



Extreme and exposed points of $\mathcal{L}(^n l_\infty^2)$ and $\mathcal{L}_s(^n l_\infty^2)$

SUNG GUEN KIM

*Department of Mathematics, Kyungpook National University
Daegu 702-701, South Korea*

sgk317@knu.ac.kr

Received November 26, 2019
Accepted April 10, 2020

Presented by Jesús M.F. Castillo

Abstract: For every $n \geq 2$ this paper is devoted to the description of the sets of extreme and exposed points of the closed unit balls of $\mathcal{L}(^n l_\infty^2)$ and $\mathcal{L}_s(^n l_\infty^2)$, where $\mathcal{L}(^n l_\infty^2)$ is the space of n -linear forms on \mathbb{R}^2 with the supremum norm, and $\mathcal{L}_s(^n l_\infty^2)$ is the subspace of $\mathcal{L}(^n l_\infty^2)$ consisting of symmetric n -linear forms. First we classify the extreme points of the closed unit balls of $\mathcal{L}(^n l_\infty^2)$ and $\mathcal{L}_s(^n l_\infty^2)$, correspondingly. As corollaries we obtain $|\text{ext } B_{\mathcal{L}(^n l_\infty^2)}| = 2^{(2^n)}$ and $|\text{ext } B_{\mathcal{L}_s(^n l_\infty^2)}| = 2^{n+1}$. We also show that $\text{exp } B_{\mathcal{L}(^n l_\infty^2)} = \text{ext } B_{\mathcal{L}(^n l_\infty^2)}$ and $\text{exp } B_{\mathcal{L}_s(^n l_\infty^2)} = \text{ext } B_{\mathcal{L}_s(^n l_\infty^2)}$.

Key words: n -linear forms, symmetric n -linear forms, extreme points, exposed points.

AMS Subject Class. (2010): 46A22.

1. INTRODUCTION

Let $n \in \mathbb{N}, n \geq 2$. We write B_E for the unit ball of a real Banach space E and the dual space of E is denoted by E^* . An element $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y + z)$ implies $x = y = z$. We denote by $\text{ext } B_E$ the set of all the extreme points of B_E . An element $x \in B_E$ is called an *exposed point* of B_E if there is a $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. We denote by $\text{exp } B_E$ the set of exposed points of B_E . We denote by $\mathcal{L}(^n E)$ the Banach space of all continuous n -linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \dots, x_n)|$. $\mathcal{L}_s(^n E)$ denote the closed subspace of all continuous symmetric n -linear forms on E .

Let us say about the history of the classifications of extreme and exposed points of the unit ball of continuous (symmetric) multilinear forms on a Banach space. Kim [1] initiated and classified $\text{ext } B_{\mathcal{L}_s(^2 l_\infty^2)}$ and $\text{exp } B_{\mathcal{L}_s(^2 l_\infty^2)}$, where $l_\infty^n = \mathbb{R}^n$ with the supremum norm. It was shown that $\text{ext } B_{\mathcal{L}_s(^2 l_\infty^2)} = \text{exp } B_{\mathcal{L}_s(^2 l_\infty^2)}$. Kim [2, 3, 4, 5] classified $\text{ext } B_{\mathcal{L}_s(^2 d_*(1,w)^2)}$, $\text{ext } B_{\mathcal{L}(^2 d_*(1,w)^2)}$, $\text{exp } B_{\mathcal{L}_s(^2 d_*(1,w)^2)}$, and $\text{exp } B_{\mathcal{L}(^2 d_*(1,w)^2)}$, where $d_*(1, w)^2 = \mathbb{R}^2$ with the octagonal norm $\|(x, y)\|_w = \max \left\{ |x|, |y|, \frac{|x|+|y|}{1+w} \right\}$. Kim [6, 7] classified $\text{ext } B_{\mathcal{L}_s(^2 \mathbb{R}_{h(w)}^2)}$



and $\text{ext } B_{\mathcal{L}(2\mathbb{R}_{h(w)}^2)}$, where $\mathbb{R}_{h(w)}^2 = \mathbb{R}^2$ with the hexagonal norm $\|(x, y)\|_{h(w)} = \max\{|y|, |x| + (1-w)|y|\}$. Kim [8, 9, 10] classified $\text{ext } B_{\mathcal{L}_s(2l_\infty^3)}$, $\text{ext } B_{\mathcal{L}_s(3l_\infty^2)}$ and $\text{ext } B_{\mathcal{L}(3l_\infty^2)}$. It was shown that every extreme point is exposed in each space. Kim [11] characterized $\text{ext } B_{\mathcal{L}(2l_\infty^n)}$ and $\text{ext } B_{\mathcal{L}_s(2l_\infty^n)}$. Recently, Kim [12] classified $\text{ext } B_{\mathcal{L}(2l_\infty^3)}$ and showed $\text{exp } B_{\mathcal{L}(2l_\infty^3)} = \text{ext } B_{\mathcal{L}(2l_\infty^3)}$.

