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Generalized representations of 3-Hom-Lie algebras

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Abstract: The propose of this paper is to extend generalized representations of 3-Lie algebras to Hom-type algebras. We introduce the concept of generalized representation of multiplicative 3-Hom-Lie algebras, develop the corresponding cohomology theory and study semi-direct products. We provide a key construction, various examples and computation of 2-cocycles of the new cohomology. Also, we give a connection between a split abelian extension of a 3-Hom-Lie algebra and a generalized semidirect product 3-Hom-Lie algebra.

Key words: 3-Hom-Lie algebra, representation, generalized representation, cohomology, abelian extension.

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Introduction

The first instances of ternary Lie algebras appeared first in Nambu's generalization of Hamiltonian mechanics [23], which was formulated algebraically by Takhtajan [29]. The structure of n-Lie algebras was studied by Filippov [15] then completed by Kasymov in [21].

The representation theory of n-Lie algebras was first introduced by Kasymov in [21]. The adjoint representation is defined by the ternary bracket in which two elements are fixed. Through fundamental objects one may also represent a 3-Lie algebra and more generally an n-Lie algebra by a Leibniz algebra [11]. The cohomology of n-Lie algebras, generalizing the Chevalley-Eilenberg Lie algebras cohomology, was introduced by Takhtajan [30] in its simplest form, later a complex adapted to the study of formal deformations was introduced by Gautheron [17], then reformulated by Daletskii and Takhtajan [11] using the notion of base Leibniz algebra of an n-Lie algebra. In [2, 3], the structure and cohomology of 3-Lie algebras induced by Lie algebras has been investigated.

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The concept of generalized representation of a 3-Lie algebra was introduced by Liu, Makhlouf and Sheng in [19]. They study the corresponding generalized semidirect product 3-Lie algebra and cohomology theory. Furthermore, they describe general abelian extensions of 3-Lie algebras using Maurer-Cartan elements. Non-abelian extensions were explored in [26].

The aim of this paper is to extend the concept of generalized representation of 3-Lie algebras to Hom-type algebras. The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov in [18] as part of a study of deformations of the Witt and the Virasoro algebras. The n-Hom-Lie algebras and various generalizations of n-ary algebras were considered in [4]. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the Hom-Jacobi identity. In particular, representations and cohomologies of Hom-Lie algebras were studied in [25], while the representations and cohomology of n-Hom-Lie algebras were first studied in [1].

The paper is organized as follows. In Section 1, we provide some basics about 3-Hom-Lie algebras, representations and cohomology. The second Section includes the new concept of generalized representation of a 3-Hom-Lie algebra, extending to Hom-type algebras the notion and results obtained in [19]. We define a corresponding semi-direct product and provide a twist procedure leading to generalized representations of 3-Hom-Lie algebras starting from generalized representations of 3-Hom-Lie algebras and algebra maps. In Section 3, we construct a new cohomology corresponding to generalized representations and show examples. In the last section we discuss abelian extensions of multiplicative 3-Hom-Lie algebras. One recovers the results in [19] when the twist map is the identity.

1. Representations of 3-Hom-Lie algebras

The aim of this section is to recall some basics about 3-Lie algebras and 3-Hom-Lie algebras. We refer mainly to [15] and [4]. In this paper, all vector spaces are considered over a field $\mathbb{K}$ of characteristic 0.

Definition 1.1. A 3-Lie algebra is a pair $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ consisting of a $\mathbb{K}$-vector space $\mathfrak{g}$ and a trilinear skew-symmetric multiplication $[\cdot,\cdot,\cdot]$ satisfying the Filippov-Jacobi identity: for x, y, z, u, v in $\mathfrak{g}$

$$[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]].$$

In this paper, we are dealing with 3-Hom-Lie algebras corresponding to the following definition.

DEFINITION 1.2. A 3-Hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot,\cdot,\cdot], \alpha)$ consisting of a $\mathbb{K}$-vector space $\mathfrak{g}$, a trilinear skew-symmetric multiplication $[\cdot,\cdot,\cdot]$ and an algebra map $\alpha : \mathfrak{g} \to \mathfrak{g}$ satisfying the Hom-Filippov-Jacobi identity: for x, y, z, u, v in $\mathfrak{g}$

$$\begin{aligned}[\alpha(u), \alpha(v), [x, y, z]] = [[u, v, x], \alpha(y), \alpha(z)] + [\alpha(x), [u, v, y], \alpha(z)] \\ + [\alpha(x), \alpha(y), [u, v, z]].\end{aligned}$$

Remark 1.3. There is more general definition of 3-Hom-Lie algebras which are given by a quadruple $(\mathfrak{g}, [\cdot,\cdot,\cdot], \alpha_1, \alpha_2)$ consisting of a $\mathbb{K}$-vector space $\mathfrak{g}$, two linear maps $\alpha_1, \alpha_2 : \mathfrak{g} \to \mathfrak{g}$ and a trilinear skew-symmetric multiplication $[\cdot,\cdot,\cdot]$ satisfying the following generalized Hom-Filippov-Jacobi identity: for x, y, z, u, v in $\mathfrak{g}$

$$\begin{aligned}[\alpha_1(u), \alpha_2(v), [x, y, z]] = [[u, v, x], \alpha_1(y), \alpha_2(z)] + [\alpha_1(x), [u, v, y], \alpha_2(z)] \\ + [\alpha_1(x), \alpha_2(y), [u, v, z]].\end{aligned}$$

We get our class of 3-Hom-Lie algebras when $\alpha_1 = \alpha_2 = \alpha$ and where α is an algebra morphism. This kind of algebras are usually called multiplicative 3-Hom-Lie algebras.

PROPOSITION 1.4. *Let* $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ *be a 3-Lie algebra and* $\alpha : \mathfrak{g} \to \mathfrak{g}$ *be a 3-Lie algebra morphism. Then* $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\alpha := \alpha \circ [\cdot,\cdot,\cdot], \alpha)$ *is a 3-Hom-Lie algebra.*

Let $(\mathfrak{g}, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom-Lie algebra, elements in $\wedge^2\mathfrak{g}$ are called *fundamental objects* of the 3-Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot], \alpha)$. There is a bilinear operation $[\cdot,\cdot]_{\mathcal{L}}$ on $\wedge^2\mathfrak{g}$, which is given by

$$[X, Y]_{\mathcal{L}} = [x_1, x_2, y_1] \wedge \alpha(y_2) + \alpha(y_1) \wedge [x_1, x_2, y_2]$$

for all $X = x_1 \wedge x_2$ and $Y = y_1 \wedge y_2$, and a linear map $\overline{\alpha}$ on $\wedge^2\mathfrak{g}$ defined by $\overline{\alpha}(X) = \alpha(x_1) \wedge \alpha(x_2)$, for simplicity, we will write $\overline{\alpha}(X) = \alpha(X)$. It is well-known that $(\wedge^2\mathfrak{g}, [\cdot,\cdot]_{\mathcal{L}}, \overline{\alpha})$ is a Hom-Leibniz algebra [1, 31].

DEFINITION 1.5. A representation of a 3-Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot], \alpha)$ on a vector space V with respect to $A \in gl(V)$ is a skew-symmetric linear map $\rho : \wedge^2\mathfrak{g} \to End(V)$ such that

$$\rho(\alpha(x_1), \alpha(x_2)) \circ A = A \circ \rho(x_1, x_2), \tag{1.1}$$

$$\begin{aligned}\rho(\alpha(x_1),\alpha(x_2))\rho(x_3, x_4) - \rho(\alpha(x_3), \alpha(x_4))\rho(x_1, x_2) \\ = \big(\rho([x_1, x_2, x_3], \alpha(x_4)) - \rho([x_1, x_2, x_4], \alpha(x_3))\big) \circ A,\end{aligned} \tag{1.2}$$

$$\begin{aligned}\rho([x_1,x_2,x_3],\alpha(x_4))\circ A-\rho(\alpha(x_2),\alpha(x_3))\rho(x_1,x_4)\\ =\rho(\alpha(x_3),\alpha(x_1))\rho(x_2,x_4)+\rho(\alpha(x_1),\alpha(x_2))\rho(x_3,x_4),\end{aligned} \tag{1.3}$$

for x_1, x_2, x_3 and x_4 in $\mathfrak{g}$.

Theorem 1.6. *Let $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ be a 3-Lie algebra, (V,ρ) be a representation, $\alpha:\mathfrak{g}\to\mathfrak{g}$ be a 3-Lie algebra morphism and $A:V\to V$ be a linear map such that $A\circ\rho(x_1,x_2)=\rho(\alpha(x_1),\alpha(x_2))\circ A$. Then $(V,\widetilde{\rho}:=A\circ\rho,A)$ is a representation of the 3-Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot,\cdot]_\alpha:=\alpha\circ[\cdot,\cdot,\cdot],\alpha)$.*

Proof. Let $x_i\in\mathfrak{g}$, where $1\le i\le 5$. Then we have

$$\begin{aligned}&\widetilde{\rho}([x_3,x_4,x_5]_\alpha,\alpha(x_1))\circ A-\widetilde{\rho}(\alpha(x_3),\alpha(x_4))\widetilde{\rho}(x_5,x_1)-\widetilde{\rho}(\alpha(x_4),\alpha(x_5))\widetilde{\rho}(x_3,x_1)\\ &\qquad -\widetilde{\rho}(\alpha(x_5),\alpha(x_3))\widetilde{\rho}(x_4,x_1)\\ &= A^2\circ(\rho([x_3,x_4,x_5],x_1)-\rho(x_3,x_4)\rho(x_5,x_1)\\ &\qquad -\rho(x_4,x_5)\rho(x_3,x_1)-\rho(x_5,x_3)\rho(x_4,x_1)) = 0.\end{aligned}$$

The second condition (1.2) is obtained similarly. ∎

The previous result allows to twist along morphisms a 3-Lie algebra with a representation to a 3-Hom-Lie algebra with a corresponding representation.

Proposition 1.7. *Let $(\mathfrak{g},[\cdot,\cdot,\cdot],\alpha)$ be a 3-Hom-Lie algebra, V be a vector space, $A\in gl(V)$ and $\rho:\wedge^2\mathfrak{g}\to gl(V)$ be a skew-symmetric linear map. Then $(V;\rho,A)$ is a representation of 3-Hom-Lie algebra $\mathfrak{g}$ if and only if there is a 3-Hom-Lie algebra structure $(\mathfrak{g}\oplus V,[\cdot,\cdot,\cdot]_\rho,\alpha_{\mathfrak{g}\oplus V})$ on the direct sum of vector spaces $\mathfrak{g}\oplus V$, defined by*

$$\begin{aligned}[x_1+v_1,x_2+v_2,x_3+v_3]_\rho=[x_1,x_2,x_3]+\rho(x_1,x_2)v_3\\ +\rho(x_3,x_1)v_2+\rho(x_2,x_3)v_1,\end{aligned}$$

and $\alpha_{\mathfrak{g}\oplus V}=\alpha+A$, for all $x_i\in\mathfrak{g}$, $v_i\in V$, $1\le i\le 3$. The obtained 3-Hom-Lie algebra is denoted by $\mathfrak{g}\ltimes_\rho V$ and called semidirect product.

Let $(\mathfrak{g},[\cdot,\cdot,\cdot],\alpha)$ be a 3-Hom-Lie algebra and (V,ρ,A) be a representation of $\mathfrak{g}$. We denote by $C^p_{\alpha,A}(\mathfrak{g},V)$ the space of all linear maps

$$\varphi:\underbrace{\wedge^2\mathfrak{g}\otimes\cdots\otimes\wedge^2\mathfrak{g}}_{(p-1)}\wedge\mathfrak{g}\to V$$

satisfying:

$$A \circ \varphi(X_1 \otimes \cdots \otimes X_{p-1}, y) = \varphi(\alpha(X_1) \otimes \cdots \otimes \alpha(X_{p-1}), \alpha(y)),$$

for all $X_1, \ldots, X_{p-1} \in \wedge^2 \mathfrak{g}$, $y \in \mathfrak{g}$. It is called the space of p-cochains.

