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A Characterization of the Essential Pseudospectra and Application to a Transport Equation

Aymen Ammar, Aref Jeribi

Département de Mathématiques, Université de Sfax, Faculté des sciences de Sfax, Route de soukra Km 3.5, B. P. 1171, 3000, Sfax, Tunisie

Aref.Jeribi@fss.rnu.tn

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Abstract: In this paper, we introduce and study the essential pseudospectra of closed, densely defined linear operators in the Banach space. We start by giving the definition and we investigate the characterization, the stability and some properties of this essential pseudospectra. The obtained results are used to describe the essential pseudospectra of transport operators.

Key words: Pseudospectra, essential spectra, compact operators, Fredholm perturbations, transport operators.

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1. Introduction

Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X,Y)$ $\big($resp. $\mathcal{C}(X,Y)\big)$ the set of all bounded (resp. closed, densely defined) linear operators from X into Y and we denote by $\mathcal{K}(X,Y)$ the subspace of compact operators from X into Y. If $T \in \mathcal{C}(X,Y)$ then $\rho(T)$ denotes the resolvent set of T, $\alpha(T)$ the dimension of the kernel $N(T)$ and $\beta(T)$ the codimension of $R(T)$ in Y. Let $\sigma(T)$ (resp. $\rho(T)$) denote the spectrum (resp. the resolvent set) of T. The set of upper semi-Fredholm operators is defined by:

$$\Phi_+(X) := \big\{T \in \mathcal{C}(X,Y) \,:\, \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } X\big\},$$

the set of lower semi-Fredholm operators defined by:

$$\Phi_-(X) := \big\{T \in \mathcal{C}(X,Y) \,:\, \beta(T) < \infty \text{ and } R(T) \text{ is closed in } X\big\}.$$

The operators in $\Phi_\pm(X,Y) := \Phi_+(X,Y) \cup \Phi_-(X,Y)$ are called semi-Fredholm operators on X into Y while $\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$ denotes the set of Fredholm operators on X into Y. If $X = Y$, the sets $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ are replaced, by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\Phi(X)$, $\Phi_+(X)$

and $\Phi_-(X)$, respectively. A complex number λ is in Φ_T if $\lambda - T$ is in $\Phi(X)$. For $T \in \Phi(X)$, the number $i(T) = \alpha(T) - \beta(T)$ is called the index of T. It is clear that if $T \in \Phi(X)$ then $i(T) < \infty$. If $T \in \Phi_+(X) \setminus \Phi(X)$ then $i(T) = -\infty$ and if $T \in \Phi_-(X) \setminus \Phi(X)$ then $i(T) = +\infty$. An operator $F \in \mathcal{L}(X,Y)$ is called a Fredholm perturbation, if $T + F \in \Phi(X,Y)$ whenever $T \in \Phi(X,Y)$. We denote by $\mathcal{F}(X)$ the set of Fredholm perturbations. A Banach space X is said to have the Dunford-Pettis property (for short property DP) if for each Banach space Y every weakly compact operator $T : X \longrightarrow Y$ takes weakly compact sets in X into norm compact sets of Y. For example it is well known that any L_1-space has the property DP and an operator $S \in \mathcal{L}(X)$ is called strictly singular if, for every infinite-dimensional subspace M of X, the restriction of S to M is not a homeomorphism. The family of weakly compact operators from X to X is denoted by $\mathcal{W}(X)$.

Let T be a closed linear operator on a Banach space X. For $x \in \mathcal{D}(T)$ the graph norm of x is defined by

$$\|x\|_T := \|x\| + \|Tx\|\,.$$

It follows from the closedness of T that $\mathcal{D}(T)$ endowed with the norm $\|\,.\,\|_T$ is a Banach space. Let X_T denote $(\mathcal{D}(T), \|\,.\,\|_T)$. In this new space the operator T satisfies $\|Tx\| \leq \|x\|_T$ and consequently T is a bounded operator from X_T into X. If $\hat{T}$ denotes the restriction of T to $\mathcal{D}(T)$, we observe that $\alpha(\hat{T}) = \alpha(T)$ and $\beta(\hat{T}) = \beta(T)$.

There are several and in general non-equivalent definitions of the essential spectrum of a bounded linear operator on a Banach space. For a self-adjoint operator in a Hilbert space, there seems to be only one reasonable way to define the essential spectrum: The set of all points of the spectrum that are not isolated eigenvalues of finite algebraic multiplicity. In this paper, we are interested by the following essential spectra:

$$\sigma_e(T) := \mathbb{C} \setminus \rho_e(T)\,,$$

$$\sigma_b(T) := \sigma(T) \setminus \sigma_d(T)\,,$$

where $\rho_e(T) := \{\lambda \in \Phi_T \,:\, i(\lambda - T) = 0\}$ and $\sigma_d(T)$ is the set of isolated points λ of the spectrum such that the corresponding Riesz projectors P_λ is finite rank operator with kernel denote by K_λ; $\sigma_e(.)$ is the Schechter essential spectrum (see for example [14, 20, 21]) and $\sigma_b(.)$ is the Browder spectrum [17]. In the 2000s, A. Jeribi and their collaborators have continued the research on the essential spectra and have applied the results to transport operators (see [6, 7, 8, 9, 10, 11, 12]).

It is well established that the spectrum of a self-adjoint operator is of crucial importance in understanding its action in various applied contexts. For highly non-self-adjoint operators, on the other hand, there is increasing evidence that the spectrum is often not very helpful, and that the pseudospectra are of more importance. The definition of pseudospectra of closed densely linear operator T for every $\varepsilon > 0$ is given by:

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{\lambda \in \mathbb{C} \,:\, \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\right\}.$$

By convention we write $\|(\lambda-T)^{-1}\| = \infty$ if $(\lambda-T)^{-1}$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma(T)$. This means that the pseudospectra can be introduced as a zone of spectral instability. The concept of the pseudospectra was introduced perhaps first time by J.M. Varah [24] and has been subsequently employed by others authors for example, H. Landau [15], L.N. Trefethen [22], D. Hinrichsen et al [5] and E.B. Davies [2]. Especially due to L.N. Trefethen, who developed this idea for matrices and operators, he used this concept to the study of interesting problems in mathematical physics.