2. THE EXTREME AND EXPOSED POINTS OF THE UNIT BALL OF $\mathcal{L}(^n l_\infty^2)$

Let $l_\infty^2 = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_\infty = \max(|x|, |y|)\}$. For $n \geq 2$, we denote

$$\mathcal{W}_n := \{(1, w_1), \dots, (1, w_n)\} : w_j = \pm 1 \text{ for } j = 1, \dots, n\}.$$

Note that \mathcal{W}_n has 2^n elements in $S_{l_\infty^2} \times \dots \times S_{l_\infty^2}$.

Recall that the Krein-Milman Theorem [13] say that every nonempty compact convex subset of a Housdorff locally convex space is the closed convex hull of its set of extreme points. Hence, the unit ball of l_∞^2 is the closed convex hull of

$$\{(1, 1), (-1, 1), (1, -1), (-1, -1)\}.$$

THEOREM 2.1. *Let $n \geq 2$ and $T \in \mathcal{L}(^n l_\infty^2)$. Then,*

$$\|T\| = \sup_{W \in \mathcal{W}_n} |T(W)|.$$

Proof. It follows that from the Krein-Milman theorem and multilinearity of T . ■

Let Z_1, \dots, Z_{2^n} be an ordering of the monomials $x_{l_1} \dots x_{l_j} y_{k_1} \dots y_{k_{n-j}}$ with $\{l_1, \dots, l_j, k_1, \dots, k_{n-j}\} = \{1, \dots, n\}$. Note that $\{Z_1, \dots, Z_{2^n}\}$ is a basis for $\mathcal{L}(^n l_\infty^2)$. Hence, $\dim(\mathcal{L}(^n l_\infty^2)) = 2^n$. If $T \in \mathcal{L}(^n l_\infty^2)$, then,

$$T = \sum_{k=1}^{2^n} a_k Z_k$$

for some $a_1, \dots, a_{2^n} \in \mathbb{R}$. By simplicity, we denote $T = (a_1, \dots, a_{2^n})^t$. Let W_1, \dots, W_{2^n} be an ordering of the elements of \mathcal{W}_n . Let

$$M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n}) = [Z_i(W_j)]$$

be the $2^n \times 2^n$ matrix. Note that, for every $T \in \mathcal{L}(^n l_\infty^2)$,

$$M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})T = (T(W_1), \dots, T(W_{2^n}))^t.$$

Here, $(\epsilon_1, \dots, \epsilon_{2^n})^t$ denote the transpose of $(\epsilon_1, \dots, \epsilon_{2^n})$.

THEOREM 2.2. *Let $n \geq 2$. Then,*

$$\begin{aligned} \text{ext } B_{\mathcal{L}(n l_\infty^2)} &= \{M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1}(\epsilon_1, \dots, \epsilon_{2^n})^t \\ &\quad : \epsilon_j = \pm 1, j = 1, \dots, 2^n\}. \end{aligned}$$

Proof. CLAIM 1: $M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})$ is invertible.

Consider the equation

$$M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})(t_1, \dots, t_{2^n})^t = (0, \dots, 0)^t. \quad (*)$$

Let a_1, \dots, a_{2^n} be a solution of (*) and let $T = \sum_{k=1}^{2^n} a_k Z_k \in \mathcal{L}(n l_\infty^2)$. Then,

$$T(W_j) = 0 \quad j = 1, \dots, 2^n.$$

By Theorem 2.1, $\|T\| = 0$, hence $T = 0$. Since Z_1, \dots, Z_{2^n} are linearly independent in $\mathcal{L}(n l_\infty^2)$, we have $a_j = 0$ for all $j = 1, \dots, 2^n$. Hence, the equation (*) has only zero solution. Therefore, $M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})$ is invertible.

CLAIM 2: $M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1}(\epsilon_1, \dots, \epsilon_{2^n})^t$ is an extreme point for $\epsilon_j = \pm 1, (j = 1, \dots, 2^n)$.