Let φ be a $(p-1)$-cochain, the coboundary operator $\delta_\rho : C^{p-1}_{\alpha,A}(\mathfrak{g}, V) \to C^p_{\alpha,A}(\mathfrak{g}, V)$ is given by

$$\begin{aligned}
&(\delta_\rho \varphi)(X_1, \ldots, X_p, z) \\
&\quad = \sum_{1 \le j < k} (-1)^j \varphi(\alpha(X_1), \ldots, \hat{X}_j, \ldots, \alpha(X_{k-1}), [X_j, X_k]_{\mathcal{L}}, \alpha(X_{k+1}), \\
&\qquad\qquad \ldots, \alpha(X_p), \alpha(z)) \\
&\qquad + \sum_{j=1}^{p} (-1)^j \varphi(\alpha(X_1), \ldots, \hat{X}_j, \ldots, \alpha(X_p), [X_j, z]) \qquad (1.4) \\
&\qquad + \sum_{j=1}^{p} (-1)^{j+1} \rho(\alpha^p(X_j)) \varphi(X_1, \ldots, \hat{X}_j, \ldots, X_p, z) \\
&\qquad + (-1)^{p+1} \Big(\rho(\alpha^p(y_p), \alpha^p(z)) \varphi(X_1, \ldots, X_{p-1}, x_p) \\
&\qquad + \rho(\alpha^p(z), \alpha^p(x_p)) \alpha(X_1, \ldots, X_{p-1}, y_p) \Big),
\end{aligned}$$

for all $X_i = (x_i, y_i) \in \wedge^2 \mathfrak{g}$, $z \in \mathfrak{g}$ and where $[X_i, z] = [x_i, y_i, z]$. An element $\varphi \in C^{p-1}_{\alpha,A}(\mathfrak{g}, V)$ is called a p-cocycle if $\delta_\rho \varphi = 0$. It is called a p-coboundary if there exists some $f \in C^{p-2}_{\alpha,A}(\mathfrak{g}, V)$ such that $\varphi = \delta_\rho f$. Denote by $Z^p_{3HL}(\mathfrak{g}; V)$ and $B^p_{3HL}(\mathfrak{g}; V)$ the sets of p-cocycles and p-coboundaries respectively. Then the p-th cohomology group is

$$H^p_{3HL}(\mathfrak{g}; V) = Z^p_{3HL}(\mathfrak{g}; V) / B^p_{3HL}(\mathfrak{g}; V). \qquad (1.5)$$

In [28], the author constructed a graded Lie algebra structure by which one can describe an n-Leibniz algebra structure as a canonical structure. Here, we give the precise formulas for the 3-Hom-Lie algebra case, generalizing the result in [19].

Set $C_{\alpha,\alpha}(\mathfrak{g}, \mathfrak{g}) = \oplus_{p \ge 0} C^p_{\alpha,\alpha}(\mathfrak{g}, \mathfrak{g})$. Let $\varphi \in C^q_{\alpha,\alpha}(\mathfrak{g}, \mathfrak{g})$, $\psi \in C^p_{\alpha,\alpha}(\mathfrak{g}, \mathfrak{g})$, $p, q \ge 0$, $X_i = x_i \wedge y_i \in \wedge^2 \mathfrak{g}$ for $i = 1, 2, \ldots, p+q$ and $x \in \mathfrak{g}$. For each subset $J = \{j_1, \ldots, j_p\}_{j_1 < \cdots < j_p} \subset N \triangleq \{1, 2, \ldots, p+q\}$, let $I = \{i_1, \ldots, i_q\}_{i_1 < \cdots < i_q} = N/J$. Define on the graded vector space $C_{\alpha,\alpha}(\mathfrak{g}, \mathfrak{g})$ the graded commutator bracket

$$[\varphi, \psi]^{3HL} = (-1)^{pq} j^\alpha_\psi(\varphi) - j^\alpha_\varphi(\psi) = (-1)^{pq} \varphi \circ_\alpha \psi - \psi \circ_\alpha \varphi,$$

where $\varphi \circ_\alpha \psi \in C^{p+q}_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g})$ is defined by

$$\begin{aligned} j^\alpha_\psi(\varphi)(X_1,\dots,X_{p+q},x) &= \varphi \circ_\alpha \psi(X_1,\dots,X_{p+q},x)\\ = \sum_{J, j_q<i_{k+1}\leq p+q} &(-1)^{(J,I)}\varphi\Big(\alpha^p(X_{i_1}),\dots,\alpha^p(X_{i_k}),\\ &\psi(X_{j_1},\dots,X_{j_p},\ )\bullet_\alpha X_{i_{k+1}},\dots,\alpha^p(X_{i_q}),\alpha^p(x)\Big)\\ &+\sum_J(-1)^{(J,I)}(-1)^q\varphi\Big(\alpha^p(X_{i_1}),\dots,\alpha^p(X_{i_q}),\psi(X_{j_1},\dots,X_{j_p},x)\Big), \end{aligned}$$

with

$$\begin{aligned} \psi(X_{j_1},\dots,X_{j_q},\ )\bullet_\alpha X_{i_{k+1}} &= \psi(X_{j_1},\dots,X_{j_p},x_{i_{k+1}})\wedge\alpha^p(y_{i_{k+1}})\\ &+\alpha^p(x_{i_{k+1}})\wedge\psi(X_{j_1},\dots,X_{j_p},y_{i_{k+1}}) \end{aligned}$$

where k is uniquely determined by the condition $j_p \leq i_{k+1}$ and if $j_p \leq i_1$ then $j_p = p$, $i_1 = p+1$ and $(-1)^{(J,I)}$ is the sign of the permutation $(J,I) = (j_1,\dots,j_p,i_1,\dots,i_q)$ of N.

We need the following lemma to establish a structure of graded Lie algebra on $C_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g})$.

LEMMA 1.8. *We have* $j_{[\varphi,\psi]^{3HL}} = -[j^\alpha_\varphi, j^\alpha_\psi]$ *for all* $\varphi,\psi \in C_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g})$, *where* $[\cdot,\cdot]$ *is the graded commutator on* $End(C_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g}))$.

Proof. Let $\varphi \in C^q_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g})$, $\psi \in C^p_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g})$, $\xi \in C^r_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g})$, $X_1,\dots,X_{p+q+r} \in \wedge^2\mathfrak{g}$ and $x \in \mathfrak{g}$

$$\begin{aligned} &[j^\alpha_\varphi, j^\alpha_\psi](\xi)(X_1,X_2,\dots,X_{p+q+r},x)\\ &\qquad= (j^\alpha_\varphi(j^\alpha_\psi\xi) - (-1)^{pq}j^\alpha_\psi(j^\alpha_\varphi\xi))(X_1,X_2,\dots,X_{p+q+r},x)\\ &\qquad= D_1 - (-1)^{pq}D_2, \end{aligned}$$

where

$$\begin{aligned} D_1 &= j^\alpha_\varphi(j^\alpha_\psi\xi)(X_1,X_2,\dots,X_{p+q+r},x),\\ D_2 &= j^\alpha_\psi(j^\alpha_\varphi\xi))(X_1,X_2,\dots,X_{p+q+r},x). \end{aligned}$$

For each subset $J = \{j_1,\dots,j_q\}_{j_1<\dots<j_q} \subset N \triangleq \{1,2,\dots,p+q+r\}$, let $I = \{i_1,\dots,i_{p+r}\}_{i_1<\dots<i_{p+r}} = N \setminus J$ and $L = \{l_1,\dots,l_p\}_{l_1<\dots<l_p} \subset I \triangleq \{i_1,i_2,\dots,i_{p+r}\}$, let $H = \{h_1,\dots,h_r\}_{h_1<\dots<h_r} = I \setminus L$.

We have for D_1 :

$$j_\varphi^\alpha(j_\psi^\alpha\xi)(X_1,X_2,\dots,X_{p+q+r},x) = (j_\psi^\alpha(\xi))\circ_\alpha \varphi(X_1,X_2,\dots,X_{p+q+r},x)$$

$$= \sum_{J,j_q<i_{k+1}\leq p+q+r}(-1)^{(J,I)}(j_\psi^\alpha(\xi))(\alpha^q(X_{i_1}),\dots,\alpha^q(X_{i_k}),\varphi(X_{j_1},\dots,X_{j_q},\ )\bullet_\alpha X_{i_{k+1}},\alpha^q(X_{i_{k+2}}),\dots,\alpha^q(X_{i_{p+r}}),\alpha^q(x))$$

$$+\sum_J(-1)^{(J,I)}(-1)^{p+r}(j_\psi^\alpha(\xi))(\alpha^q(X_{i_1}),\dots,\alpha^q(X_{i_{p+r}}),\varphi(X_{j_1},\dots,X_{j_q},x))$$

$$= \sum_{J,j_q<i_{k+1}\leq p+q+r}\ \sum_{\substack{L,l_p<h_{m+1}\leq i_{p+r},\\ i_{k+1}=h_n,n\leq m}}(-1)^{(J,I)}(-1)^{(L,H)}\xi(\alpha^{p+q}(X_{h_1}),\dots,\alpha^{p+q}(X_{h_{n-1}}),\alpha^p(\varphi(X_{j_1},\dots,X_{j_q},\ )\bullet_\alpha X_{i_{k+1}}),$$

$$\dots,\alpha^{p+q}(X_{h_m}),\psi(\alpha^q(X_{l_1}),\dots,\alpha^q(X_{l_p}),\ )\bullet_\alpha \alpha^q(X_{h_{m+1}}),\alpha^{p+q}(X_{h_{m+2}}),\dots,\alpha^{p+q}(X_{h_r}),\alpha^{p+q}(x))$$

$$+\sum_{J,j_q<i_{k+1}\leq p+q+r}\ \sum_{\substack{L,l_p<h_{m+1}\leq i_{p+r},\\ i_{k+1}=h_{m+1}}}(-1)^{(J,I)}(-1)^{(L,H)}\xi(\alpha^{p+q}(X_{h_1}),\dots,\alpha^{p+q}(X_{h_m}),\psi(\alpha^q(X_{l_1}),\dots,\alpha^q(X_{l_p}),\ )$$

$$\bullet_\alpha(\varphi(X_{j_1},\dots,X_{j_q},\ )\bullet_\alpha X_{i_{k+1}}),\alpha^{p+q}(X_{h_{m+2}}),\dots,\alpha^{p+q}(X_{h_r}),\alpha^{p+q}(x))$$

$$+\sum_{J,j_q<i_{k+1}\leq p+q+r}\ \sum_{\substack{L,l_p<h_{m+1}\leq i_{p+r},\\ i_{k+1}=h_n,n>m+1}}(-1)^{(J,I)}(-1)^{(L,H)}\xi(\alpha^{p+q}(X_{h_1}),\dots,\alpha^{p+q}(X_{h_m}),\psi(\alpha^q(X_{l_1}),\dots,\alpha^q(X_{l_p}),\ )$$

$$\bullet_\alpha \alpha^{p+q}(X_{h_{m+1}}),\alpha^{p+q}(X_{h_{m+2}}),\dots,\alpha^{p+q}(X_{h_{n-1}}),\alpha^p(\varphi(X_{j_1},\dots,X_{j_q},\ )$$

$$\bullet_\alpha X_{i_{k+1}}),\alpha^{p+q}(X_{h_{n+1}},\dots,\alpha^{p+q}(X_{h_r}),\alpha^{p+q}(x))$$

$$+\sum_{J, j_q < i_{k+1} \leq p+q+r} \sum_{\substack{L, l_p < h_{m+1} \leq i_{p+r}, \\ i_{k+1} = l_s, s \leq p}} (-1)^{(J,I)}(-1)^{(L,H)}\xi(\alpha^{p+q}(X_{h_1}), \dots, \alpha^{p+q}(X_{h_m}), \psi(\alpha^q(X_{l_1}), \dots, \alpha^q(X_{l_{s-1}}),$$

$$\varphi(X_{j_1}, \dots, X_{j_q}, X_{j_q}, \ ) \bullet_\alpha X_{i_{k+1}}, \alpha^q(X_{l_{s+1}}), \dots, \alpha^q(X_{l_p}), \ ) \bullet_\alpha \alpha^q(X_{h_{m+1}}), \alpha^{p+q}(X_{h_{m+2}}), \dots, \alpha^{p+q}(X_{h_r}), \alpha^{p+q}(x))$$

