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On angular localization of spectra of perturbed operators

M.I. Gil'

Department of Mathematics, Ben Gurion University of the Negev
P.O. Box 653, *Beer-Sheva* 84105, *Israel*

gilmi@bezeqint.net

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Presented by Manuel González

Abstract: Let A and $\tilde{A}$ be bounded operators in a Hilbert space. We consider the following problem: let the spectrum of A lie in some angular sector. In what sector the spectrum of $\tilde{A}$ lies if A and $\tilde{A}$ are "close"? Applications of the obtained results to integral operators are also discussed.

Key words: Operators, spectrum, angular location, perturbations, integral operator.

AMS *Subject Class.* (2010): 47A10, 47A55, 47B10.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a complex separable Hilbert space with a scalar product $(.\,,.)$, the norm $\|.\| = \sqrt{(.\,,.)}$ and unit operator I. By $\mathcal{B}(\mathcal{H})$ we denote the set of bounded operators in $\mathcal{H}$. For an $A \in \mathcal{B}(\mathcal{H})$, A^* is the adjoint operator, $\|A\|$ is the operator norm and $\sigma(A)$ is the spectrum.

We consider the following problem: let A and $\tilde{A}$ be "close" operators and $\sigma(A)$ lie in some angular sector. In what sector $\sigma(\tilde{A})$ lies?

Not too much works are devoted to the angular localizations of spectra. The papers [5, 6, 7, 8] should be mentioned. In particular, in the papers by E.I. Jury, N.K. Bose and B.D.O. Anderson [5, 6] it is shown that the test to determine whether all eigenvalues of a complex matrix of order n lie in a certain sector can be replaced by an equivalent test to find whether all eigenvalues of a real matrix of order $4n$ lie in the left half plane. The results from [5] have been applied by G.H. Hostetter [4] to obtain an improved test for the zeros of a polynomial in a sector. In [7] M.G. Krein announces two theorems concerning the angular localization of the spectrum of a multiplicative operator integral. In the paper [8] G.V. Rozenblyum studies the asymptotic behavior of the distribution functions of eigenvalues that appear in a fixed angular region of the complex plane for operators that are close to normal. As applications, he calculates the asymptotic behavior of the spectrum of two classes of oper-

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ators: elliptic pseudo-differential operators acting on the sections of a vector bundle over a manifold with a boundary, and operators of elliptic boundary value problems for pseudo-differential operators. It should be noted that in the just pointed papers the perturbations of an operator whose spectrum lie in a given sector are not considered. Below we give bounds for the spectral sector of a perturbed operator.

Without loss of the generality it is assumed that

$$\beta(A) := \inf \operatorname{Re} \sigma(A) > 0. \tag{1.1}$$

If this condition does not hold, instead of A we can consider perturbations of the operator $A_1 = A + Ic$ with a constant $c > |\beta(A)|$.

For a $Y \in \mathcal{B}(\mathcal{H})$ we write $Y > 0$ if Y is positive definite, i.e., $\inf_{x\in\mathcal{H},\|x\|=1} (Yx, x) > 0$. Let $Y > 0$. Define the *angular Y-characteristic* $\tau(A, Y)$ of A by

$$\cos \tau(A, Y) := \inf_{x\in\mathcal{H},\|x\|=1} \frac{\operatorname{Re}(YAx, x)}{|(YAx, x)|}.$$

The set

$$S(A, Y) := \{z \in \mathbb{C} : |\arg z| \leq \tau(A, Y)\}$$

will be called the *Y-spectral sector of A*.