Let

$$T := M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1}(\epsilon_1, \dots, \epsilon_{2^n})^t.$$

Since

$$M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})T = (\epsilon_1, \dots, \epsilon_{2^n})^t,$$

$T(W_j) = \epsilon_j$ for $j = 1, \dots, 2^n$. By Theorem 2.1,

$$\|T\| = \sup_{1 \leq j \leq 2^n} |T(W_j)| = \sup_{1 \leq j \leq 2^n} |\epsilon_j| = 1.$$

Suppose that $T = \frac{1}{2}(T_1 + T_2)$ for some $T_k \in B_{\mathcal{L}(n l_\infty^2)}$ ($k = 1, 2$). We may write

$$T_1 = M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1}(\epsilon_1, \dots, \epsilon_{2^n})^t + (\delta_1, \dots, \delta_{2^n})^t$$

and

$$T_2 = M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1}(\epsilon_1, \dots, \epsilon_{2^n})^t - (\delta_1, \dots, \delta_{2^n})^t$$

for some $\delta_j \in \mathbb{R}$ ($j = 1, \dots, 2^n$). Note that

$$(T_k(W_1), \dots, T_k(W_{2^n}))^t = M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})T_k \quad \text{for } k = 1, 2.$$

Therefore,

$$\begin{aligned} (T_1(W_1), \dots, T_1(W_{2^n}))^t &= (\epsilon_1, \dots, \epsilon_{2^n})^t \\ &\quad + M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})(\delta_1, \dots, \delta_{2^n})^t \end{aligned}$$

and

$$\begin{aligned} (T_2(W_1), \dots, T_2(W_{2^n}))^t &= (\epsilon_1, \dots, \epsilon_{2^n})^t \\ &\quad - M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})(\delta_1, \dots, \delta_{2^n})^t. \end{aligned}$$

Hence, for $j = 1, \dots, 2^n$,

$$T_1(W_j) = \epsilon_j + (Z_1(W_j), \dots, Z_{2^n}(W_j))(\delta_1, \dots, \delta_{2^n})^t,$$

and

$$T_2(W_j) = \epsilon_j - (Z_1(W_j), \dots, Z_{2^n}(W_j))(\delta_1, \dots, \delta_{2^n})^t.$$

It follows that, for $j = 1, \dots, 2^n$,

$$\begin{aligned} 1 &\geq \max\{|T_1(W_j)|, |T_2(W_j)|\} \\ &= |\epsilon_j| + |(Z_1(W_j), \dots, Z_{2^n}(W_j))(\delta_1, \dots, \delta_{2^n})^t| \\ &= 1 + |(Z_1(W_j), \dots, Z_{2^n}(W_j))(\delta_1, \dots, \delta_{2^n})^t|, \end{aligned}$$

which shows that

$$(Z_1(W_j), \dots, Z_{2^n}(W_j))(\delta_1, \dots, \delta_{2^n})^t = 0 \quad \text{for } j = 1, \dots, 2^n.$$

Hence,

$$M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})(\delta_1, \dots, \delta_{2^n})^t = 0.$$

Therefore,

$$\begin{aligned} (\delta_1, \dots, \delta_{2^n})^t &= M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1}(0, \dots, 0)^t \\ &= (0, \dots, 0)^t. \end{aligned}$$

Hence, $T_k = T$ for $k = 1, 2$. Therefore, T is extreme.

Suppose that $T \in \text{ext } B_{\mathcal{L}(n)l_\infty^2}$. Note that

$$(T(W_1), \dots, T(W_{2^n}))^t = M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})T.$$

CLAIM 3: $|T(W_j)| = 1$ for all $j = 1, \dots, 2^n$.

If not. There exists $1 \leq j_0 \leq 2^n$ such that $|T(W_{j_0})| < 1$. Let $\delta_0 > 0$ such that $|T(W_{j_0})| + \delta_0 < 1$. Let

$$T_1 = M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1} \\ \times (T(W_1), \dots, T(W_{j_0-1}), T(W_{j_0}) + \delta_0, T(W_{j_0+1}), \dots, T(W_{2^n}))^t$$

and

$$T_2 = M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1} \\ \times (T(W_1), \dots, T(W_{j_0-1}), T(W_{j_0}) - \delta_0, T(W_{j_0+1}), \dots, T(W_{2^n}))^t.$$