$$+\sum_{J, j_q < i_{k+1} \leq p+q+r} \sum_{L, i_{k+1} = h_n} (-1)^{(J,I)}(-1)^{(L,H)}(-1)^r\xi(\alpha^{p+q}(X_{h_1}), \dots, \alpha^{p+q}(X_{h_{n-1}}), \alpha^p(\varphi(X_{j_1}, \dots, X_{j_q}, \ ) \bullet_\alpha X_{i_{k+1}}),$$

$$\alpha^{p+q}(X_{h_{n+1}}), \dots, \alpha^{p+q}(X_{h_r}), \psi(\alpha^q(X_{l_1}), \dots, \alpha^q(X_{l_p}), \alpha^q(x)))$$

$$+\sum_{J, j_q < i_{k+1} \leq p+q+r} \sum_{L, i_{k+1} = l_s} (-1)^{(J,I)}(-1)^{(L,H)}(-1)^r\xi(\alpha^{p+q}(X_{h_1}), \dots, \alpha^{p+q}(X_{h_r}), \psi(\alpha^q(X_{l_1}), \dots, \alpha^q(X_{l_{s-1}}),$$

$$\varphi(X_{j_1}, \dots, X_{j_q}, \ ) \bullet_\alpha X_{i_{k+1}}, \alpha^q(X_{l_{s+1}}), \dots, \alpha^q(X_{l_p}), \alpha^q(x)))$$

$$+\sum_{J} \sum_{L, l_p < h_{m+1} \leq i_{p+r}} (-1)^{(J,I)}(-1)^{p+r}(-1)^{(L,H)}\xi(\alpha^{p+q}(X_{h_1}), \dots, \alpha^{p+q}(X_{h_m}), \psi(\alpha^q(X_{l_1}), \dots, \alpha^q(X_{l_p}), \ ) \bullet_\alpha (\alpha^q(X_{h_{m+1}}),$$

$$\alpha^{p+q}(X_{h_{m+2}}), \dots, \alpha^{p+q}(X_{h_r}), \alpha^p(\varphi(X_{j_1}, \dots, X_{j_q}, x)))$$

$$+\sum_{J}\sum_{L}(-1)^{(J,I)}(-1)^{p+r}(-1)^{(L,H)}(-1)^r\xi(\alpha^{p+q}(X_{h_1}), \dots, \alpha^{p+q}(X_{h_r}), \psi(\alpha^q(X_{l_1}), \dots, \alpha^{p+q}(X_{l_p}), \varphi(X_{j_1}, \dots, X_{j_q}, x))).$$

Similarly one can compute D_2.

Let $A=\{a_1,a_2,\dots,a_{p+q}\}_{a_1<a_2<\cdots<a_{p+q}}\subseteq N=\{1,\dots,p+q+r\}$, $H=\{h_1,\dots,h_r\}_{h_1<h_2<\cdots<h_r}=N\setminus A$, and $J=\{j_1,\dots,j_q\}_{j_1<\cdots<j_q}\subseteq A$ and $L=\{l_1,\dots,l_p\}_{l_1<\cdots<l_p}=A\setminus J$. We have:

$$
\begin{aligned}
&j_{[\varphi,\psi]^{3HL}}(\xi)(X_1,\dots,X_{p+q+r},x)\\
&=\sum_{A,a_{p+q}<h_{m+1}\leq p+q+r}(-1)^{(A,H)}\xi(\alpha^{p+q}(X_{h_1}),\dots,\alpha^{p+q}(X_{h_m}),[\varphi,\psi]^{3HL}_{\alpha}(X_{a_1},\dots,X_{a_{p+q}},\ )\\
&\qquad\bullet_\alpha X_{h_{m+1}},\alpha^{p+q}(X_{h_{m+2}}),\dots,\alpha^{p+q}(X_{h_r}),\alpha^{p+q}(x))\\
&\quad+\sum_{A}(-1)^{(A,H)}(-1)^r\xi(\alpha^{p+q}(X_{h_1}),\dots,\alpha^{p+q}(X_{h_r}),[\varphi,\psi]^{3HL}_{\alpha}(X_{a_1},\dots,X_{a_{p+q}},x))\\
&=\sum_{A,a_{p+q}<h_{m+1}\leq p+q+r}\ \sum_{L,lp<j_{t+1}\leq a_{p+q}}(-1)^{pq}(-1)^{(A,H)}(-1)^{(L,J)}\xi(\alpha^{p+q}(X_{h_1}),\dots,\alpha^{p+q}(X_{h_m}),\varphi(\alpha^p(X_{j_1}),\dots,\alpha^p(X_{j_t}),\\
&\qquad\psi(X_{l_1},\dots,X_{l_p},\ )\bullet_\alpha X_{j_{t+1}},\alpha^p(X_{j_{t+2}})\dots,\alpha^p(X_{j_q}),\ )\bullet_\alpha \alpha^p(X_{h_{m+1}}),\alpha^{p+q}(X_{h_{m+2}}),\dots,\alpha^{p+q}(X_{h_r}),\alpha^{p+q}(x))\\
&\quad-\sum_{A,a_{p+q}<h_{m+1}\leq p+q+r}\ \sum_{J,jq<l_{s+1}\leq a_{p+q}}(-1)^{(A,H)}(-1)^{(J,L)}\xi(\alpha^{p+q}(X_{h_1}),\dots,\alpha^{p+q}(X_{h_m}),\psi(\alpha^q(X_{l_1}),\dots,\alpha^q(X_{l_s}),\\
&\qquad\varphi(X_{j_1},\dots,X_{j_q},\ )\bullet_\alpha X_{l_{s+1}},\alpha^q(X_{l_{s+2}})\dots,\alpha^q(X_{l_p}),\ )\bullet_\alpha \alpha^q(X_{h_{m+1}}),\dots,\alpha^{p+q}(X_{h_r}),\alpha^{p+q}(x))\\
&\quad+\sum_{A,a_{p+q}<h_{m+1}\leq p+q+r}\ \sum_{L}(-1)^{(A,H)}(-1)^{(L,J)}(-1)^{pq}(-1)^q\xi(\alpha^{p+q}(X_{h_1}),\dots,\alpha^{p+q}(X_{h_m}),\varphi(\alpha^p(X_{j_1}),\dots,\alpha^p(X_{j_q},\ )\\
&\qquad\bullet_\alpha(\psi(X_{l_1},\dots,X_{l_p},\ )\bullet_\alpha X_{h_{m+1}}),\alpha^{p+q}(X_{h_{m+2}}),\dots,\alpha^{p+q}(X_{h_r}),\alpha^{p+q}(x))
\end{aligned}
$$

$$-\sum_{A,a_{p+q}<h_{m+1}\leq p+q+r}\sum_{J}(-1)^{(A,H)}(-1)^{(J,L)}(-1)^{pq}(-1)^{P}\xi(\alpha^{p+q}(X_{h_1}),\dots,\alpha^{p+q}(X_{h_m}),\psi(\alpha^{q}(X_{l_1}),\dots,\alpha^{q}(X_{l_p},\ )$$

$$\bullet_\alpha\,(\varphi(X_{j_1},\dots,X_{j_q},\ )\bullet_\alpha X_{h_{m+1}}),\alpha^{p+q}(X_{h_{m+2}}),\dots,\alpha^{p+q}(X_{h_r}),\alpha^{p+q}(x))$$

$$+\sum_{A}\sum_{L,l_p<j_{t+1}\leq a_{p+q}}(-1)^{r}(-1)^{pq}(-1)^{(A,H)}(-1)^{(L,J)}(\xi(\alpha^{p+q}(X_{h_1}),\dots,\alpha^{p+q}(X_{h_r}),\varphi(\alpha^{p}(X_{j_1}),\dots,\alpha^{p}(X_{j_t}),$$

$$\psi(X_{l_1},\dots,X_{l_p},\ )\bullet_\alpha X_{j_{t+1}},\alpha^{p}(X_{j_{t+2}}),\dots,\alpha^{p}(X_{j_q}),\alpha^{p}(x)))$$

$$+\sum_{A}\sum_{L}(-1)^{r}(-1)^{q}(-1)^{pq}(-1)^{(A,H)}(-1)^{(L,J)}(\xi(\alpha^{p+q}(X_{h_1}),\dots,\alpha^{p+q}(X_{h_r}),\varphi(\alpha^{p}(X_{j_1}),\dots,\alpha^{p}(X_{j_q}),\psi(X_{l_1},\dots,X_{l_p},x))$$

$$-\sum_{A}\sum_{J,J_q<l_{s+1}\leq a_{p+q}}(-1)^{r}(-1)^{(A,H)}(-1)^{(J,L)}(\xi(\alpha^{p+q}(X_{h_1}),\dots,\alpha^{p+q}(X_{h_r}),\psi(\alpha^{q}(X_{l_1}),\dots,\alpha^{q}(X_{l_s}),$$

$$\varphi(X_{j_1},\dots,X_{j_q},\ )\bullet_\alpha X_{l_{s+1}},\alpha^{q}(X_{l_{s+2}}),\dots,\alpha^{q}(X_{l_p}),\alpha^{q}(x)))$$

$$-\sum_{A}\sum_{J}(-1)^{p}(-1)^{r}(-1)^{(A,H)}(-1)^{(J,L)}(\xi(\alpha^{p+q}(X_{h_1}),\dots,\alpha^{p+q}(X_{h_r}),\psi(\alpha^{q}(X_{l_1}),\dots,\alpha^{q}(X_{l_p}),\varphi(X_{j_1},\dots,X_{j_q},x)).$$

By a straightforward verification, we obtain $D_1-(-1)^{pq}D_2=j^{\alpha}_{[\varphi,\psi]^{3HL}}(\xi)(X_1,\dots,X_{p+q+r},x)$. Hence the proof. ∎

THEOREM 1.9. *The pair $(C_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g}),[\cdot,\cdot]^{3HL})$ is a graded Lie algebra.*

Proof. Let $\varphi\in C^q_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g})$, $\psi\in C^p_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g})$ and $\phi\in C^r_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g})$.
(1) Skew-symmetry:

$$\begin{aligned}[\varphi,\psi]^{3HL}&=(-1)^{pq}j^\alpha_\varphi(\psi)-j^\alpha_\psi(\varphi)\\&=(-1)^{pq+1}\big((-1)^{pq}j^\alpha_\psi(\varphi)-j^\alpha_\varphi(\psi)\big)\\&=-(-1)^{pq}[\psi,\varphi]^{3HL}.\end{aligned}$$

(2) Graded Jacobi identity:

$$\begin{aligned}&\circlearrowleft_{\varphi,\psi,\phi}(-1)^{qr}\big[\varphi,[\psi,\phi]^{3HL}\big]^{3HL}\\&\quad=(-1)^{qp}j_{[\psi,\phi]^{3HL}}(\varphi)-(-1)^{qr}j^\alpha_\varphi\big([\psi,\phi]^{3HL}\big)\\&\qquad+(-1)^{pr}j_{[\phi,\varphi]^{3HL}}(\psi)-(-1)^{pq}j^\alpha_\psi\big([\phi,\varphi]^{3HL}\big)\\&\qquad+(-1)^{rq}j_{[\varphi,\psi]^{3HL}}(\phi)-(-1)^{rp}j^\alpha_\phi\big([\varphi,\psi]^{3HL}\big)\\&\quad=(-1)^{qp}j_{[\psi,\phi]^{3HL}}(\varphi)-(-1)^{qr}j^\alpha_\varphi\big((-1)^{rp}j^\alpha_\phi(\psi)-j^\alpha_\psi(\phi)\big)\\&\qquad+(-1)^{pr}j_{[\phi,\varphi]^{3HL}}(\psi)-(-1)^{pq}j^\alpha_\psi\big((-1)^{qr}j^\alpha_\varphi(\phi)-j^\alpha_\phi(\varphi)\big)\\&\qquad+(-1)^{rq}j_{[\varphi,\psi]^{3HL}}(\phi)-(-1)^{rp}j^\alpha_\phi\big((-1)^{qp}j^\alpha_\psi(\varphi)-j^\alpha_\varphi(\psi)\big).\end{aligned}$$