In [1], F. Abdmouleh, A. Ammar and A. Jeribi defined the notion of pseudo-Browder essential spectra of densely closed, linear operators in the Banach space by:

$$\sigma_{b,\varepsilon}(T) = \sigma_b(T) \cup \left\{\lambda \in \mathbb{C} \,:\, \|R_b(\lambda, T)\| > \frac{1}{\varepsilon}\right\},$$

where $R_b(\lambda, T) = \big((\lambda - T)|_{K_\lambda}\big)^{-1}\big(I - P_\lambda\big) + P_\lambda$ and by convention we write $\|R_b(\lambda, T)\| = \infty$ if $R_b(\lambda, T)$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma_b(T)$.

The notion of essential pseudospectra can be extended by devoting our studies on the essential spectrum and we have by

$$\sigma_{e,\varepsilon}(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma_\varepsilon(A + K)\,.$$

In the following we characterize the essential pseudospectra by: $\lambda \notin \sigma_{e,\varepsilon}(A)$ if and only if, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have

$$\lambda \in \Phi_{A+D} \qquad \text{and} \qquad i(A + D - \lambda) = 0\,.$$

This is equivalent to say that

$$\sigma_{e,\varepsilon}(A) = \bigcup_{\|D\|<\varepsilon} \sigma_e(A + D)\,.$$

When dealing with essential pseudospectra of closed, densely defined linear operators on Banach spaces, one of the main problems consists of studying the invariance of the essential pseudospectra of these operators subjected to various kinds of perturbation. Let $\varepsilon > 0$ and linear operator A on a Banach space X, the question is what are the conditions that we must impose on the operator $K \in \mathcal{C}(X)$ such that $\sigma_\varepsilon(A+K) = \sigma_\varepsilon(A)$. If K is a compact operator on the Banach space X, then the result follows from the definitions of $\sigma_\varepsilon(.)$ and if K is Fredholm perturbation that $\sigma_\varepsilon(A+K) = \sigma_\varepsilon(A)$. In fact, if for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $A(B+D) \in \mathcal{F}(X)$ and $(B+D)A \in \mathcal{F}(X)$ then

$$\sigma_{e,\varepsilon}(A+B)\backslash\{0\} = \big[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\big]\backslash\{0\}\,.$$

Finally, we will apply the results described above to investigate the essential pseudospectra of the following integro-differential operator:

$$\begin{aligned} A_H\psi(x,\xi) &= -\xi\frac{\partial\psi}{\partial x} - \sigma(\xi)\psi(x,\xi) + \int_{-1}^{1}\kappa(x,\xi,\xi')\psi(x,\xi')\,\mathrm{d}\xi' \\ &= T_H\psi + K\psi\,, \end{aligned} \tag{1.1}$$

with general boundary conditions where T_H denotes the streaming operator and K stands for the collision one (the integral part of A_H), $x \in [-a,a]$, $a > 0$ and $\xi \in [-1,1]$. This operator describes the transport of particles (neutrons, photons, molecules of gas, etc). in a plane parallel domain with a width of $2a$ mean free paths. The function $\psi(x,\xi)$ represents the number (or probability) density of gas particles having the position x and the direction cosine of propagation ξ. (The variable ξ may be thought of as the cosine of the angle between the velocity of particles and the x direction.) The functions $\sigma(.)$ and $\kappa(.\,,.)$ are called, respectively, the collision frequency and scattering kernel. The boundary conditions are modeled by $\psi_{|\Gamma_-} = H\psi_{|\Gamma_+}$, where Γ_- (resp. Γ_+) is the incoming (resp. outgoing) part of the phase space boundary, $\psi_{|\Gamma_-}$ (resp. $\psi_{|\Gamma_+}$) is the restriction of ψ to Γ_- (resp. Γ_+) and H is a linear bounded operator from a suitable function space on Γ_+ to a similar one on Γ_-. There is a wealth of literature treating the transport equation in slab geometry with different boundary conditions (see [7, 8, 9]).

We organize our paper in the following way: In Section 2 we present the notion of the essential pseudospectra. The main results of this section are Theorem 2.1 and Theorem 2.2. Finally, in Section 3 we apply the results obtained in the last section to investigate the essential pseudospectra of one-dimensional transport operator.

2. Main results

Definition 2.1. Let X be a Banach space, $\varepsilon > 0$ and $A \in \mathcal{C}(X)$. We define the essential pseudospectra of the operator A by

$$\sigma_{e,\varepsilon}(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma_\varepsilon(A+K)\,.$$

Remark 2.1. It follows from Definition 2.1 and the properties of pseudospectra (see, for example [3, 23]) that

(i) $\sigma_{e,\varepsilon}(A) \subset \sigma_\varepsilon(A)\,;$

(ii) $\bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(A) = \sigma_e(A)\,;$

(iii) if $\varepsilon_1 < \varepsilon_2$ then $\sigma_e(A) \subset \sigma_{e,\varepsilon_1}(A) \subset \sigma_{e,\varepsilon_2}(A)\,;$

(iv) $\sigma_{e,\varepsilon}(A+K) = \sigma_{e,\varepsilon}(A)$ for all $K \in \mathcal{K}(X)\,.$

The following theorem gives a characterization of the essential pseudospectra by means of Fredholm operator.