Hence,

$$T_1(W_{j_0}) = T(W_{j_0}) + \delta_0, T_2(W_{j_0}) \\ = T(W_{j_0}) - \delta_0, T_1(W_j) = T_2(W_j) = T(W_j) \quad (j \neq j_0).$$

Obviously, $T \neq T_k$ for $k = 1, 2$. By Theorem 2.1, $\|T_k\| = 1$ for $k = 1, 2$ and $T = \frac{1}{2}(T_1 + T_2)$, which is a contradiction. Therefore,

$$T = M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1}(T(W_1), \dots, T(W_{2^n}))^t$$

with $|T(W_j)| = 1$ for all $j = 1, \dots, 2^n$. ■

Kim [10] characterized $\text{ext } B_{\mathcal{L}(^3 l_\infty^2)}$. Notice that using Wolfram Mathematica 8 and Theorem 2.2, we can exclusively describe $\text{ext } B_{\mathcal{L}(^n l_\infty^2)}$ for a given $n \geq 2$.

For every $T \in \mathcal{L}(^n l_\infty^2)$, we let

$$\text{Norm}(T) := \{[(1, w_1), \dots, (1, w_n)] \in \mathcal{W}_n : |T((1, w_1), \dots, (1, w_n))| = \|T\|\}.$$

We call $\text{Norm}(T)$ the set of *the norming points* of T .

COROLLARY 2.3. (a) Let $n \geq 2$. $\text{ext } B_{\mathcal{L}(^n l_\infty^2)}$ has exactly $2^{(2^n)}$ elements.

(b) Let $n \geq 2$ and $T \in \mathcal{L}(^n l_\infty^2)$ with $\|T\| = 1$. Then $T \in \text{ext } B_{\mathcal{L}(^n l_\infty^2)}$ if and only if $\text{Norm}(T) = \mathcal{W}_n$.

THEOREM 2.4. ([4]) Let E be a real Banach space such that $\text{ext } B_E$ is finite. Suppose that $x \in \text{ext } B_E$ satisfies that there exists an $f \in E^*$ with $f(x) = 1 = \|f\|$ and $|f(y)| < 1$ for every $y \in \text{ext } B_E \setminus \{\pm x\}$. Then $x \in \text{exp } B_E$.

THEOREM 2.5. *Let $n \geq 2$. Then, $\exp B_{\mathcal{L}(n)l_\infty^2} = \text{ext } B_{\mathcal{L}(n)l_\infty^2}$.*

Proof. Let $T \in \text{ext } B_{\mathcal{L}(n)l_\infty^2}$ and let

$$f := \frac{1}{2^n} \sum_{1 \leq j \leq 2^n} \text{sign}(T(W_j)) \delta_{W_j} \in \mathcal{L}(n)l_\infty^2{}^*.$$

Note that $1 = \|f\| = f(T)$. Let $S \in \text{ext } B_{\mathcal{L}(n)l_\infty^2}$ be such that $|f(S)| = 1$. We will show that $S = T$ or $S = -T$. It follows that

$$\begin{aligned} 1 &= |f(S)| = \left| \frac{1}{2^n} \sum_{1 \leq j \leq 2^n} \text{sign}(T(W_j)) S(W_j) \right| \\ &\leq \frac{1}{2^n} \sum_{1 \leq j \leq 2^n} |S(W_j)| \\ &\leq 1, \end{aligned}$$

which shows that

$$S(W_j) = \text{sign}(T(W_j)) \quad (1 \leq j \leq 2^n)$$

or

$$S(W_j) = -\text{sign}(T(W_j)) \quad (1 \leq j \leq 2^n).$$

Suppose that

$$S(W_j) = -\text{sign}(T(W_j)) \quad (1 \leq j \leq 2^n).$$

It follows that

$$\begin{aligned} S &= M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1} (S(W_1), \dots, S(W_{2^n}))^t \\ &= M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1} (-\text{sign}(T(W_1)), \dots, -\text{sign}(T(W_{2^n})))^t \\ &= M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1} (-T(W_1), \dots, -T(W_{2^n}))^t \\ &= -T. \end{aligned}$$

Note that if $S(W_j) = \text{sign}(T(W_j))$ ($1 \leq j \leq 2^n$), then $S = T$. By Theorem 2.4, T is exposed. ■

3. THE EXTREME AND EXPOSED POINTS OF THE UNIT BALL OF $\mathcal{L}_s(^n l_\infty^2)$

Let $n \geq 2$ and

$$\begin{aligned} \mathcal{U}_n := \{ & [(1, 1), (1, 1), \dots, (1, 1)], [(1, -1), (1, 1), \dots, (1, 1)], \\ & [(1, -1), (1, -1), (1, 1), \dots, (1, 1)], \\ & [(1, -1), (1, -1), (1, -1), (1, 1), \dots, (1, 1)], \\ & \dots, [(1, -1), (1, -1), \dots, (1, -1), (1, 1)], \\ & [(1, -1), (1, -1), \dots, (1, -1), (1, -1)] \}. \end{aligned}$$

Note that \mathcal{U}_n has $n + 1$ elements in $S_{l_\infty^2} \times \dots \times S_{l_\infty^2}$.