Organizing these terms leads to

$$\begin{aligned}&\circlearrowleft_{\varphi,\psi,\phi}(-1)^{qr}\big[\varphi,[\psi,\phi]^{3HL}\big]^{3HL}\\&\quad=(-1)^{pq}j_{[\psi,\phi]^{3HL}}(\varphi)+(-1)^{pq}\big(j^\alpha_\psi(j^\alpha_\phi(\varphi))-(-1)^{rp}j^\alpha_\phi(j^\alpha_\psi(\varphi))\big)\\&\qquad+(-1)^{rp}j_{[\phi,\varphi]^{3HL}}(\psi)+(-1)^{rp}\big(j^\alpha_\phi(j^\alpha_\varphi(\psi))-(-1)^{qr}j^\alpha_\varphi(j^\alpha_\phi(\psi))\big)\\&\qquad+(-1)^{qr}j_{[\varphi,\psi]^{3HL}}(\phi)+(-1)^{qr}\big(j^\alpha_\varphi(j^\alpha_\psi(\phi))-(-1)^{qp}j^\alpha_\psi(j^\alpha_\varphi(\phi))\big)\\&\quad=(-1)^{pq}\big([j^\alpha_\psi,j^\alpha_\phi]+j_{[\psi,\phi]^{3HL}}\big)(\varphi)\\&\qquad+(-1)^{rp}\big([j^\alpha_\phi,j^\alpha_\varphi]+j_{[\phi,\varphi]^{3HL}}\big)(\psi)\\&\qquad+(-1)^{qr}\big([j^\alpha_\varphi,j^\alpha_\psi]+j_{[\varphi,\psi]^{3HL}}\big)(\phi).\end{aligned}$$

Using the previous lemma, we get

$$\circlearrowleft_{\varphi,\psi,\phi}(-1)^{qr}\big[\varphi,[\psi,\phi]^{3HL}\big]^{3HL}=0.$$

∎

Remark 1.10. The pair $(C_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g}),\circ_\alpha)$ is a right symmetric graded algebra.

The previous structure of graded Lie algebra is useful to describe 3-Hom-Lie algebra structures as well as coboundary operators.

COROLLARY 1.11. *The maps $\pi:\wedge^3\mathfrak{g}\to\mathfrak{g}$ and $\alpha:\mathfrak{g}\to\mathfrak{g}$ define a 3-Hom-Lie structure if and only if $[\pi,\pi]^{3HL}=0$.*

Let $\mathfrak{g}$ be a 3-Hom-Lie algebra. Given $x_1,x_2\in\mathfrak{g}$, define $ad:\wedge^2\mathfrak{g}\to gl(\mathfrak{g})$ by

$$ad_{x_1,x_2}y=[x_1,x_2,y].$$

Then, the pair (ad,α) defines a representation of the 3-Hom-Lie algebra $\mathfrak{g}$ on itself, which we call adjoint representation of $\mathfrak{g}$. The coboundary operator associated to this representation is denoted by $\delta_{\mathfrak{g}}$.

COROLLARY 1.12. *If $\pi:\wedge^3\mathfrak{g}\to\mathfrak{g}$ is a 3-Hom-Lie bracket, then we have*

$$[\pi,\varphi]^{3HL}=\delta_g(\varphi)\qquad \text{for all } \varphi\in C^p_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g}),\ p\geq 0.$$

2. Generalized representations of 3-Hom-Lie algebras

In this section, we provide the Hom-type version of generalized representations of a 3-Lie algebras introduced in [19]. First, we show that a representation of a 3-Hom-Lie algebra will give rise to a canonical structure.

Let $\mathfrak{g}$ be a 3-Hom-Lie algebra and V be a vector space. Let $\rho:\wedge^2\mathfrak{g}\to gl(V)$ be a linear map. Then, it induces a linear map $\overline{\rho}:\wedge^3(\mathfrak{g}\oplus V)\to\mathfrak{g}\oplus V$ defined by

$$\overline{\rho}(x+u,y+v,z+w)=\rho(x,y)(w)+\rho(y,z)(u)+\rho(z,x)(v)$$

for all $x,y,z\in\mathfrak{g},\ u,v,w\in V$. Consider the graded Lie algebra given in Theorem 1.9 associated to the vector space $\mathfrak{g}\oplus V$.

PROPOSITION 2.1. *A linear map $\rho:\wedge^2\mathfrak{g}\to gl(V)$ is a representation on a vector space V of the 3-Hom-Lie algebra $\mathfrak{g}$ with respect to $A\in gl(V)$ if and only if $\pi+\overline{\rho}$ is a canonical structure in the graded Lie algebra associated to $\mathfrak{g}\oplus V$, i.e.*

$$[\pi+\overline{\rho},\pi+\overline{\rho}]^{3HL}=0.$$

Proof. By Proposition 1.7, $\rho : \wedge^2\mathfrak{g} \to gl(V)$ is a representation of $\mathfrak{g}$ if and only if $\mathfrak{g}\oplus V$ is a 3-Hom-Lie algebra, where the 3-Hom-Lie structure is exactly given by

$$\begin{aligned}[x+u, y+v, z+w]_\rho &= [x,y,z] + \rho(x,y)(w) + \rho(y,z)(u) + \rho(z,x)(v)\\ &= (\pi+\bar{\rho})(x+u, y+v, z+w),\end{aligned}$$

and $\alpha_{\mathfrak{g}\oplus V} = \alpha + A$. Thus, by Lemma 1.11, $\rho : \wedge^2\mathfrak{g} \to gl(V)$ is a representation of $\mathfrak{g}$ if and only if $\pi+\bar{\rho}$ is a canonical structure. ∎

The concept of representation of 3-Lie algebras introduced by Liu, Makhlouf and Sheng [19] is generalized to Hom-type algebras as follows.

Definition 2.2. A *generalized representation* of a 3-Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot], \alpha)$ with respect to $A \in gl(V)$ consists of linear maps $\rho : \wedge^2\mathfrak{g} \to gl(V)$, $\nu : \mathfrak{g} \to \mathrm{Hom}(\wedge^2 V, V)$, such that

$$[\pi+\overline{\rho}+\overline{\nu}, \pi+\overline{\rho}+\overline{\nu}]^{3HL} = 0,$$

where $\overline{\nu} : \wedge^3(\mathfrak{g}\oplus V) \to (\mathfrak{g}\oplus V)$ is induced by ν via

$$\overline{\nu}(x+u, y+v, z+w) = \nu(x)(v\wedge w) + \nu(y)(w\wedge u) + \nu(z)(u\wedge v)$$

for all $x,y,z \in \mathfrak{g}$, $u,v,w \in V$. We will refer to a generalized representation by $(V;\rho,\nu,A)$.

Remark 2.3. If $\nu = 0$, then we recover the usual definition of a representation of a 3-Hom-Lie algebra on a vector space V. If the dimension of the vector space V is 1, then ν must be zero. In this case, we only have the usual representation.

Given linear maps $\rho : \wedge^2\mathfrak{g} \to End(V)$, $\nu : \mathfrak{g} \to \mathrm{Hom}(\wedge^2 V, V)$, and $A : V \to V$, define a trilinear bracket operation on $\mathfrak{g}\oplus V$ by

$$\begin{aligned}[x+u, y+v, z+w]_{(\rho,\nu)} = [x,y,z] &+ \rho(x,y)(w) + \rho(y,z)(u)\\ &+ \rho(z,x)(v) + \nu(x)(v\wedge w)\\ &+ \nu(y)(w\wedge u) + \nu(z)(u\wedge v).\end{aligned} \tag{2.1}$$

Theorem 2.4. *Let $(\mathfrak{g}, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom-Lie algebra and $(V;\rho,\nu,A)$ a generalized representation of $\mathfrak{g}$ with respect to A. Then $(\mathfrak{g}\oplus V, [\cdot,\cdot,\cdot]_{(\rho,\nu)}, \alpha_{\mathfrak{g}\oplus V} = \alpha + A)$ is 3-Hom-Lie algebra, where $[\cdot,\cdot,\cdot]_{(\rho,\nu)}$ is given by* (2.1). *We call the 3-Hom-Lie algebra $(\mathfrak{g}\oplus V, [\cdot,\cdot,\cdot]_{(\rho,\nu)}, \alpha_{\mathfrak{g}\oplus V} = \alpha + A)$* the generalized semidirect product *of $\mathfrak{g}$ and V.*

Proof. It follows from $[x+u, y+v, z+w](\rho, \nu) = (\pi+\overline{\rho}+\overline{\nu})(x+u, y+v, z+w)$ and Lemma 1.11. ∎

In the following, we give a characterization of a generalized representation of a 3-Hom-Lie algebra.

PROPOSITION 2.5. *Let $\rho : \wedge^2\mathfrak{g} \to End(V)$, $\nu : \mathfrak{g} \to \mathrm{Hom}(\wedge^2 V, V)$ and $A : V \to V$ be linear maps. They give rise to a generalized representation of a 3-Hom-Lie algebra $\mathfrak{g}$ with respect to A if and only if for all $x_i \in \mathfrak{g}$, $v_j \in V$, the following equalities hold:*

$$\begin{aligned}\rho(\alpha(x_1),\alpha(x_2))\rho(x_3,x_4) = {} & \rho([x_1,x_2,x_3],\alpha(x_4))\circ A \\ & - \rho([x_1,x_2,x_4],\alpha(x_3))\circ A \\ & + \rho(\alpha(x_3),\alpha(x_4))\rho(x_1,x_2),\end{aligned} \tag{2.2}$$

$$\begin{aligned}\rho([x_1,x_2,x_3],\alpha(x_4))\circ A = {} & \rho(\alpha(x_2),\alpha(x_3))\rho(x_1,x_4) \\ & + \rho(\alpha(x_3),\alpha(x_1))\rho(x_2,x_4) \\ & + \rho(\alpha(x_1),\alpha(x_2))\rho(x_3,x_4),\end{aligned} \tag{2.3}$$

$$\begin{aligned}\rho(\alpha(x_1),\alpha(x_2))\nu(x_3)(v_1,v_2) = {} & \nu([x_1,x_2,x_3])(A(v_1),A(v_2)) \\ & + \nu(\alpha(x_3))(\rho(x_1,x_2)v_1,A(v_2)) \\ & + \nu(\alpha(x_3))(A(v_2),\rho(x_1,x_2)v_1),\end{aligned} \tag{2.4}$$

$$\begin{aligned}\nu(\alpha(x_1))(A(v_1),\rho(x_2,x_3)v_2) = {} & \nu(\alpha(x_3))(A(v_2),\rho(x_2,x_1)v_1) \\ & + \nu(\alpha(x_2))(\rho(x_3,x_1)v_1,A(v_2)) \\ & + \rho(\alpha(x_2),\alpha(x_3))\nu(x_1)(v_1,v_2),\end{aligned} \tag{2.5}$$

$$\begin{aligned}\nu(\alpha(x_1))(A(v_1),\nu(x_2)(v_2,v_3)) = {} & \nu(\alpha(x_2))(\nu(x_1)(v_1,v_2),A(v_3)) \\ & + \nu(\alpha(x_2))(A(v_2),\nu(x_1)(v_1,v_3)),\end{aligned} \tag{2.6}$$

$$\nu(\alpha(x_1))(\nu(x_2)(v_1,v_2),A(v_3)) = \nu(\alpha(x_2))(\nu(x_1)(v_1,v_2),A(v_3)). \tag{2.7}$$

Proof. The quadruple $(V;\rho,\nu,A)$ is a generalized representation if and only if $[\pi+\overline{\rho}+\overline{\nu}, \pi+\overline{\rho}+\overline{\nu}]^{3HL} = 0$. By straightforward computations,

$$[\pi+\bar{\rho}+\bar{\nu}, \pi+\bar{\rho}+\bar{\nu}]^{3HL}(x_1,x_2,x_3,x_4,v) = 0$$

is equivalent to (2.2); and

$$[\pi+\bar{\rho}+\bar{\nu}, \pi+\bar{\rho}+\bar{\nu}]^{3HL}(x_1,v,x_2,x_3,x_4) = 0$$

is equivalent to (2.3). Other identities can be proved similarly. The details are omitted. ∎

Remark 2.6. By (2.2) and (2.3), the map ρ in a generalized representation $(V;\rho,\nu,A)$ gives rise to a usual representation in the sense of Definition 1.5. Conversely, for any representation ρ, $(V;\rho,\nu=0,A)$ is a generalized representation.