Theorem 2.1. *Let X be a Banach space, $\varepsilon > 0$ and $A \in \mathcal{C}(X)$. Then, $\lambda \notin \sigma_{e,\varepsilon}(A)$ if and only if, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have*

$$\lambda \in \Phi_{A+D} \qquad \text{and} \qquad i(A+D-\lambda) = 0\,.$$

Proof. Let $\lambda \notin \sigma_{e,\varepsilon}(A)$. Using [3, Theorem 9.2.13 (ii), p. 259] we infer that there exists a compact operator K on X such that

$$\lambda \notin \bigcup_{\|D\|<\varepsilon} \sigma(A+K+D)\,.$$

So, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $\lambda \in \rho(A+D+K)$. Therefore,

$$A+D+K-\lambda \in \Phi(X) \qquad \text{and} \qquad i(A+D+K-\lambda) = 0\,,$$

for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. It comes from [21, Theorem 7.26, p. 172] that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have

$$A+D-\lambda \in \Phi(X) \qquad \text{and} \qquad i(A+D-\lambda) = 0\,.$$

Conversely, we suppose for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $(A+D-\lambda) \in \Phi(X)$ and $i(A+D-\lambda) = 0$. Without loss of generality, we may assume $\lambda = 0$.

Let $n = \alpha(A+D-\lambda) = \beta(A+D-\lambda)$, $(x_1, \dots, x_n)$ be basis for $N(A+D-\lambda)$ and $(y'_1, \dots, y'_n)$ be bases for the $R(A + D - \lambda)^\circ$ (where $y \in R(A + D - \lambda)^\circ$ if $y'_j(y) = 0$ for all $1 \leq j \leq n$). Then, by [21, Lemma 4.14, p. 92] there exist $x'_1, \dots, x'_n \in X^*$ (the adjoint space of X) and $y_1, \dots, y_n \in Y$ such that

$$x'_j(x_k) = \delta_{jk}\,, \qquad y_j(y_k) = \delta_{jk}\,, \qquad 1 \leq j\,,\ k \leq n\,. \tag{2.1}$$

Let

$$Kx = \sum_{k=1}^{n} x'_k(x) y_k\,, \qquad x \in X\,; \tag{2.2}$$

K is an operator of finite rank on X. First we will proved for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ that

$$N(A + D) \cap N(F) = \{0\}\,, \qquad R(A + D) \cap R(F) = \{0\}\,. \tag{2.3}$$

Let $x \in N(A + D)$, then

$$x = \sum_{k=1}^{n} \alpha_k x_k\,,$$

therefore $x_j(x) = \alpha_j$, $1 \leq j \leq n$. On the other hand, if $N(F)$ then $x'_j(x) = 0$, $1 \leq j \leq n$. This proves the first relation in equation (2.3).

The second inclusion is similar. In fact, if $y \in R(F)$, then

$$y = \sum_{k=1}^{n} \alpha_k y_k\,,$$

and hence

$$y_j(y) = \alpha_j\,, \qquad 1 \leq j \leq n\,.$$

But, if $y \in R(A + D)$, then

$$y'_j(y) = 0\,, \qquad 1 \leq j \leq n\,.$$

This gives the second relation in equation (2.3). On the other hand K is compact operator. We deduce from [21, Theorem 7.26, p. 172] that $0 \in \Phi_{A+F+D}$ and $i(A + D + F) = 0$. If $x \in N(A + D + F)$ then $(A + D)x$ is in $R(A + D) \cap R(F)$ this implies that $x \in N(A + D) \cap N(F)$ hence $x = 0$. Thus $\alpha(A + D + F) = 0$. In the same way, we will prove that $R(A + D + F) = X$. Hence, $0 \in \rho(A + D + F)$. This implies that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $0 \notin \sigma(A + D + K)$. Also, $0 \notin \bigcap_{K \in \mathcal{K}(X)} \sigma_\varepsilon(A + K)$. So, $0 \notin \sigma_{e,\varepsilon}(A)$. ∎

Remark 2.2. It follows, immediately, from Theorem 2.1 and [21, Theorem 7.27, p. 172] that

$$\sigma_{e,\varepsilon}(A) = \bigcup_{\|D\|<\varepsilon} \sigma_e(A+D)\,.$$

THEOREM 2.2. *Let X be a Banach space, $\varepsilon > 0$ and $A \in \mathcal{C}(X)$. Then*

$$\sigma_{e,\varepsilon}(A) := \bigcap_{F\in\mathcal{F}(X)} \sigma_\varepsilon(A+F)\,.$$

Proof. Let $\mathcal{O} := \bigcap_{F\in\mathcal{F}(X)} \sigma_\varepsilon(A+F)$. Since, $\mathcal{K}(X) \subset \mathcal{F}(X)$ we infer that $\mathcal{O} \subset \sigma_{\varepsilon,e}(A)$. Conversely, let $\lambda \notin \mathcal{O}$ then there exist $F \in \mathcal{F}(X)$ such that $\lambda \notin \sigma_\varepsilon(A+F)$. Thus, by [3, Theorem 9.2.13 (ii), p. 259] we see that $\lambda \in \rho(A+D+F)$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. So,

$$A+D+F-\lambda \in \Phi(X) \qquad \text{and} \qquad i(A+D+F-\lambda) = 0\,.$$

The use of [13, Lemma 2.1 (i)] makes us conclude that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$,

$$A+D-\lambda \in \Phi(X) \qquad \text{and} \qquad i(A+D-\lambda) = 0\,.$$

Finally, Theorem 2.1 shows that $\lambda \notin \sigma_{e,\varepsilon}(A)$. ∎

Remark 2.3. (i) It follows, from Theorem 2.2 that $\sigma_{e,\varepsilon}(A+F) = \sigma_{e,\varepsilon}(A)$ for all $F \in \mathcal{F}(X)$.

(ii) It is proved in [16, Section 3] that if X is a Banach space with the property DP, then $\mathcal{W}(X) \subset \mathcal{F}(X)$. Thus the essential pseudospectra is invariant under weakly compact perturbations on this class of Banach spaces.