THEOREM 3.1. *Let $n \geq 2$ and $T \in \mathcal{L}_s(^n l_\infty^2)$. Then,*

$$\|T\| = \sup_{U \in \mathcal{U}_n} |T(U)|.$$

Proof. It follows that from Theorem 2.1 and symmetry of T . \blacksquare

For $j = 0, \dots, n$, we let

$$F_j = \sum_{\{l_1, \dots, l_j, k_1, \dots, k_{n-j}\} = \{1, \dots, n\}} x_{l_1} \cdots x_{l_j} y_{k_1} \cdots y_{k_{n-j}}.$$

Then, $\{F_0, \dots, F_n\}$ is a basis for $\mathcal{L}_s(^n l_\infty^2)$. Hence, $\dim(\mathcal{L}_s(^n l_\infty^2)) = n + 1$. If $T \in \mathcal{L}_s(^n l_\infty^2)$, then,

$$T = \sum_{j=0}^n b_j F_j$$

for some $b_0, \dots, b_n \in \mathbb{R}$. By simplicity, we denote $T = (b_0, \dots, b_n)^t$. For $j = 0, \dots, n$, we let

$$U_j = [(1, u_1), \dots, (1, u_n)] \in \mathcal{U}_n,$$

where $u_k = -1$ for $1 \leq k \leq j$ and $u_k = 1$ for $j + 1 \leq k \leq n$. Let

$$M(F_0, \dots, F_n; U_0, \dots, U_n) = [F_i(U_j)]$$

be the $(n + 1) \times (n + 1)$ matrix. Note that, for every $T \in \mathcal{L}_s(^n l_\infty^2)$,

$$M(F_0, \dots, F_n; U_0, \dots, U_n)T = (T(U_0), \dots, T(U_n))^t.$$

By analogous arguments in the claim 1 of Theorem 2.2, $M(F_0, \dots, F_n; U_0, \dots, U_n)$ is invertible.

THEOREM 3.2. *Let $n \geq 2$. Then,*

$$\begin{aligned} \text{ext } B_{\mathcal{L}_s(nl_\infty^2)} &= \{M(F_0, \dots, F_n; U_0, \dots, U_n)^{-1}(\epsilon_0, \dots, \epsilon_n)^t \\ &\quad : \epsilon_j = \pm 1, j = 0, \dots, n\}. \end{aligned}$$

Proof. It follows by Theorem 3.1 and analogous arguments in the claims 2 and 3 of Theorem 2.2. ■

Notice that using Wolfram Mathematica 8 and Theorem 3.2, we can exclusively describe $\text{ext } B_{\mathcal{L}_s(nl_\infty^2)}$ for a given $n \geq 2$.

For every $T \in \mathcal{L}_s(nl_\infty^2)$, we let

$$\text{Norm}(T) := \{[(1, u_1), \dots, (1, u_n)] \in \mathcal{U}_n : |T((1, u_1), \dots, (1, u_n))| = \|T\|\}.$$

We call $\text{Norm}(T)$ the set of *the norming points* of T .

COROLLARY 3.3. (a) *Let $n \geq 2$. $\text{ext } B_{\mathcal{L}_s(nl_\infty^2)}$ has exactly 2^{n+1} elements.*

(b) *Let $n \geq 2$ and $T \in \mathcal{L}_s(nl_\infty^2)$ with $\|T\| = 1$. Then $T \in \text{ext } B_{\mathcal{L}_s(nl_\infty^2)}$ if and only if $\text{Norm}(T) = \mathcal{U}_n$.*

THEOREM 3.4. *Let $n \geq 2$. Then, $\text{exp } B_{\mathcal{L}_s(nl_\infty^2)} = \text{ext } B_{\mathcal{L}_s(nl_\infty^2)}$.*