DEFINITION 2.7. Let $(V_1;\rho_1,\nu_1,A_1)$ and $(V_2;\rho_2,\nu_2,A_2)$ be two generalized representations of a 3-Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot,\cdot],\alpha)$. They are said to be *equivalent* if there exists an isomorphism of vector spaces $T:V_1\to V_2$ such that

$$T\rho_1(x,y)(u)=\rho_2(x,y)(Tu),\ \ T\nu_1(x)(u,v)=\nu_2(x)(Tu,Tv),\ \ T\circ A_1=A_2\circ T$$

for all $x,y\in\mathfrak{g}$, $u,v\in V_1$. In terms of diagrams, we have

$$\begin{array}{ccc} \wedge^2\mathfrak{g}\times V_1 & \xrightarrow{\rho_1} & V_1 \\ {\scriptstyle id\times T}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \wedge^2\mathfrak{g}\times V_2 & \xrightarrow{\rho_2} & V_2 \end{array}\ ,\qquad \begin{array}{ccc} \mathfrak{g}\times\wedge^2 V_1 & \xrightarrow{\nu_1} & V_1 \\ {\scriptstyle id\times\wedge^2 T}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \mathfrak{g}\times\wedge^2 V_2 & \xrightarrow{\nu_2} & V_2 \end{array}\ ,\qquad \begin{array}{ccc} V_1 & \xrightarrow{A_1} & V_1 \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ V_2 & \xrightarrow{A_2} & V_2 \end{array}\ .$$

In the following, we provide a series of examples to illustrate the new concept of generalized representation and also a procedure to twist a generalized representation along linear maps.

EXAMPLE 2.8. Let $\mathfrak{g}$ be an abelian 3-Hom-Lie algebra. Define $\rho=0$, and $\nu=\xi\otimes\pi$, where $\xi\in\mathfrak{g}^*$ and $\pi\in\mathrm{Hom}(\wedge^2V\otimes V)$ is a Hom-Lie algebra structure on (V,π,A). Then $(V;\rho,\nu,A)$ is a generalized representation. In fact, since $\mathfrak{g}$ is abelian and $\rho=0$, (2.2)-(2.5) hold naturally. Since π satisfies the Hom-Jacobi identity, (2.6) and (2.7) also hold.

PROPOSITION 2.9. *Let $(\mathfrak{g},[\cdot,\cdot,\cdot],\alpha)$ be a 3-Hom-Lie algebra, (V,ρ,ν,A) be a generalized representation, $\beta:\mathfrak{g}\to\mathfrak{g}$ be an algebra morphism and $B:V\to V$ a linear map such that*

$$\begin{aligned} B\circ\rho(x_1,x_2)&=\rho(\beta(x_1),\beta(x_2))\circ B,\\ B\circ\nu(x)&=\nu(\beta(x))\circ(B\otimes B),\\ B\circ A&=A\circ B. \end{aligned}$$

Then $(V, \widetilde{\rho}, \widetilde{\nu}, B)$ *is a generalized representation of 3-Hom-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\beta, \beta\circ\alpha)$ *where*

$$[\cdot,\cdot,\cdot]_\beta = [\cdot,\cdot,\cdot]\circ\beta^{\otimes 3}, \qquad \widetilde{\rho} = B\circ\rho,\ \widetilde{\nu}(x) = B\circ\nu(x).$$

Proof. We have to show that $\widetilde{\rho}$ and $\widetilde{\nu}$ satisfy Eqs. (2.2)-(2.7).
Let $x_1, x_2, x_3 \in \mathfrak{g}$ and $v_1, v_2 \in V$,

$$\begin{aligned}
&\widetilde{\rho}(\beta\circ\alpha(x_1), \beta\circ\alpha(x_2))\widetilde{\nu}(x_3)(v_1, v_2) - \widetilde{\nu}([x_1, x_2, x_3]_\beta)(B\circ A(v_1), B\circ A(v_2))\\
&\qquad - \widetilde{\nu}(\beta\circ\alpha(x_3))(\widetilde{\rho}(x_1, x_2)v_1, B\circ A(v_2))\\
&\qquad - \widetilde{\nu}(\beta\circ\alpha(x_3))(B\circ A(v_2),\ widetilde\rho(x_1, x_2)v_1)\\
&\quad = B\circ\rho(\beta\circ\alpha(x_1), \beta\circ\alpha(x_2))\circ B\circ\nu(x_3)(v_1, v_2)\\
&\qquad - B\circ\nu(\beta\circ[x_1, x_2, x_3])\circ(B\otimes B)(A(v_1), A(v_2))\\
&\qquad - B\circ\nu(\beta\circ\alpha(x_3))(B\circ\rho(x_1, x_2)v_1, B\circ A(v_2))\\
&\qquad - B\circ\nu(\beta\alpha(x_3))(B\circ A(v_2), B\circ\rho(x_1, x_2)v_1))\\
&\quad = B^2\circ(\rho(\alpha(x_1), \alpha(x_2))\circ\nu(x_3)(v_1, v_2) - \nu([x_1, x_2, x_3])(A(v_1), A(v_2))\\
&\qquad - \nu(\alpha(x))(\rho(x_1, x_2)v_1, A(v_2)) - \nu(\alpha(x_3))[2pt](A(v_2), \rho(x_1, x_2)v_1)) = 0,
\end{aligned}$$

$$\begin{aligned}
&\widetilde{\nu}(\beta\circ\alpha(x_1))(B\circ A(v_1), \widetilde{\rho}(x_2, x_3)v_2) - \widetilde{\nu}(\beta\circ\alpha(x_3))(B\circ A(v_2), \widetilde{\rho}(x_2, x_1)v_1)\\
&\qquad - \widetilde{\nu}(\beta\circ\alpha(x_2))(\widetilde{\rho}(x_3, x_1)v_1, B\circ A(v_2))\\
&\qquad - \widetilde{\rho}(\beta\circ\alpha(x_2), \beta\circ\alpha(x_3))\widetilde{\nu}(x_1)(v_1, v_2)\\
&\quad = B\circ\nu(\beta\circ\alpha(x_1))(B\circ A(v_1), B\circ\rho(x_2, x_3)v_2)\\
&\qquad - B\circ\nu(\beta\circ\alpha(x_3))(B\circ A(v_2), B\circ\rho(x_2, x_1)v_1)\\
&\qquad - B\circ\nu(\beta\circ\alpha(x_2))(B\circ\rho(x_3, x_1)v_1, B\circ A(v_2))\\
&\qquad - B\circ\rho(\beta\circ\alpha(x_2), \beta\circ\alpha(x_3))\circ B\circ\nu(x_1)(v_1, v_2)\\
&\quad = B^2\circ(\nu(\alpha(x_1))(A(v_1), \rho(x_2, x_3)v_2) - \nu(\alpha(x_3))(v_2, \rho(x_2, x_1)v_1)\\
&\qquad - \nu(\alpha(x_2))(\rho(x_3, x_1)v_1, A(v_2)) - \rho(\alpha(x_2), \alpha(x_3))\circ\nu(x_1)(v_1, v_2)) = 0.
\end{aligned}$$

Then identities (2.4) and (2.5) are proved. One similarly proves identities (2.6) and (2.7). ∎

COROLLARY 2.10. *Let* $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ *be a 3-Lie algebra,* (V, ρ, ν) *be a generalized representation,* $\alpha : \mathfrak{g}\to\mathfrak{g}$ *be an algebra morphism and* $A : V \to V$ *be a* *linear map such that for all* $x_1, x_2 \in \mathfrak{g}$ *and* $v_1, v_2 \in V$*,*

$$A \circ \rho(x_1, x_2) = \rho(\alpha(x_1), \alpha(x_2)) \circ A,$$
$$A \circ \nu(x)(v_1, v_2) = \nu(\alpha(x)) \circ (A \otimes A)(v_1, v_2).$$

Then $(V, \widetilde{\rho} := A \circ \rho, \widetilde{\nu} := A \circ \nu, A)$ *is a generalized representation of the 3-Hom-Lie algebra* $(L, [\cdot,\cdot,\cdot]_\alpha := \alpha \circ [\cdot,\cdot,\cdot], \alpha)$*.*

Example 2.11. Let $\mathfrak{g}$ be the 3-dimensional 3-Lie algebra defined with respect to a basis $\{e_1, e_2, e_3\}$ by the skew-symmetric bracket $[e_1, e_2, e_3] = e_1$. Let V be a 2-dimensional vector space and $\{v_1, v_2\}$ its basis. We have a representation defined by the following maps (ρ, μ), given with respect to previous bases by

$$\rho(e_1, e_2)(v_1) = 0, \qquad \rho(e_1, e_2)(v_2) = v_1, \qquad \rho(e_1, e_3)(v_1) = 0,$$
$$\rho(e_1, e_3)(v_2) = r_1 v_1, \qquad \rho(e_2, e_3)(v_1) = v_1, \qquad \rho(e_2, e_3)(v_2) = r_2 v_1,$$
$$\nu(e_1)(v_1, v_2) = 0, \qquad \nu(e_2)(v_1, v_2) = s v_1, \qquad \nu(e_3)(v_1, v_2) = s r_1 v_1,$$

where r_1, r_2, s are parameters in $\mathbb{K}$.

Let $\alpha : \mathfrak{g} \to \mathfrak{g}$ be an algebra morphism and $A \in gl(V)$ defined respectively by:

$$\alpha(e_1) = \lambda e_1, \qquad \alpha(e_2) = e_2, \qquad \alpha(e_3) = e_3,$$
$$A(v_1) = \lambda v_1, \qquad A(v_2) = r_2 v_1 + v_2,$$

where λ is a parameter in $\mathbb{K}$. They satisfy

$$A \circ \rho(x_1, x_2) = \rho(\alpha(x_1), \alpha(x_2)) \circ A,$$
$$A \circ \nu(x)(u_1, u_2) = \nu(\alpha(x)) \circ (A \otimes A)(u_1, u_2),$$

where x, x_1, x_2 are in $\mathfrak{g}$ and u_1, u_2 in V.

Then, using the Twist procedure, $(V; \widetilde{\rho}, \widetilde{\nu}, A)$ is a generalized representation of the 3-Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\alpha, \alpha)$. More precisely, we have

$$[e_1, e_2, e_3]_\alpha = [\alpha(e_1), \alpha(e_2), \alpha(e_3)] = \lambda e_1,$$
$$\widetilde{\rho}(e_1, e_2)(v_1) = 0, \qquad \widetilde{\rho}(e_1, e_2)(v_2) = \lambda v_1,$$
$$\widetilde{\rho}(e_1, e_3)(v_1) = 0, \qquad \widetilde{\rho}(e_1, e_3)(v_2) = r_1 r_2 (\lambda - 1) v_1 - r_1 v_2,$$
$$\widetilde{\rho}(e_2, e_3)(v_1) = \lambda v_1, \qquad \widetilde{\rho}(e_2, e_3)(v_2) = r_2 \lambda v_1,$$
$$\widetilde{\nu}(e_1)(v_1, v_2) = 0, \qquad \widetilde{\nu}(e_2)(v_1, v_2) = s\lambda v_1, \qquad \widetilde{\nu}(e_3)(v_1, v_2) = s r_1 \lambda v_1.$$

EXAMPLE 2.12. Let $\mathfrak{g}$ be the 4-dimensional 3-Lie algebra defined, with respect to a basis $\{e_1, e_2, e_3, e_4\}$, by the skew-symmetric brackets

$$[e_1, e_2, e_4] = e_3, \qquad [e_1, e_3, e_4] = e_2, \qquad [e_2, e_3, e_4] = e_1.$$

Every generalized representation $(V; \rho, \nu)$, on a 2-dimensional vector space V with trivial ρ, of $\mathfrak{g}$ is given by one of the following maps ν defined, with respect to a basis $\{v_1, v_2\}$ of V, by

$$\nu(e_1)(v_1, v_2) = 0, \qquad \nu(e_2)(v_1, v_2) = 0,$$
$$\nu(e_3)(v_1, v_2) = 0, \qquad \nu(e_4)(v_1, v_2) = s_1 v_1 + s_2 v_2,$$

where s_1, s_2 are parameters in $\mathbb{K}$, and $s_1 s_2 \neq 0$.