COROLLARY 2.1. *Let X be a Banach space and $\mathfrak{I}(X)$ be any subset of $\mathcal{L}(X)$. If $\mathcal{K}(X) \subset \mathfrak{I}(X) \subset \mathcal{F}(X)$, then*

$$\sigma_{\varepsilon,e}(A) = \bigcap_{J\in\mathfrak{I}(X)} \sigma_\varepsilon(A+J)\,.$$

Remark 2.4. It comes from Corollary 2.1 that $\sigma_{e,\varepsilon}(A+J) = \sigma_{e,\varepsilon}(A)$ for all $J \in \mathfrak{I}(X)$ such that $\mathcal{K}(X) \subset \mathfrak{I}(X) \subset \mathcal{F}(X)$.

THEOREM 2.3. *Let $\varepsilon > 0$, A and B be two elements of $\mathcal{C}(X)$ such that $0 \notin \sigma_e(A) \cup \sigma_e(B)$. Assume that there exist A_0 and B_0 be two bounded operators in the Banach space X such that*

$$AA_0 = I - F_1\,, \tag{2.4}$$

$$BB_0 = I - F_2\,, \tag{2.5}$$

with $F_i \in \mathcal{F}(X)$, $i = 1, 2$. If the difference $A_0 - B_0 \in \mathcal{F}(X)$ then

$$\sigma_{e,\varepsilon}(A) = \sigma_{e,\varepsilon}(B)\,.$$

Proof. Using equations (2.4) and (2.5) we infer that for any scalar λ

$$(A + D - \lambda)A_0 - (B + D - \lambda)B_0 = F_2 - F_1 + (D - \lambda)(A_0 - B_0)\,. \tag{2.6}$$

Let $\lambda \notin \sigma_{e,\varepsilon}(A)$ then $A+D-\lambda$ is Fredholm operator and $i(A+D-\lambda) = 0$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Since $A+D$ is closed and $\mathcal{D}(A+D) = \mathcal{D}(A)$ endowed with the graph norm is a Banach space denoted by X_{A+D} and using [20, Corollary 7.6, p. 160] we obtain

$$\widehat{A+D} - \lambda \in \Phi^b(X_{A+D}, X)\,.$$

Moreover, $F_1 \in \mathcal{F}(X)$. Using equation (2.4) and [11, Theorem 2.1] find that $A_0 \in \Phi^b(X, X_{A+D})$. Thus

$$\big(\widehat{A+D} - \lambda\big)A_0 \in \Phi^b(X)\,. \tag{2.7}$$

Next, if the difference $A_0 - B_0 \in \mathcal{F}(X)$, applying equation (2.6) we get

$$(A + D - \lambda)A_0 - (B + D - \lambda)B_0 \in \mathcal{F}(X)\,.$$

Also, it follows from equation (2.7) that $\big(\widehat{B+D} - \lambda\big)B_0 \in \Phi^b(X)$ and

$$i\big[\big(\widehat{B+D} - \lambda\big)B_0\big] = i\big[\big(\widehat{A+D} - \lambda\big)A_0\big] = 0\,. \tag{2.8}$$

Since $B \in \mathcal{C}(X)$, using equation (2.5) and arguing as in the last part we conclude that

$$B_0 \in \Phi^b(X, X_{B+D})\,.$$

Thus, since $(B + D - \lambda)B_0$ is Fredholm operator the use of [20, Theorem 7.12, p. 162] shows that $\widehat{B+D} - \lambda \in \Phi^b(X_{B+D}, X)$. This implies that $B + D - \lambda$

is Fredholm operator. On the other hand, $0 \notin \sigma_e(A) \cup \sigma_e(B)$ then $i(A) = i(B) = 0$. Therefore, using equations (2.4)−(2.5) and [11, Theorem 2.1] we have that $i(A_0) = i(B_0) = 0$. This together with equation (2.6) shows that

$$i(A + D - \lambda) = i(B + D - \lambda) = 0\,.$$

Thus, $\lambda \notin \sigma_{e,\varepsilon}(B)$. This proves that $\sigma_{e,\varepsilon}(B) \subset \sigma_{e,\varepsilon}(A)$. The opposite inclusion follows by symmetry. ∎

LEMMA 2.1. *Let X be a Banach space, $\varepsilon > 0$, A and B two elements of $\mathcal{C}(X)$. Assume that for all bounded operator D such that $\|D\| < \varepsilon$ the operator B is $(A + D)$-compact, then*

$$\sigma_{e,\varepsilon}(A) = \sigma_{e,\varepsilon}(A + B)\,.$$

Proof. Let $\lambda \notin \sigma_{e,\varepsilon}(A)$ then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $A + D - \lambda$ is a Fredholm operator and $i(A + D - \lambda) = 0$. Since, B is (A+D)-compact, applying [21, Theorem 3.3, p. 12], we get

$$\lambda \in \Phi_{A+B+D} \qquad \text{and} \qquad i(A + B + D - \lambda) = 0\,.$$

Therefore $\lambda \notin \sigma_{e,\varepsilon}(A + B)$. We conclude that

$$\sigma_{e,\varepsilon}(A + B) \subset \sigma_{e,\varepsilon}(A)\,.$$

Conversely, let $\lambda \notin \sigma_{e,\varepsilon}(A + B)$ then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $A + B + D - \lambda$ is a Fredholm operator and $i(A + B + D - \lambda) = 0$. On the other hand, B is $(A + D)$-compact, using [21, Theorem 2.12, p. 9] we infer that B is $(A + B + D)$-compact, then

$$\lambda \in \Phi_{A+D} \qquad \text{and} \qquad i(A + D - \lambda) = 0\,.$$

So, $\lambda \notin \sigma_{e,\varepsilon}(A)$. This proves that

$$\sigma_{e,\varepsilon}(A) \subset \sigma_{e,\varepsilon}(A + B)\,.$$

∎

The following theorem gives a relation between the essential pseudospectra of the sum of the two bounded linear operators and essential pseudospectra of each of these operators.