LEMMA 1.1. *For an $A \in \mathcal{B}(\mathcal{H})$, let condition* (1.1) *hold and Y be a positive definite operator, such that $(YA)^* + YA > 0$. Then $\sigma(A)$ lies in the Y-spectral sector of A.*

Proof. Take a ray $z = re^{it}$ $(0 < r < \infty)$ intersecting $\sigma(A)$, and take the point $z_0 = r_0e^{it}$ on it with the maximum modulus. By the theorem on the boundary point of the spectrum [1, Section I.4.3, p. 28] there exists a normed sequence $\{x_n\}$, such that $Ax_n - z_0x_n \to 0$, $(n \to \infty)$. Hence,

$$\frac{\operatorname{Re}(YAx_n, x_n)}{|(YAx_n, x_n)|} = \frac{\operatorname{Re} r_0e^{it}(Yx_n, x_n)}{r_0|(Yx_n, x_n)|} + \epsilon_n = \cos t + \epsilon_n$$

with $\epsilon_n \to 0$ as $n \to \infty$. So z_0 is in $S(A, Y)$. This proves the lemma. ∎

EXAMPLE 1.2. Let $A = A^* > 0$. Then condition (1.1) holds. For any $Y > 0$ commuting with A (for example $Y = I$) we have $(YA)^* + YA = 2YA$ and $\operatorname{Re}(YAx, x) = |(YAx, x)|$. Thus $\cos \tau(A, Y) = 1$ and $S(A, Y) = \{z \in \mathbb{C} : \arg z = 0\}$.

So Lemma 1.1 is sharp.

Remark 1.3. Suppose A has a bounded inverse. Recall that the quantity $\operatorname{dev}(A)$ defined by

$$\cos \operatorname{dev}(A) := \inf_{x\in\mathcal{H}, x\neq 0} \frac{\operatorname{Re}(Ax,x)}{\|Ax\|\|x\|}$$

is called the angular deviation of A, cf. [1, Chapter 1, Exercise 32]. For example, for a positive definite operator A one has

$$\cos \operatorname{dev}(A) = \frac{2\sqrt{\lambda_M \lambda_m}}{\lambda_M + \lambda_m},$$

where λ_m and λ_M are the minimum and maximum of the spectrum of A, respectively (see [1, Chapter 1, Exercise 33]). Besides, in Exercise 32 it is pointed that the spectrum of A lies in the sector $|\arg z| \leq \operatorname{dev}(A)$. Since $|(Ax,x)| \leq \|Ax\|\|x\|$, Lemma 1.1 refines the just pointed assertion.

2. The main result

Let A be a bounded linear operator in $\mathcal{H}$, whose spectrum lies in the open right half-plane. Then by the Lyapunov theorem, cf. [1, Theorem I.5.1], there exists a positive definite operator $X \in \mathcal{B}(\mathcal{H})$ solving the Lyapunov equation

$$2\operatorname{Re}(AX) = XA + A^*X = 2I. \tag{2.1}$$

So $\operatorname{Re}(XAx,x) = ((XA + A^*X)x,x)/2 = (x,x)$ $(x \in \mathcal{H})$ and

$$\cos\tau(A,X) = \inf_{x\in\mathcal{H},\|x\|=1} \frac{(x,x)}{|(XAx,x)|} = \frac{1}{\sup_{x\in\mathcal{H},\|x\|=1}|(XAx,x)|} \geq \frac{1}{\|AX\|}.$$

Put

$$J(A) = 2\int_0^\infty \|e^{-At}\|^2 dt.$$

Now we are in a position to formulate our main result.

Theorem 2.1. *Let* $A, \tilde{A} \in \mathcal{B}(\mathcal{H})$, *condition* (1.1) *hold and* X *be a solution of* (2.1). *Then with the notation* $q = \|A - \tilde{A}\|$ *one has*

$$\cos\tau(\tilde{A},X) \geq \cos\tau(A,X)\frac{(1-qJ(A))}{(1+qJ(A))},$$

provided

$$qJ(A) < 1.$$

The proof of this theorem is based on the following lemma.