Let $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a 3-Lie algebra morphism and $A : V \to V$ be a linear map, defined respectively by

$$\alpha(e_1) = a_1 e_1, \quad \alpha(e_2) = -a_1 e_2, \quad \alpha(e_3) = a_1 e_3, \quad \alpha(e_4) = \frac{-1}{a_1} e_4,$$
$$A(v_1) = -a_1 v_1, \qquad A(v_2) = a_2 v_1 + \frac{a_2 s_2 - a_1 s_1}{s_1} v_2,$$

where a_1, a_2, s_1, s_2 are parameters in $\mathbb{K}$ such that, $a_1 s_1 s_2 \neq 0$.

They satisfy $A \circ \nu(x) = \nu(\alpha(x)) \circ (A \otimes A)$. Therefore, using the Twist procedure, $(V; \widetilde{\rho}, \widetilde{\nu}, A)$ is a generalized representation of the 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot]_\alpha, \alpha)$ with trivial $\widetilde{\rho}$. Namely, we have

$$[e_1, e_2, e_4]_\alpha = a_1 e_3, \qquad [e_1, e_3, e_4]_\alpha = -a_1 e_2, \qquad [e_2, e_3, e_4]_\alpha = a_1 e_1,$$

and

$$\widetilde{\nu}(e_1)(v_1, v_2) = 0, \qquad \widetilde{\nu}(e_2)(v_1, v_2) = 0, \qquad \widetilde{\nu}(e_3)(v_1, v_2) = 0,$$
$$\widetilde{\nu}(e_4)(v_1, v_2) = (a_2 s_2 - a_1 s_1) v_1 + \left(\frac{a_2 s_2^2}{s_1} - a_1 s_2 \right) v_2.$$

3. New cohomology complex of 3-Hom-Lie algebras

Based on the generalized representations defined in the previous section, we introduce a new type of cohomology for 3-Hom-Lie algebras.

Let $(\mathfrak{g}, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom-Lie algebra and $(V; \rho, \nu, A)$ be a generalized representation of $\mathfrak{g}$. We set $\widetilde{C}^p_{\alpha+A,A}(\mathfrak{g} \oplus V, V)$ to be the set of $(p+1)$-Hom-cochains, which are defined as a subset of $C^p_{\alpha+A,A}(\mathfrak{g} \oplus V, V)$ such that

$$C^p_{\alpha+A,A}(\mathfrak{g} \oplus V, V) = \widetilde{C}^p_{\alpha+A,A}(\mathfrak{g} \oplus V, V) \oplus C^p_A(V, V).$$

Elements of $C^p_{\alpha+A,A}(\mathfrak{g}\oplus V, V)$ are of the form

$$\varphi : \wedge^2(\mathfrak{g}\oplus V)\otimes \overset{(p \text{ times})}{\cdots} \otimes \wedge^2 (\mathfrak{g}\oplus V) \wedge (\mathfrak{g}\oplus V) \to V.$$

By direct calculation, we have

$$[\pi + \bar{\rho} + \bar{\nu}, \widetilde{C}^{\bullet}_{\alpha+A,A}(\mathfrak{g}\oplus V, V)] \subseteq \widetilde{C}^{\bullet+1}_{\alpha+A,A}(\mathfrak{g}\oplus V, V).$$

Define $d : \widetilde{C}^p_{\alpha+A,A}(\mathfrak{g}\oplus V, V) \to \widetilde{C}^{p+1}_{\alpha+A,A}(\mathfrak{g}\oplus V, V)$ by

$$d(\varphi) := [\pi + \bar{\rho} + \bar{\nu}, \varphi]^{3HL}, \qquad \varphi \in \widetilde{C}^p_{\alpha+A,A}(\mathfrak{g}\oplus V, V).$$

THEOREM 3.1. *Let $(V;\rho,\nu,A)$ be a generalized representation of a 3-Hom-Lie algebra $\mathfrak{g}$. Then $d\circ d = 0$. Thus, we obtain a new cohomology complex, where the space of p-Hom-cochains is given by $\widetilde{C}^{p-1}_{\alpha+A,A}(\mathfrak{g}\oplus V, V)$.*

Proof. By the graded Jacobi identity, for any $\varphi \in \widetilde{C}^{p-1}_{\alpha+A,A}(\mathfrak{g}\oplus V, V)$, one obtains

$$\begin{aligned} d\circ d(\varphi) &:= [\pi + \bar{\rho} + \bar{\nu}, [\pi + \bar{\rho} + \bar{\nu}, \varphi]^{3HL}]^{3HL} \\ &= \frac{1}{2}[[\pi + \bar{\rho} + \bar{\nu}, \pi + \bar{\rho} + \bar{\nu}]^{3HL}, \varphi]^{3HL} \;=\; 0. \end{aligned}$$

∎

An element $\varphi \in \widetilde{C}^{p-1}_{\alpha+A,A}(\mathfrak{g}\oplus V, V)$ is called a p-cocycle if $d(\varphi) = 0$; it is called a p-coboundary if there exists $f \in \widetilde{C}^{p-2}_{\alpha+A,A}(\mathfrak{g}\oplus V, V)$ such that $\varphi = d(f)$.

Denote by $\mathcal{Z}^p_{3HL}(\mathfrak{g}; V)$ and $\mathcal{B}^p_{3HL}(\mathfrak{g}; V)$ the sets of p-cocycles and p-coboundaries respectively. By Theorem 3.1, we have $\mathcal{B}^p_{3HL}(\mathfrak{g}; V) \subset \mathcal{Z}^p(\mathfrak{g}; V)$. We define the p-th cohomolgy group $\mathcal{H}^p_{3HL}(\mathfrak{g}; V)$ to be $\mathcal{Z}^p_{3HL}(\mathfrak{g}; V)/\mathcal{B}^p_{3HL}(\mathfrak{g}; V)$.

The following proposition provides a relationship between this new cohomology and the one given by (1.5).

PROPOSITION 3.2. *There is a forgetful map from $\mathcal{H}^p_{3HL}(\mathfrak{g}; V)$ to $H^p_{3HL}(\mathfrak{g}; V)$.*

Proof. It is obvious that $C^p_{\alpha,A}(\mathfrak{g}, V) \subseteq \widetilde{C}^p_{\alpha+A,A}(\mathfrak{g}\oplus V, V)$. By direct calculation, for $X_i \in \wedge^2\mathfrak{g}$, $z \in \mathfrak{g}$, we have

$$d(\varphi)(X_1,\dots,X_{p+1},z) = \delta_\rho(\varphi)(X_1,\dots,X_{p+1},z), \qquad \varphi \in C^p_{\alpha,A}(\mathfrak{g}, V),$$

where δ_ρ is the coboundary operator given by (1.4). Thus, the natural projection from $\widetilde{C}^p_{\alpha+A,A}(\mathfrak{g}\oplus V, V)$ to $C^p_{\alpha,A}(\mathfrak{g}, V)$ induces a forgetful map from $\mathcal{H}^p_{3HL}(\mathfrak{g}; V)$ to $H^p_{3HL}(\mathfrak{g}; V)$. ∎

In the sequel, we give some characterization of low dimensional cocycles.

PROPOSITION 3.3. *A linear map $\varphi \in \mathrm{Hom}(\mathfrak{g}, V)$ is a 1-cocycle if only if for all $x_1, x_2, x_3 \in \mathfrak{g}$, $v \in V$, the following identities hold:*

$$\varphi \circ \alpha = A \circ \varphi,$$
$$\nu(x_1)(\varphi(x_2), v) - \nu(x_2)(\varphi(x_1), v) = 0,$$
$$\varphi([x_1, x_2, x_3]) - \rho(\alpha(x_1), \alpha(x_2))(\varphi(x_3))$$
$$-\rho(\alpha(x_2), \alpha(x_3))(\varphi(x_1)) - \rho(\alpha(x_3), \alpha(x_1))(\varphi(x_2)) = 0.$$

Proof. For $\varphi \in \mathrm{Hom}(\mathfrak{g}, V)$ satisfying $\varphi \circ \alpha = A \circ \varphi$, we have

$$d(\varphi)(x_1, x_2, v) = \nu(x_1)(\varphi(x_2), v) - \nu(x_2)(\varphi(x_1), v),$$

and

$$\begin{aligned} d(\varphi)(x_1, x_2, x_3) &= \delta_\rho(\varphi)(x_1, x_2, x_3) \\ &= \rho(\alpha(x_1), \alpha(x_2))(\varphi(x_3)) + \rho(\alpha(x_2), \alpha(x_3))(\varphi(x_1)) \\ &\quad + \rho(\alpha(x_3), \alpha(x_1))(\varphi(x_2)) - \varphi([x_1, x_2, x_3]). \end{aligned}$$

∎

PROPOSITION 3.4. *A 2-cochain $\varphi_1 + \varphi_2 + \varphi_3 \in \widetilde{C}^1_{\alpha,A}(\mathfrak{g} \oplus V, V)$, where $\varphi_1 \in \mathrm{Hom}(\wedge^2 V \wedge \mathfrak{g}, V)$, $\varphi_2 \in \mathrm{Hom}(\wedge^2 \mathfrak{g} \wedge V, V)$, $\varphi_3 \in \mathrm{Hom}(\wedge^3 \mathfrak{g}, V)$, is a 2-cocycle if and only if for all $x_i \in \mathfrak{g}, v_j \in V$ and $v \in V$, the following identities hold:*

$$\begin{aligned} 0 = &-\rho(\alpha(x_1), \alpha(x_2))(\varphi_3(x_3, x_4, x_5)) - \varphi_3(\alpha(x_1), \alpha(x_2), [x_3, x_4, x_5]) \\ &+ \rho(\alpha(x_4), \alpha(x_5))(\varphi_3(x_1, x_2, x_3)) + \varphi_3([x_1, x_2, x_3], \alpha(x_4), \alpha(x_5)) \\ &+ \rho(\alpha(x_5), \alpha(x_3))(\varphi_3(x_1, x_2, x_4)) + \varphi_3(\alpha(x_3), [x_1, x_2, x_4], \alpha(x_5)) \\ &+ \rho(\alpha(x_3), \alpha(x_4))(\varphi_3(x_1, x_2, x_5)) + \varphi_3(\alpha(x_3), \alpha(x_4), [x_1, x_2, x_5]), \end{aligned} \tag{3.1}$$

$$\begin{aligned} 0 = &\ \nu(\alpha(x_4))(\varphi_3(x_1, x_2, x_3), A(v)) + \nu(\alpha(x_3))(A(v), \varphi_3(x_1, x_2, x_4)) \\ &+ \rho(\alpha(x_1), \alpha(x_2))(\varphi_2(x_3, x_4, v)) - \rho(\alpha(x_3), \alpha(x_4))(\varphi_2(x_1, x_2, v)) \\ &- \varphi_2([x_1, x_2, x_3], \alpha(x_4), A(v)) - \varphi_2(\alpha(x_3), [x_1, x_2, x_4], A(v)), \end{aligned} \tag{3.2}$$

$$\begin{aligned} 0 = &\ \nu(\alpha(x_1))(A(v), \varphi_3(x_2, x_3, x_4)) + \rho(\alpha(x_3), \alpha(x_4))(\varphi_2(x_1, x_2, v)) \\ &- \rho(\alpha(x_2), \alpha(x_4))(\varphi_2(x_1, x_3, v)) + \rho(\alpha(x_2), \alpha(x_3))(\varphi_2(x_1, x_4, v)) \\ &+ \varphi_2(\alpha(x_3), \alpha(x_4), \rho(x_1, x_2)(v)) - \varphi_2(\alpha(x_2), \alpha(x_4), \rho(x_1, x_3)(v)) \\ &+ \varphi_2(\alpha(x_2), \alpha(x_3), \rho(x_1, x_4)(v)) - \varphi_2(\alpha(x_1), [x_2, x_3, x_4], A(v)), \end{aligned} \tag{3.3}$$