THEOREM 2.4. *Let X be a Banach space, $\varepsilon > 0$, A and B be two elements of $\mathcal{L}(X)$. If for all bounded operator D such $\|D\| < \varepsilon$ we have $A(B+D) \in \mathcal{F}(X)$, then*

$$\sigma_{e,\varepsilon}(A+B)\backslash\{0\} \subseteq \big[\sigma_e(A) \cup \sigma_{\varepsilon,e}(B)\big]\backslash\{0\}\,.$$

If, further $(B+D)A \in \mathcal{F}(X)$, then

$$\sigma_{e,\varepsilon}(A+B)\backslash\{0\} = \big[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\big]\backslash\{0\}\,.$$

Proof. For $\lambda \in \mathbb{C}$, we can write

$$(\lambda - A)(\lambda - B - D) = A(B+D) + \lambda(\lambda - A - B - D) \tag{2.9}$$

and

$$(\lambda - B - D)(\lambda - A) = (B+D)A + \lambda(\lambda - A - B - D)\,. \tag{2.10}$$

Let $\lambda \notin \big[\sigma_e(A) \cup \sigma_{\varepsilon,e}(B)\big]\backslash\{0\}$. Then, $(\lambda - A) \in \Phi(X)$ and for all $\|D\| < \varepsilon$, $(\lambda - B - D) \in \Phi(X)$. It follows from [20, Theorem 5.7, p. 106] that

$$(\lambda - A)(\lambda - B - D) \in \Phi(X)\,.$$

Since $A(B+D) \in \mathcal{F}(X)$, applying equation (2.9), we have $(\lambda - A - B - D) \in \Phi(X)$ then $\lambda \notin \sigma_{e,\varepsilon}(A+B)$. Therefore

$$\sigma_{e,\varepsilon}(A+B)\backslash\{0\} \subseteq \big[\sigma_e(A) \cup \sigma_{\varepsilon,e}(B)\big]\backslash\{0\}\,. \tag{2.11}$$

Now, we prove the inverse inclusion of equation (2.11).

Suppose $\lambda \notin \sigma_{e,\varepsilon}(A+B)\backslash\{0\}$, then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have $(\lambda - A - B - D) \in \Phi(X)$. Since $A(B+D) \in \mathcal{F}(X)$, $(B+D)A \in \mathcal{F}(X)$ then by equations (2.9) and (2.10), we have

$$(\lambda - A)(\lambda - B - D) \in \Phi(X) \qquad \text{and} \qquad (\lambda - B - D)(\lambda - A) \in \Phi(X)\,.$$

Applying [19, Theorem 6 (iii), p. 151], it is clear that $(\lambda - A) \in \Phi(X)$ and for all $\|D\| < \varepsilon$ we have $(\lambda - B - D) \in \Phi(X)$. Therefore $\lambda \notin \sigma_{e,S}(A) \cup \sigma_{e,S}(B)$. This proves

$$\sigma_{e,\varepsilon}(A+B)\backslash\{0\} = \big[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\big]\backslash\{0\}\,.$$

∎

3. Application to transport equation

In this section, we will apply the results of Section 2 to study the essential pseudospectra of the transport operator equation (1.1). Let

$$X_p = L_p\Big([-a,a]\times[-1,1],\, \mathrm{d}x\,\mathrm{d}\xi\Big), \qquad a>0 \ \text{ and } \ p\in[1,\infty).$$

We consider the boundary spaces:

$$\begin{aligned} X_p^o &:= L_p\Big(\{-a\}\times[-1,0], |\xi|\,\mathrm{d}\xi\Big)\times L_p\Big(\{a\}\times[0,1], |\xi|\,\mathrm{d}\xi\Big) \\ &:= X_{1,p}^o \times X_{2,p}^o \end{aligned}$$

and

$$\begin{aligned} X_p^i &:= L_p\Big(\{-a\}\times[0,1], |\xi|\,\mathrm{d}\xi\Big)\times L_p\Big(\{a\}\times[-1,0], |\xi|\,\mathrm{d}\xi\Big) \\ &:= X_{1,p}^i \times X_{2,p}^i\,, \end{aligned}$$

respectively equipped with the norms

$$\begin{aligned} \|\psi^o\|_{X_p^o} &= \left(\|\psi_1^o\|^p_{X_{1,p}^o} + \|\psi_2^o\|^p_{X_{2,p}^o}\right)^{\frac{1}{p}} \\ &= \left[\int_{-1}^0 |\psi(-a,\xi)|^p|\xi|\,\mathrm{d}\xi + \int_0^1 |\psi(a,\xi)|^p|\xi|\,\mathrm{d}\xi\right]^{\frac{1}{p}} \end{aligned}$$

and

$$\begin{aligned} \|\psi^i\|_{X_p^i} &= \left(\|\psi_1^i\|^p_{X_{1,p}^i} + \|\psi_2^i\|^p_{X_{2,p}^i}\right)^{\frac{1}{p}} \\ &= \left[\int_0^1 |\psi(-a,\xi)|^p|\xi|\,\mathrm{d}\xi + \int_{-1}^0 |\psi(a,\xi)|^p|\xi|\,\mathrm{d}\xi\right]^{\frac{1}{p}}. \end{aligned}$$

Let $\mathcal{W}_p$ the space defined by:

$$\mathcal{W}_p = \left\{\psi\in X_p \,:\, \xi\frac{\partial\psi}{\partial x}\in X_p\right\}.$$

It is well-known that any function ψ in $\mathcal{W}_p$ possesses traces on the spatial boundary $\{-a\}\times(-1,0)$ and $\{a\}\times(0,1)$ which respectively belong to the spaces X_p^o and X_p^i (see [4]). They are denoted, respectively, by ψ^o and ψ^i. Let H be the boundary operator

$$\begin{cases} H : X_p^o \longrightarrow X_p^i \,, \\ H\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \end{cases}$$

where for all $H_{11} \in \mathcal{L}(X_{1,p}^o, X_{1,p}^i)$, $H_{12} \in \mathcal{L}(X_{2,p}^o, X_{1,p}^i)$, $H_{21} \in \mathcal{L}(X_{1,p}^o, X_{2,p}^i)$ and $H_{22} \in \mathcal{L}(X_{2,p}^o, X_{2,p}^i)$.

