varphi = \varphi_{\Omega,K}$, then*

$$N_{\varphi,\alpha}(z) = O\Big((1-|z|)^{(\alpha+\mu_K-1)/\beta}\Big) \quad (z \to 1-)\,.$$

*Proof.* Since $\beta < (1-\alpha)/2$, there exists $\gamma > 2/(1-\alpha)$ such that $\Omega(t^\gamma)$ is concave. Note that

$$\mathcal{D}_\alpha(\varphi^n_{\Omega,K}) = \mathcal{D}_\alpha(\varphi_{n\Omega,K})\,.$$

Thus, by Lemma 4.1 and (4.2)

$$\begin{aligned}
\mathcal{D}_\alpha(\varphi^n_{\Omega,K}) &= \mathcal{D}_\alpha(\varphi_{n\Omega,K}) \\
&\leq c_1 n^2 \int_0^{2\pi} \Omega'(t)^2 t^\alpha |K_t| \mathrm{e}^{-2n\Omega(t)}\,\mathrm{d}t \\
&= c_1 n^2 \int_0^{2\pi} t^{2\beta-2+\alpha+\mu_K} \mathrm{e}^{-2nt^\beta}\,\mathrm{d}t \\
&\leq c_2 n^2 \int_0^1 u^{(\beta+\alpha+\mu_K-1)/\beta} \mathrm{e}^{-nu}\,\mathrm{d}u \\
&= O\Big(1/n^{(\alpha+\mu_K-1)/\beta}\Big).
\end{aligned}$$

The proof now follows from the Theorem 2.4. ∎

4.2. Examples of Hilbert-Schmidt composition operators. Now we shall give some examples of operators in the Hilbert Schmidt class. Let $\mathcal{H}$ be a Hilbert space. We denote by $\mathcal{S}_2(\mathcal{H})$ the class of Hilbert Schmidt operators.

We need the following lemma.

LEMMA 4.5. *Let $0 < \alpha < 1$ and let $\varphi$ be a holomorphic self-map of $\mathbb{D}$. The following statements are equivalent:*

(i) $C_\varphi \in \mathcal{S}_2(\mathcal{D}_\alpha)$;

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\mathcal{D}_\alpha(\varphi^n)}{(1+n)^{1-\alpha}} < \infty$;

(iii) $\displaystyle\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2+\alpha}}\, \mathrm{d}A_\alpha(z) < \infty$;

(iv) $\displaystyle\int_{\mathbb{D}} \frac{N_{\varphi_\alpha}(z)}{(1-|z|^2)^{2+\alpha}}\, \mathrm{d}A(z) < \infty$.

*Proof.* We first prove the equivalence (i) and (ii). Let $e_n = z^n/(1+n)^{\frac{1-\alpha}{2}}$. Since $(e_n)_{n=0}^{\infty}$ is an orthonormal basis of $\mathcal{D}_\alpha$ and $C_\varphi(e_n) = \varphi^n/(1+n)^{\frac{1-\alpha}{2}}$, then $C_\varphi \in S_2(\mathcal{D}_\alpha)$ if and only if

$$\sum_{n=1}^{\infty} \|C_\varphi(e_n)\|_\alpha^2 = \sum_{n\geq 1} \frac{|\varphi(0)|^{2n}}{(1+n)^{1-\alpha}} + \sum_{n=1}^{\infty} \frac{\mathcal{D}_\alpha(\varphi^n)}{(1+n)^{1-\alpha}} < \infty .$$

Note that

$$\sum_{n\geq 1} \frac{|\varphi(0)|^{2n}}{(1+n)^{1-\alpha}} \asymp \frac{|\varphi(0)|^2}{(1-|\varphi(0)|^2)^\alpha} < \infty .$$

Now we prove the equivalence (ii) and (iii). We have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\mathcal{D}_\alpha(\varphi^n)}{(1+n)^{1-\alpha}} &\asymp \int_{\mathbb{D}} \sum_{n=1}^{\infty} (1+n)^{1+\alpha} |\varphi(z)|^{2n-2} |\varphi'(z)|^2 \, \mathrm{d}A_\alpha(z) \\ &\asymp \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2+\alpha}}\, \mathrm{d}A_\alpha(z) .\end{aligned}$$

Finally the equivalence (iii) and (iv) follows from the change of variable Lemma 2.1. ∎

The following result was obtained in [5] for the Dirichlet space, $\alpha = 0$.

THEOREM 4.6. *Let $0 < \alpha < 1$. Let $K$ be a generalized Cantor set satisfying (4.2), and let $\Omega : [0, 2\pi] \to \mathbb{R}^+$ be an increasing function such that $t \to \Omega(t^\gamma)$ is concave for some $\gamma > 2/(1-\alpha)$. If*

$$\int_0^1 \frac{\Omega'(t)^2}{\Omega(t)^{2+\alpha}}\, t^\alpha |K_t| \, \mathrm{d}t < \infty , \tag{4.4}$$

*then* $C_{\varphi_{\Omega,K}} \in S_2(\mathcal{D}_\alpha)$.

*Proof.* By Lemma 4.5 and Lemma 4.1, we have

$$\mathcal{D}_\alpha(\varphi_{\Omega,K}) \le c \int_0^{2\pi} \Omega'(t)^2 \, \mathrm{e}^{-2\Omega(t)} \, t^\alpha |K_t| \, \mathrm{d}t \, .$$

Since $\varphi^n_{\Omega,K} = \varphi_{n\Omega,K}$, we obtain

$$\begin{aligned}\int_{\mathbb{D}} \frac{|\varphi'_{\Omega,K}(z)|^2}{(1-|\varphi_{\Omega,K}(z)|^2)^{2+\alpha}} \, \mathrm{d}A_\alpha(z) &\asymp \sum_{n=1}^\infty \frac{\mathcal{D}_\alpha(\varphi_{n\Omega,K})}{n^{1-\alpha}} \\ &\le c_1 \int_0^1 \Omega'(t)^2 \, t^\alpha |K_t| \sum_{n=1}^\infty n^\alpha \mathrm{e}^{-2n\Omega(t)} \, \mathrm{d}t \\ &\le c_2 \int_0^1 \frac{\Omega'(t)^2}{\big[1-\mathrm{e}^{-2\Omega(t)}\big]^{2+\alpha}} \, t^\alpha |K_t| \, \mathrm{d}t \, ,\end{aligned}$$

where $c_1$ and $c_2$ are positives constants. Noting that

$$1 - \mathrm{e}^{-2\Omega(t)} \asymp \Omega(t) \, ,$$

we get the result. ∎


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