$$\begin{aligned}0 &= \nu(\alpha(x_3))(A(v_2), \varphi_2(x_1, x_2, v_1)) + \nu(\alpha(x_3))(\varphi_2(x_1, x_2, A(v_2)), v_1) \\ &\quad + \varphi_2(\alpha(x_1), \alpha(x_2), \nu(x_3)(v_1, v_2)) + \rho(\alpha(x_1), \alpha(x_2))(\varphi_1(v_1, v_2, x_3)) \\ &\quad - \varphi_1(\rho(x_1, x_2)(v_1), A(v_2), \alpha(x_3)) - \varphi_1(A(v_1), \rho(x_1, x_2)(v_2), \alpha(x_3)) \\ &\quad - \varphi_1(A(v_1), A(v_2), [x_1, x_2, x_3]),\end{aligned} \tag{3.4}$$

$$\begin{aligned}0 &= \nu(\alpha(x_3))(A(v_2), \varphi_2(x_2, x_1, v_1)) + \nu(\alpha(x_2))(\varphi_2(x_3, x_1, v_1), A(v_2)) \\ &\quad - \nu(\alpha(x_1))(A(v_1), \varphi_2(x_2, x_3, v_2)) + \varphi_2(\alpha(x_2), \alpha(x_3), \nu(x_1)(v_1, v_2)) \\ &\quad + \rho(\alpha(x_2), \alpha(x_3))(\varphi_1(v_1, v_2, x_1)) + \varphi_1(\rho(x_1, x_2)(v_1), A(v_2), \alpha(x_3)) \\ &\quad - \varphi_1(A(v_1), \rho(x_2, x_3)(v_2), \alpha(x_1)) + \varphi_1(A(v_2), \rho(x_1, x_3)(v_1), \alpha(x_2)),\end{aligned} \tag{3.5}$$

$$\begin{aligned}0 &= \varphi_2(\alpha(x_1), \alpha(x_3), \nu(x_2)(v_1, v_2)) - \varphi_2(\alpha(x_2), \alpha(x_3), \nu(x_1)(v_1, v_2)) \\ &\quad - \varphi_2(\alpha(x_1), \alpha(x_2), \nu(x_3)(v_1, v_2)) + \rho(\alpha(x_1), \alpha(x_3))(\varphi_1(v_1, v_2, x_2)) \\ &\quad - \rho(\alpha(x_1), \alpha(x_2))(\varphi_1(v_1, v_2, x_3)) - \rho(\alpha(x_2), \alpha(x_3))(\varphi_1(v_1, v_2, x_1)) \\ &\quad + \varphi_1(A(v_1), A(v_2), [x_1, x_2, x_3]),\end{aligned} \tag{3.6}$$

$$\begin{aligned}0 &= -\nu(\alpha(x_2))(\varphi_1(v_1, v_2, x_1), A(v_3)) - \nu(\alpha(x_2))(A(v_2), \varphi_1(v_1, v_3, x_1)) \\ &\quad + \nu(\alpha(x_1))(A(v_1), \varphi_1(v_2, v_3, x_2)) - \varphi_1(\nu(x_1)(v_1, v_2), A(v_3), \alpha(x_2)) \\ &\quad - \varphi_1(A(v_2), \nu(x_1)(v_1, v_3), \alpha(x_2)) + \varphi_1(A(v_1), \nu(x_2)(v_2, v_3), \alpha(x_1)),\end{aligned} \tag{3.7}$$

$$\begin{aligned}0 &= \nu(\alpha(x_2))(\varphi_1(v_1, v_2, x_1), A(v_3)) - \nu(\alpha(x_1))(A(v_3), \varphi_1(v_1, v_2, x_2)) \\ &\quad + \varphi_1(\nu(x_1)(v_1, v_2), A(v_3), \alpha(x_2)) - \varphi_1(\nu(x_2)(v_1, v_2), A(v_3), \alpha(x_1)).\end{aligned} \tag{3.8}$$

Proof. For $\varphi_3 \in \mathrm{Hom}(\wedge^3\mathfrak{g}, V)$, we have

$$\begin{aligned}d(\varphi_3)(x_1, x_2, x_3, x_4, x_5) &= \rho(\alpha(x_1), \alpha(x_2))(\varphi_3(x_3, x_4, x_5)) \\ &\quad + \varphi_3(\alpha(x_1), \alpha(x_2), [x_3, x_4, x_5]) \\ &\quad - \rho(\alpha(x_4), \alpha(x_5))(\varphi_3(x_1, x_2, x_3)) \\ &\quad - \varphi_3([x_1, x_2, x_3], \alpha(x_4), \alpha(x_5)) \\ &\quad - \rho(\alpha(x_5), \alpha(x_3))(\varphi_3(x_1, x_2, x_4)) \\ &\quad - \varphi_3(\alpha(x_3), [x_1, x_2, x_4], \alpha(x_5)) \\ &\quad - \rho(\alpha(x_3), \alpha(x_4))(\varphi_3(x_1, x_2, x_5)) \\ &\quad - \varphi_3(\alpha(x_3), \alpha(x_4), [x_1, x_2, x_5]), \\ d(\varphi_3)(x_1, x_2, x_3, x_4, v) &= \nu(\alpha(x_4))(\varphi_3(x_1, x_2, x_3), A(v)) \\ &\quad + \nu(\alpha(x_3))(A(v), \varphi_3(x_1, x_2, x_4)), \\ d(\varphi_3)(x_1, v, x_2, x_3, x_4) &= \nu(\alpha(x_1))(A(v), \varphi_3(x_2, x_3, x_4)).\end{aligned}$$

For $\varphi_2 \in \mathrm{Hom}(\wedge^2\mathfrak{g} \wedge V, V)$, we have

$$\begin{aligned}
d(\varphi_2)(x_1,x_2,x_3,x_4,v) &= \rho(\alpha(x_1),\alpha(x_2))(\varphi_2(x_3,x_4,v))\\
&\quad - \rho(\alpha(x_3),\alpha(x_4))(\varphi_2(x_1,x_2,v))\\
&\quad - \varphi_2([x_1,x_2,x_3],\alpha(x_4),A(v))\\
&\quad - \varphi_2(\alpha(x_3),[x_1,x_2,x_4],A(v)),\\
d(\varphi_2)(x_1,v,x_2,x_3,x_4) &= \rho(\alpha(x_3),\alpha(x_4))(\varphi_2(x_1,x_2,v))\\
&\quad - \rho(\alpha(x_2),\alpha(x_4))(\varphi_2(x_1,x_3,v))\\
&\quad + \rho(\alpha(x_2),\alpha(x_3))(\varphi_2(x_1,x_4,v))\\
&\quad + \varphi_2(\alpha(x_3),\alpha(x_4),\rho(x_1,x_2)(v))\\
&\quad - \varphi_2(\alpha(x_2),\alpha(x_4),\rho(x_1,x_3)(v))\\
&\quad + \varphi_2(\alpha(x_2),\alpha(x_3),\rho(x_1,x_4)(v))\\
&\quad - \alpha_2(x_1,[x_2,x_3,x_4],v),\\
d(\varphi_2)(x_1,x_2,v_1,v_2,x_3) &= \nu(\alpha(x_3))(A(v_2),\varphi_2(x_1,x_2,v_1))\\
&\quad + \nu(\alpha(x_3))(\varphi_2(x_1,x_2,v_2),A(v_1))\\
&\quad + \varphi_2(\alpha(x_1),\alpha(x_2),\nu(x_3)(v_1,v_2)),\\
d(\varphi_2)(x_1,v_1,x_2,v_2,x_3) &= \nu(\alpha(x_3))(A(v_2),\varphi_2(x_2,x_1,v_1))\\
&\quad + \nu(\alpha(x_2))(\varphi_2(x_3,x_1,v_1),A(v_2))\\
&\quad - \nu(\alpha(x_1))(A(v_1),\varphi_2(x_2,x_3,v_2))\\
&\quad + \varphi_2(\alpha(x_2),\alpha(x_3),\nu(x_1)(v_1,v_2)),\\
d(\varphi_2)(v_1,v_2,x_1,x_2,x_3) &= \varphi_2(\alpha(x_1),\alpha(x_3),\nu(x_2)(v_1,v_2))\\
&\quad - \varphi_2(\alpha(x_2),\alpha(x_3),\nu(x_1)(v_1,v_2))\\
&\quad - \varphi_2(\alpha(x_1),\alpha(x_2),\nu(x_3)(v_1,v_2)).
\end{aligned}$$

For $\varphi_1 \in \mathrm{Hom}(\wedge^2 V \wedge \mathfrak{g}, V)$, we have

$$\begin{aligned}
d(\varphi_1)(x_1,x_2,v_1,v_2,x_3) &= \rho(\alpha(x_1),\alpha(x_2))(\varphi_1(v_1,v_2,x_3))\\
&\quad - \varphi_1(\rho(x_1,x_2)(v_1),A(v_2),\alpha(x_3))\\
&\quad - \varphi_1(A(v_1),\rho(x_1,x_2)(v_2),\alpha(x_3))\\
&\quad - \varphi_1(A(v_1),A(v_2),[x_1,x_2,x_3]),\\
d(\varphi_1)(x_1,v_1,x_2,v_2,x_3) &= \rho(\alpha(x_2),\alpha(x_3))(\varphi_1(v_1,v_2,x_1))\\
&\quad + \varphi_1(\rho(x_1,x_2)(v_1),A(v_2),\alpha(x_3))\\
&\quad - \varphi_1(A(v_1),\rho(x_2,x_3)(v_2),\alpha(x_1))\\
&\quad + \varphi_1(A(v_2),\rho(x_1,x_3)(v_1),\alpha(x_2)),
\end{aligned}$$

$$\begin{aligned}
d(\varphi_1)(v_1,v_2,x_1,x_2,x_3) &= \rho(\alpha(x_1),\alpha(x_3))(\varphi_1(v_1,v_2,x_2)) \\
&\quad - \rho(\alpha(x_1),\alpha(x_2))(\varphi_1(v_1,v_2,x_3)) \\
&\quad - \rho(\alpha(x_2),\alpha(x_3))(\varphi_1(v_1,v_2,x_1)) \\
&\quad + \varphi_1(A(v_1),A(v_2),[x_1,x_2,x_3]), \\
d(\varphi_1)(x_1,v_1,v_2,v_3,x_2) &= -\nu(\alpha(x_2))(\varphi_1(v_1,v_2,x_1),A(v_3)) \\
&\quad - \nu(\alpha(x_2))(A(v_2),\varphi_1(v_1,v_3,x_1)) \\
&\quad + \nu(\alpha(x_1))(A(v_1),\varphi_1(v_2,v_3,x_2)) \\
&\quad - \varphi_1(\nu(x_1)(v_1,v_2),A(v_3),\alpha(x_2)) \\
&\quad - \varphi_1(A(v_2),\nu(x_1)(v_1,v_3),\alpha(x_2)) \\
&\quad + \varphi_1(A(v_1),\nu(x_2)(v_2,v_3),\alpha(x_1)), \\
d(\varphi_1)(v_1,v_2,x_1,x_2,v_3) &= \nu(\alpha(x_2))(\varphi_1(v_1,v_2,x_1),A(v_3)) \\
&\quad - \nu(\alpha(x_1))(A(v_3),\varphi_1(v_1,v_2,x_2)) \\
&\quad + \varphi_1(\nu(x_1)(v_1,v_2),A(v_3),\alpha(x_2)) \\
&\quad - \varphi_1(\nu(x_2)(v_1,v_2),A(v_3),\alpha(x_1)).
\end{aligned}$$

Thus, $d(\varphi_1+\varphi_2+\varphi_3)=0$ if and only if Eqs. (3.1)-(3.8) hold. ∎

In the following we provide an example of computation of 2-cocycles of a 3-dimensional 3-Hom-Lie algebra.