We define the streaming operator T_H by

$$\begin{cases} T_H : \mathcal{D}(T_H) \subseteq X_p & \longrightarrow & X_p \\ \qquad\qquad \psi & \longmapsto & T_H\psi(x,\xi) = -\xi \dfrac{\partial \psi}{\partial x}(x,\xi) - \sigma(\xi)\psi(x,\xi) \,, \\ \mathcal{D}(T_H) = \big\{\psi \in \mathcal{W}_p \,:\, \psi^o = H\psi^i\big\} \,, \end{cases}$$

where $\sigma(.) \in L^\infty(-1,1)$, $\psi^o = (\psi_1^o, \psi_2^o)^\perp$ and $\psi^i = (\psi_1^i, \psi_2^i)^\perp$ with ψ_1^o, ψ_2^o, ψ_1^i and ψ_2^i given by

$$\begin{aligned} \psi_1^i &: \xi \in (0,1) \longmapsto \psi(-a,\xi) \,, \\ \psi_2^i &: \xi \in (-1,0) \longmapsto \psi(a,\xi) \,, \\ \psi_1^0 &: \xi \in (-1,0) \longmapsto \psi(-a,\xi) \,, \\ \psi_2^0 &: \xi \in (0,1) \longmapsto \psi(a,\xi) \,. \end{aligned}$$

Finally, we denote by K the following partially integral operator on X_p

$$\begin{cases} K : X_p & \longrightarrow & X_p \\ \quad \psi & \longmapsto & \displaystyle\int_{-1}^{1} \kappa(x,\xi,\xi')\psi(x,\xi')\,\mathrm{d}\xi' \,, \end{cases}$$

where the scattering kernel $\kappa(.,.,.)$ is a measurable function from $[-a,a] \times [-1,1] \times [-1,1]$ to $\mathbb{R}$. Observe that the operator K acts only on the variable ξ, so x may be viewed merely as a parameter in $[-a,a]$. Hence we may consider K as a function $K : x \in [-a,a] \longmapsto K(x) \in \mathcal{Z}$ where $\mathcal{Z} := \mathcal{L}\big(L_p\big([-1,1], d\xi\big)\big)$.

Let us now consider the resolvent equation for T_H

$$(\lambda - T_H)\psi = \varphi \,,$$

where φ is a given element of X_p and the unknown ψ must be sought in $\mathcal{D}(T_H)$.

Let λ^* be the real defined by

$$\lambda^* := \lim \inf_{|\xi| \longrightarrow 0} \sigma(\xi)\,.$$

For $\mathrm{Re}\,\lambda + \lambda^* > 0$, the solution formally given by:

$$\begin{cases} \psi(x,\xi) \;=\; \psi(-a,\xi)\, e^{-\frac{(\lambda+\sigma(\xi))|a+x|}{|\xi|}} \\ \qquad\qquad + \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{x} e^{\frac{-(\lambda+\sigma(\xi))|x-\tilde{x}|}{|\xi|}} \psi(x,\xi')\,\mathrm{d}x'\,, \qquad 0<\xi<1\,, \\ \psi(x,\xi) \;=\; \psi(a,\xi)\, e^{-\frac{(\lambda+\sigma(\xi))|a-x|}{|\xi|}} \\ \qquad\qquad + \dfrac{1}{|\xi|}\displaystyle\int_{x}^{a} e^{\frac{-(\lambda+\sigma(\xi))|x-\tilde{x}|}{|\xi|}} \psi(x',\xi)\,\mathrm{d}x'\,, \qquad -1<\xi<0\,. \end{cases} \tag{3.1}$$

Accordingly, $\psi(a,\xi)$ and $\psi(-a,\xi)$ are given by

$$\begin{cases} \psi(a,\xi) \;=\; \psi(-a,\xi)\, e^{\frac{-2a(\lambda+\sigma(\xi))}{|\xi|}} \\ \qquad\qquad + \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{a} e^{2a\frac{-(\lambda+\sigma(\xi))|a+x|}{|\xi|}} \psi(x,\xi)\,\mathrm{d}x'\,, \qquad 0<\xi<1\,, \\ \psi(-a,\xi) \;=\; \psi(a,\xi)\, e^{-\frac{2a(\lambda+\sigma(\xi))}{|\xi|}} \\ \qquad\qquad + \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{a} e^{\frac{-(\lambda+\sigma(\xi))|a-x|}{|\xi|}} \psi(x,\xi)\,\mathrm{d}x\,, \qquad -1<\xi<0\,. \end{cases} \tag{3.2}$$

In the sequel we shall need the operators

$$\begin{cases} M_\lambda : X_p^i \longrightarrow X_p^o\,, \qquad M_\lambda u := \big(M_\lambda^+ u, M_\lambda^- u\big) \quad \text{with} \\ \qquad (M_\lambda^+ u)(-a,\xi) := u(-a,\xi) e^{\frac{-2a}{|\xi|}(\lambda+\sigma(\xi))}\,, \qquad -1<\xi<0\,, \\ \qquad (M_\lambda^- u)(a,\xi) := u(a,\xi)\, e^{\frac{-2a}{|\xi|}(\lambda+\sigma(\xi))}\,, \qquad 0<\xi<1\,; \end{cases}$$