Proof. Let $T \in \text{ext } B_{\mathcal{L}_s(nl_\infty^2)}$ and let

$$f := \frac{1}{n+1} \sum_{0 \leq j \leq n} \text{sign}(T(U_j)) \delta_{U_j} \in \mathcal{L}_s(nl_\infty^2)^*.$$

Note that $1 = \|f\| = f(T)$. By analogous arguments in the proof of Theorem 2.5, f exposes T . Therefore, T is exposed. ■

QUESTIONS. (a) Let $n \geq 2$ and $\epsilon_1, \dots, \epsilon_{2^n}$ be fixed with $\epsilon_j = \pm 1$, ($j = 1, \dots, 2^n$). Is it true that

$$\begin{aligned} \text{ext } B_{\mathcal{L}(nl_\infty^2)} &= \{M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1}(\epsilon_1, \dots, \epsilon_{2^n})^t \\ &\quad : Z_1, \dots, Z_{2^n}, W_1, \dots, W_{2^n} \text{ are any ordering}\}? \end{aligned}$$

(b) By Theorem 2.2, $M(Z_1, \dots, Z_{2^n}; W_1, \dots, W_{2^n})^{-1}(\epsilon_1, \dots, \epsilon_{2^n})^t$ is extreme if $Z_1, \dots, Z_{2^n}, W_1, \dots, W_{2^n}$ are any ordering. Similarly, we may ask the following: Let $n \geq 2$ and $\delta_0, \dots, \delta_n$ be fixed with $\delta_k = \pm 1$, ($k = 0, \dots, n$). Is it true that

$$\begin{aligned} \text{ext } B_{\mathcal{L}_s(nl_\infty^2)} &= \{M(F_0, \dots, F_n; U_0, \dots, U_n)^{-1}(\delta_0, \dots, \delta_n)^t \\ &\quad : F_0, \dots, F_n, U_0, \dots, U_n \text{ are any ordering}\}? \end{aligned}$$

ACKNOWLEDGEMENTS

The author is thankful to the referee for the careful reading and considered suggestions leading to a better presented paper.

REFERENCES

- [1] S.G. KIM, The unit ball of $\mathcal{L}_s(^2 l_\infty^2)$, *Extracta Math.* **24** (2009), 17–29.
- [2] S.G. KIM, The unit ball of $\mathcal{L}_s(^2 d_*(1, w)^2)$, *Kyungpook Math. J.* **53** (2013), 295–306.
- [3] S.G. KIM, Extreme bilinear forms of $\mathcal{L}(^2 d_*(1, w)^2)$, *Kyungpook Math. J.* **53** (2013), 625–638.
- [4] S.G. KIM, Exposed symmetric bilinear forms of $\mathcal{L}_s(^2 d_*(1, w)^2)$, *Kyungpook Math. J.* **54** (2014), 341–347.
- [5] S.G. KIM, Exposed bilinear forms of $\mathcal{L}(^2 d_*(1, w)^2)$, *Kyungpook Math. J.* **55** (2015), 119–126.
- [6] S.G. KIM, The unit ball of $\mathcal{L}(^2 \mathbb{R}_{h(w)}^2)$, *Bull. Korean Math. Soc.* **54** (2017), 417–428.
- [7] S.G. KIM, Extremal problems for $\mathcal{L}_s(^2 \mathbb{R}_{h(w)}^2)$, *Kyungpook Math. J.* **57** (2017), 223–232.
- [8] S.G. KIM, The unit ball of $\mathcal{L}_s(^2 l_\infty^3)$, *Comment. Math.* **57** (2017), 1–7.
- [9] S.G. KIM, The geometry of $\mathcal{L}_s(^3 l_\infty^2)$, *Commun. Korean Math. Soc.* **32** (2017), 991–997.
- [10] S.G. KIM, The geometry of $\mathcal{L}(^3 l_\infty^2)$ and optimal constants in the Bohnenblust-Hill inequality for multilinear forms and polynomials, *Extracta Math.* **33**(1) (2018), 51–66.
- [11] S.G. KIM, Extreme bilinear forms on \mathbb{R}^n with the supremum norm, *Period. Math. Hungar.* **77** (2018), 274–290.
- [12] S.G. KIM, The unit ball of the space of bilinear forms on \mathbb{R}^3 with the supremum norm, *Commun. Korean Math. Soc.* **34**(2) (2019), 487–494.
- [13] M.G. KREIN, D.P. MILMAN, On extreme points of regular convex sets, *Studia Math.* **9** (1940), 133–137.