EXAMPLE 3.5. Let $(\mathfrak{g},[\cdot,\cdot,\cdot],\alpha)$ be the 3-dimensional 3-Hom-Lie algebra defined, with respect to a basis $\{e_1,e_2,e_3\}$, by $[e_1,e_2,e_3]=a_1e_1$, $\alpha(e_1)=a_1e_1$, $\alpha(e_2)=a_2e_2$, $\alpha(e_3)=\frac{1}{a_2}e_3$. Let V be a 2-dimensional vector space, $\{v_1,v_2\}$ its basis and $A\in gl(V)$ defined by: $A(v_1)=a_1v_1$, $A(v_2)=\frac{a_2a_3}{a_1}v_2$, where a_1,a_2,a_3 are parameters in $\mathbb{K}$.

We consider the generalized representation (V,ρ,ν,A), where ρ and ν are defined with respect to the basis by

$$\begin{aligned}
&\rho(e_1,e_2)(v_1)=0, && \rho(e_1,e_2)(v_2)=0, && \rho(e_1,e_3)(v_1)=0,\\
&\rho(e_1,e_3)(v_2)=r_2a_1v_1, && \rho(e_2,e_3)(v_1)=a_1v_1, && \rho(e_2,e_3)(v_2)=\frac{r_1a_2a_3}{a_1}v_2,\\
&\nu(e_1)(v_1,v_2)=0, && \nu(e_2)(v_1,v_2)=0, && \nu(e_3)(v_1,v_2)=s_1a_1v_1,
\end{aligned}$$

with s_1,r_1,r_2 parameters in $\mathbb{K}$ and $a_1a_2s_1\neq 0$.

We have the following 2-cocycles: $\varphi_1 = 0$, $\varphi_3 = 0$ and φ_2 defined as

$$\begin{aligned} \varphi_2(e_1, e_2, v_1) &= 0, & \varphi_2(e_1, e_2, v_2) &= 0, \\ \varphi_2(e_1, e_3, v_1) &= c_1 v_1, & \varphi_2(e_1, e_3, v_2) &= 0, \\ \varphi_2(e_2, e_3, v_1) &= c_2 v_1, & \varphi_2(e_2, e_3, v_2) &= c_3 v_2, \end{aligned}$$

where c_1, c_2, c_3 are parameters in $\mathbb{K}$.

4. Abelian extensions of 3-Hom-Lie algebras

In this section, we show that associated to any abelian extension, there is a generalized representation and a 2-cocycle.

Definition 4.1. Let $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}}, \alpha)$, $(V, [\cdot,\cdot,\cdot]_V, A)$ and $(\hat{\mathfrak{g}}, [\cdot,\cdot,\cdot]_{\hat{\mathfrak{g}}}, \alpha_{\hat{\mathfrak{g}}})$ be 3-Hom-Lie algebras and $i : V \to \hat{\mathfrak{g}}$, $p : \hat{\mathfrak{g}} \to \mathfrak{g}$ be morphisms of 3-Hom-Lie algebras. The following sequence of 3-Hom-Lie algebras is a short exact sequence if $\mathrm{Im}(i) = \mathrm{Ker}(p)$, $\mathrm{Ker}(i) = 0$ and $\mathrm{Im}(p) = \mathfrak{g}$:

$$0 \to V \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0\,,$$

where $A(V) = \alpha_{\hat{\mathfrak{g}}}(V)$. In this case, we call $\hat{\mathfrak{g}}$ an extension of $\mathfrak{g}$ by V, and denote it by $E_{\hat{\mathfrak{g}}}$. It is called an abelian extension if V is an abelian ideal of $\hat{\mathfrak{g}}$, i.e., $[u, v, w]_V = 0$ for all $u, v, w \in V$. A section σ of $p : \hat{\mathfrak{g}} \to \mathfrak{g}$ consists of a linear maps $\sigma : \mathfrak{g} \to \hat{\mathfrak{g}}$ such that $p \circ \sigma = id_{\mathfrak{g}}$ and $\sigma \circ \alpha = \alpha_{\hat{\mathfrak{g}}} \circ \sigma$.

Definition 4.2. Two extensions of 3-Hom-Lie algebras,

$$E_{\hat{\mathfrak{g}}} : 0 \to V \xrightarrow{i_1} \hat{\mathfrak{g}} \xrightarrow{p_1} \mathfrak{g} \to 0 \qquad \text{and} \qquad E_{\tilde{\mathfrak{g}}} : 0 \to V \xrightarrow{i_2} \tilde{\mathfrak{g}} \xrightarrow{p_2} \mathfrak{g} \to 0,$$

are equivalent if there exists a morphism of 3-Hom-Lie algebras $\phi : \hat{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i_1} & \hat{\mathfrak{g}} & \xrightarrow{p_1} & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}_V} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{Id}_{\mathfrak{g}}} & & \\ 0 & \longrightarrow & V & \xrightarrow{i_2} & \widetilde{\mathfrak{g}} & \xrightarrow{p_2} & \mathfrak{g} & \longrightarrow & 0 \end{array}$$

A linear map $\sigma : \mathfrak{g} \to \hat{\mathfrak{g}}$ is called a splitting of $\mathfrak{g}$ if it satisfies $p \circ \sigma = id_{\mathfrak{g}}$. If there exists a splitting which is also a homomorphism between 3-Hom-Lie algebras, we say that the abelian extension is split. Let $\hat{\mathfrak{g}}$ be a split abelian extension and $\sigma : \mathfrak{g} \to \hat{\mathfrak{g}}$ the corresponding splitting. Define $\rho : \wedge^2\mathfrak{g} \to \mathfrak{gl}(V)$ and $\nu : \mathfrak{g} \to \mathrm{Hom}(\wedge^2 V, V)$ by

$$\begin{aligned}\rho(x,y)(u) &= [\sigma(x),\sigma(y),u]_{\hat{\mathfrak{g}}},\\ \nu(x)(u,v) &= [\sigma(x),u,v]_{\hat{\mathfrak{g}}}.\end{aligned}$$

Then, we can transfer the 3-Hom-Lie algebra structure on $\hat{\mathfrak{g}}$ to that on $\mathfrak{g}\oplus V$ in terms of ρ and ν.

Note that the Hom-Filippov-Jacobi identity gives the character of ρ and ν:

$$\begin{aligned}[x+u,y+v,z+w]_{(\rho,\nu)} = [x,y,z] &+ \rho(x,y)(w) + \rho(y,z)(u) + \rho(z,x)(v)\\ &+ \nu(x)(v\wedge w) + \nu(y)(w\wedge u) + \nu(z)(u\wedge v).\end{aligned}$$

However, by Theorem 2.4, it is straightforward to obtain the following proposition.

PROPOSITION 4.3. *Any split abelian extension of* 3*-Hom-Lie algebras is isomorphic to a generalized semidirect of product* 3*-Hom-Lie algebra.*

Now, for non-split abelian extensions, we can further define $\omega : \wedge^3\mathfrak{g} \to V$ by

$$\omega(x,y,z) = [\sigma(x),\sigma(y),\sigma(z)]_{\hat{\mathfrak{g}}} - \sigma[x,y,z]_{\mathfrak{g}}.$$

Then, we also transfer the 3-Hom-Lie algebra structure on $\hat{\mathfrak{g}}$ to that on $\mathfrak{g}\oplus V$ in terms of ρ,ν and ω:

$$\begin{aligned}[x_1+v_1,x_2+v_2,x_3+v_3]_{(\rho,\nu,\omega)} = [x_1,x_2,x_3]_{\mathfrak{g}} &+ \rho(x_1,x_2)(v_3)\\ &+ \rho(x_3,x_1)(v_2) + \rho(x_2,x_3)(v_1)\\ &+ \nu(x_1)(v_2,v_3) + \nu(x_2)(v_3,v_1)\\ &+ \nu(x_3)(v_1,v_2) + \omega(x_1,x_2,x_3).\end{aligned}$$

THEOREM 4.4. *With above notations,* $(\mathfrak{g}\oplus V, [\cdot,\cdot,\cdot]_{(\rho,\nu,\omega)}, \alpha_{\mathfrak{g}\oplus V})$ *is a* 3*-Hom-Lie algebra if and only if for all* $x_1,x_2,x_3,x_4,x_5 \in \mathfrak{g}$ *and* $v,v_1,v_2,v_3 \in V$*, Eqs.* (2.4)-(2.7) *and the following identities hold:*

$$
\begin{aligned}
0 = &-\rho(\alpha(x_1),\alpha(x_2))(\omega(x_3,x_4,x_5)) - \omega(\alpha(x_1),\alpha(x_2),[x_3,x_4,x_5]) \\
&+ \rho(\alpha(x_4),\alpha(x_5))(\omega(x_1,x_2,x_3)) + \omega([x_1,x_2,x_3],\alpha(x_4),\alpha(x_5)) \\
&+ \rho(\alpha(x_5),\alpha(x_3))(\omega(x_1,x_2,x_4)) + \omega(\alpha(x_3),[x_1,x_2,x_4],\alpha(x_5)) \\
&+ \rho(\alpha(x_3),\alpha(x_4))(\omega(x_1,x_2,x_5)) + \omega(\alpha(x_3),\alpha(x_4),[x_1,x_2,x_5]),
\end{aligned} \tag{4.1}
$$

$$
\begin{aligned}
0 = &\,\nu(\alpha(x_1))(A(v),\omega(x_2,x_3,x_4)) + \rho([x_2,x_3,x_4],\alpha(x_1))(A(v)) \\
&+ \rho(\alpha(x_3),\alpha(x_4))\rho(x_1,x_2)(v) - \rho(\alpha(x_2),\alpha(x_4))\rho(x_1,x_3)(v) \\
&+ \rho(\alpha(x_2),\alpha(x_3))\rho(x_1,x_4)(v),
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
0 = &\,\rho(\alpha(x_1),\alpha(x_2))\rho(x_3,x_4)(v) - \rho(\alpha(x_3),\alpha(x_4))\rho(x_1,x_2)(v) \\
&- \rho([x_1,x_2,x_3],\alpha(x_4))(A(v)) - \nu(\alpha(x_4))(A(v),\omega(x_1,x_2,x_3)) \\
&- \rho(\alpha(x_3),[x_1,x_2,x_4])(A(v)) + \nu(\alpha(x_3))(A(v),\omega(x_1,x_2,x_4)).
\end{aligned} \tag{4.3}
$$

The Fundamental Identity gives the character of ρ, ν and ω.

Proof. The triple $(\mathfrak{g}\oplus V,[\cdot,\cdot,\cdot]_{(\rho,\nu,\omega)},\alpha_{\mathfrak{g}\oplus V})$ defines a 3-Hom-Lie algebra if and only if the Hom-Filippov-Jacobi identity holds on all elements of $\mathfrak{g}\oplus V$. Condition (4.1) is obtained using the Hom-Filippov-Jacobi identity on $\{x_1,x_2,x_3,x_4,x_5\}$ elements of $\mathfrak{g}$.

Similarly, elements $\{x_1,v,x_2,x_3,x_4\}$ gives Eq. (4.2), $\{x_1,x_2,v,x_3,x_4\}$ gives Eq. (4.3), $\{x_1,x_2,v_1,v_2,x_3\}$ gives Eq. (2.4), $\{v_1,x_1,v_2,x_2,x_3\}$ gives Eq. (2.5), $\{v_1,x_1,v_2,v_3,x_2\}$ gives Eq. (2.6) and $\{v_1,v_2,v_3,x_1,x_2\}$ gives Eq. (2.7).

Conversely, if Eqs. (2.4)-(2.7) and Eqs. (4.1)-(4.3) hold, it is straightforward to see that for all $e_1,\dots,e_5\in\mathfrak{g}\oplus V$, the Hom-Filippov-Jacobi identity holds. Thus, $(\mathfrak{g}\oplus V,[\cdot,\cdot,\cdot]_{(\rho,\nu,\omega)},\alpha_{\mathfrak{g}\oplus V})$ is a 3-Hom-Lie algebra. ∎

Remark 4.5. The triple (ρ,ν,ω) is a cochain with respect to the new cohomology defined in Section 3 but it is not necessarily a 2-cocycle.

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