$$\begin{cases} B_\lambda : X_p^i \longrightarrow X_p\,, \qquad B_\lambda = \chi_{(-1,0)}(\xi) B_\lambda^+ u + \chi_{(0,1)}(\xi) B_\lambda^- u \quad \text{with} \\ \qquad (B_\lambda^- u)(x,\xi) := u(-a,\xi)\, e^{\frac{(\lambda+\sigma(\xi))}{|\xi|}|a-x|}\,, \qquad 0<\xi<1\,, \\ \qquad (B_\lambda^+ u)(x,\xi) := u(-a,\xi)\, e^{\frac{(\lambda+\sigma(\xi))}{|\xi|}|a-x|}\,, \qquad 1<\xi<0\,; \end{cases}$$

$$\begin{cases} G_\lambda : X_p \longrightarrow X_p^0, \qquad G_\lambda u := (G_\lambda^+\varphi, G_\lambda^-\varphi) \quad \text{with} \\ \qquad G_\lambda^+\varphi := \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{a} e^{\frac{-(\lambda+\sigma(\xi))}{|\xi|}|a-x|}\varphi(x,\xi)\,\mathrm{d}x\,, \qquad 0<\xi<1\,, \\ \qquad G_\lambda^-\varphi := \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{a} e^{\frac{-(\lambda+\sigma(\xi))}{|\xi|}|a+x|}\varphi(x,\xi)\,\mathrm{d}x\,, \qquad 1<\xi<0\,; \end{cases}$$

and

$$\begin{cases} C_\lambda : X_p \longrightarrow X_p\,, \qquad C_\lambda\varphi = \chi_{(-1,0)}(\xi)C_\lambda^+\varphi + \chi_{(0,1)}(\xi)C_\lambda^-\varphi \quad \text{with} \\ \qquad C_\lambda^-\varphi := \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{x} e^{\frac{-(\lambda+\sigma(\xi))}{|\xi|}|x'-x|}\varphi(x',\xi)\,\mathrm{d}x'\,, \qquad 0<\xi<1\,, \\ \qquad C_\lambda^+\varphi := \dfrac{1}{|\xi|}\displaystyle\int_{x}^{a} e^{\frac{-(\lambda+\sigma(\xi))}{|\xi|}|x'-x|}\varphi(x',\xi)\,\mathrm{d}x'\,, \qquad -1<\xi<0\,; \end{cases}$$

where $\chi_{(-1,0)}(.)$ and $\chi_{(0,1)}(.)$ denote, respectively the characteristic functions of the intervals $(-1,0)$ and $(0,1)$. The operators M_λ, B_λ, G_λ and C_λ are bounded on their respective spaces. In fact, their norms are bounded above, respectively, by $e^{-2a(\operatorname{Re}\lambda+\lambda^*)}$, $[p(\operatorname{Re}\lambda+\lambda^*)]^{\frac{-1}{p}}$, $[p(\operatorname{Re}\lambda+\lambda^*)]^{\frac{-1}{q}}$ and $[p(\operatorname{Re}\lambda+\lambda^*)]^{-1}$ where q denotes the conjugate of p.

Now, we may write equation (3.2) abstractly in the space X_p^o in the operator form

$$\psi^o = M_\lambda H\psi^o + G_\lambda\varphi\,.$$

Let λ_0 be the real defined by

$$\begin{cases} -\lambda^*\,, & \text{if } \|H\| \le 1\,, \\ -\lambda^* + \frac{1}{2a}\log(\|H\|)\,, & \text{if } \|H\| > 1\,. \end{cases}$$

It follows from estimate of M_λ that, for $\operatorname{Re}\lambda > \lambda_0$, $\|M_\lambda H\| < 1$ and consequently

$$\psi^o = \sum_{n\ge 0}(M_\lambda H)^n G_\lambda\varphi\,. \tag{3.3}$$

On the other hand, equation (3.1) can be rewritten in the form

$$\psi = B_\lambda H\psi^o + C_\lambda\varphi\,.$$

Substituting equation (3.3) into the above equation we get

$$\psi = \sum_{n\geq 0} B_\lambda H(M_\lambda H)^n G_\lambda \varphi + C_\lambda \varphi \,.$$

Finally, the resolvent set of the operator T_H contains $\{\lambda \in \mathbb{C} \,:\, \operatorname{Re}\lambda > \lambda_0\}$ and for $\operatorname{Re}\lambda > \lambda_0$ we have

$$(\lambda - T_H)^{-1} = \sum_{n\geq 0} B_\lambda H(M_\lambda H)^n G_\lambda + C_\lambda \,. \tag{3.4}$$

Theorem 3.1. *Let $\varepsilon > 0$ and H be a bounded arbitrary boundary operator. Then there exists $C > 0$ such that*

$$\sigma_{e,\varepsilon}(T_H) \subset \Big\{\lambda \in \mathbb{C} \,:\, \operatorname{Re}\lambda \leq -\lambda^* + \varepsilon\Big(1 + C\|H\|(p)^{-\frac{1}{p}}\Big)\Big\} \,.$$

Proof. For $\operatorname{Re}\lambda > \lambda_0$, the operator $(\lambda - T_H)^{-1}$ exists and is given by equation (3.4). So

$$(\lambda - T_H)^{-1} = B_\lambda H(I - M_\lambda H)^{-1} G_\lambda + C_\lambda \,.$$

Moreover, we have

$$\|(\lambda - T_H)^{-1}\| \leq \|B_\lambda\|\|H\|\|(I - M_\lambda H)^{-1}\|\|G_\lambda\| + \|C_\lambda\| \,.$$

But the operator $(I - M_\lambda H)^{-1}$ is uniformly bounded on the half plane $\{\lambda \in \mathbb{C} \,:\, \operatorname{Re}\lambda > \lambda_0\}$. Therefore for $\operatorname{Re}\lambda > \lambda_0$ there exists $C > 0$ such that $\|(I - M_\lambda H)^{-1}\| < C$.

Furthermore we have

$$\|(\lambda - T_H)^{-1}\| \leq \frac{\Big(1 + C\|H\|(p)^{\frac{-1}{p}}\Big)}{\operatorname{Re}\lambda + \lambda^*} \,. \tag{3.5}$$

Let $\lambda \in \sigma_{\varepsilon,e}(A)$, it follows from Remark 2.1 (i) that $\frac{1}{\varepsilon} < \|(\lambda - T_H)^{-1}\|$. Using equation (3.5) we infer that $\operatorname{Re}\lambda \leq -\lambda^* + \varepsilon\left(1 + C\|H\|(p)^{-\frac{1}{p}}\right)$. ∎

Remark 3.1. If H is strictly singular, then we have

$$\big\{\lambda \in \mathbb{C} \,:\, \operatorname{Re}\lambda \leq -\lambda^*\big\} \subset \sigma_{e,\varepsilon}(T_H) \,.$$

Corollary 3.1. *Let $\varepsilon > 0$. Then we have*

$$\big\{\lambda \in \mathbb{C} \,:\, \operatorname{Re}\lambda \leq -\lambda^*\big\} \subset \sigma_{e,\varepsilon}(T_0) \subset \big\{\lambda \in \mathbb{C} \,:\, \operatorname{Re}\lambda \leq -\lambda^* + \varepsilon\big\} \,.$$

In the following we will make the assumption:

$$(\mathcal{H}) : \begin{cases} K \text{ is a measurable, i.e.,} \\ \{x \in [-a,a] \,:\, K(x) \in \mathcal{O}\} \text{ is measurable if } \mathcal{O} \subset \mathcal{Z} \text{ is open,} \\ \text{there exists a compact subset } T \subset \mathcal{Z} \,:\, K(x) \in T \text{ a.e.} \\ \text{and finally } K(x) \in \mathcal{K}\big(L_p([-1,1]\,,\ \mathrm{d}\xi)\big) \text{ a.e.} \end{cases}$$

where $\mathcal{K}\big(L_p([-1,1]\,,\ \mathrm{d}\xi)\big)$ denotes the set of all compact operators on $L_p\big([-1,1]\,,\ \mathrm{d}\xi\big)$.

Definition 3.1. A collision operator K is said to be regular if it satisfies the assumption $(\mathcal{H})$.

Proposition 3.1. ([18, Lemma 2.1]) *If the collision operator K is regular, then, for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -\lambda^*$, the operator $(\lambda - T_H)^{-1}K$ is compact on X_p $(1 < p < \infty)$ and weakly compact on X_1.*

Theorem 3.2. *Let $\varepsilon > 0$, we assume the operator K is regular on X_p. If $\|K\| < \lambda^*$, then we have*

$$\sigma_{e,\varepsilon}(A_H^{-1}) = \sigma_{e,\varepsilon}(T_H^{-1})\,.$$

Proof. Let $\bar{\lambda}$ the leading eigenvalue of A_H. In [25], the strip $-\lambda^* < \bar{\lambda} \le -\lambda^* + \|K\|$ contains at most isolated points of $\sigma(A_H)$. Now using that

$$\sigma(T_H) = \sigma C(T_H) = \big\{\lambda \in \mathbb{C} \,:\ \operatorname{Re}\lambda \le -\lambda^*\big\}\,,$$

if $\|K\| < \lambda^*$, then $0 \in \rho(T_H) \cap \rho(A_H)$. Since, $\lambda^* > \|K\|$ we have, $\|T_H^{-1}K\| \le \frac{\|K\|}{\lambda^*}$, hence $\|T_H^{-1}K\| < 1$. Therefore, we have

$$A_H^{-1} = T_H^{-1} + \sum_{n\ge 1} \big[(T_H)^{-1}K\big]^n\, T_H^{-1}.$$

Let, $\Xi = \sum_{n\ge1} \big[(T_H)^{-1}K\big]^n\, T_H^{-1}$, then, $\sigma_{e,\varepsilon}(A_H^{-1}) = \sigma_{e,\varepsilon}(T_H^{-1} + \Xi)$. Using the assumption $(\mathcal{H})$ we infer that Ξ is compact on X_p $(1 < p < \infty)$ and weakly compact on X_1. So, it follows from Remark 2.3 (i), we deduce that

$$\sigma_{e,\varepsilon}(A_H^{-1}) = \sigma_{e,\varepsilon}(T_H^{-1})\,.$$

∎

Corollary 3.2. *Assume that the hypotheses of Theorem* 3.2 *are satisfied. Then, there exists* $\varepsilon' > 0$ *such that*

$$\sigma_{e,\varepsilon}(A_H^{-1}) \subset \left\{\lambda \in \mathbb{C} : \operatorname{Re}\frac{1}{\lambda} \leq \varepsilon' - \lambda^*\right\}.$$

Theorem 3.3. *Assume that the hypotheses of Theorem* 3.2 *are satisfied. Then, there exists* $\varepsilon_1 > 0$ *such that*

$$\sigma_{e,\varepsilon}(A_H) \subset \left\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq \varepsilon_1 - \lambda^*\right\}.$$

Proof. Let $\lambda \notin \sigma_{e,\varepsilon}(A_H)$. Using Theorem 2.1, we find that there exists $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $\lambda \in \sigma(A_H + D)$. We can write

$$(\lambda - A_H - D) = -\lambda\big(\lambda^{-1} - {A_H}^{-1} + \lambda^{-1} D {A_H}^{-1}\big) A_H\,.$$

Since, A_H is one to one and onto, then

$$\alpha(\lambda - A_H - D) = \alpha(\lambda^{-1} - {A_H}^{-1} - D')$$

and

$$R(\lambda - A_H - D) = R(\lambda^{-1} - {A_H}^{-1} - D')\,,$$

where $D' = -\lambda^{-1} D {A_H}^{-1}$. This shows that there exists $\varepsilon_0 > 0$ such that $\lambda^{-1} \in \sigma_{e,\varepsilon_0}(A_H^{-1})$. Finally, the use of Corollary 3.2 completes the proof. ∎

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