LEMMA 2.2. *Let* $A, \tilde{A} \in \mathcal{B}(\mathcal{H})$, *condition* (1.1) *hold and* X *be a solution of* (2.1). *If, in addition,*

$$q\|X\| < 1, \tag{2.2}$$

then

$$\cos\tau(\tilde{A}, X) \geq \cos\tau(A, X)\frac{(1 - \|X\|q)}{(1 + \|X\|q)}.$$

Proof. Put $E = \tilde{A} - A$. Then $q = \|E\|$ and due to (2.1), with $x \in \mathcal{H}$, $\|x\| = 1$, we obtain

$$\begin{aligned} \operatorname{Re}(X(A+E)x, x) &\geq \operatorname{Re}(XAx, x) - |(XEx, x)| \\ &= (x, x) - |(XEx, x)| \\ &\geq (x, x) - \|X\|\|E\|\|x\|^2 = 1 - \|X\|q. \end{aligned} \tag{2.3}$$

In addition,

$$\begin{aligned} |(X(A+E)x, x)| &\leq |(XAx, x)| + \|X\|\|E\|\|x\|^2 \\ &= |(XAx, x)|\left(1 + \frac{\|X\|q}{|(XAx, x)|}\right) \quad (\|x\| = 1). \end{aligned}$$

But

$$|(XAx, x)| \geq |\operatorname{Re}(XAx, x)| = \operatorname{Re}(XAx, x) = (x, x) = 1.$$

Hence

$$|(X(A+E)x, x)| \leq |(XAx, x)|\left(1 + \frac{\|X\|q}{\operatorname{Re}(XAx, x)}\right) \leq |(XAx, x)|(1 + \|X\|q).$$

Now (2.3) yields.

$$\frac{\operatorname{Re}(X\tilde{A}x, x)}{|(X\tilde{A}x, x)|} \geq \frac{(1 - \|X\|q)}{|(XAx, x)|(1 + \|X\|q)} \quad (\|x\| = 1),$$

provided (2.2) holds. Since

$$\cos\tau(\tilde{A}, X) = \inf_{x \in \mathcal{B}, \|x\|=1} \frac{\operatorname{Re}(X\tilde{A}x, x)}{|(X\tilde{A}x, x)|},$$

we arrive at the required result. ∎

Proof of Theorem 2.1 Note that X is representable as

$$X = 2\int_0^\infty e^{-A^*t}e^{-At}dt$$

[1, Section 1.5]. Hence, we easily have $\|X\| \le J(A)$. Now the latter lemma proves the theorem. ∎

3. Operators with Hilbert-Schmidt Hermitian components

In this section we obtain an estimate for $J(A)$ ($A \in \mathcal{B}(\mathcal{H})$) assuming that $A \in \mathcal{B}(\mathcal{H})$ and

$$A_I := (A - A^*)/i \quad \text{is a Hilbert-Schmidt operator,} \tag{3.1}$$

i.e., $N_2(A_I) := (\text{trace}(A_I^2))^{1/2} < \infty$. Numerous integral operators satisfy this condition. Introduce the quantity (the departure from normality)

$$g_I(A) := \left[2N_2^2(A_I) - 2\sum_{k=1}^\infty |\operatorname{Im}\lambda_k(A)|^2\right]^{1/2} \le \sqrt{2}N_2(A_I),$$

where $\lambda_k(A)$ ($k = 1, 2, \ldots$) are the eigenvalues of A taken with their multiplicities and ordered as $|\operatorname{Im}\lambda_{k+1}(A)| \le |\operatorname{Im}\lambda_k(A)|$. If A is normal, then $g_I(A) = 0$, cf. [2, Lemma 9.3].

Lemma 3.1. *Let conditions* (1.1) *and* (3.1) *hold. Then* $J(A) \le \hat{J}(A)$, *where*

$$\hat{J}(A) := \sum_{j,k=0}^\infty \frac{g_I^{j+k}(A)(k+j)!}{2^{j+k}\beta^{j+k+1}(A)(j!\,k!)^{3/2}}.$$

Proof. By [2, Theorem 10.1] we have

$$\|e^{-At}\| \le \exp\big[-\beta(A)t\big]\sum_{k=0}^\infty \frac{g_I^k(A)t^k}{(k!)^{3/2}} \quad (t \ge 0).$$

Then

$$J(A) \le 2\int_0^\infty \exp[-2\beta(A)t]\left(\sum_{k=0}^\infty \frac{g_I^k(A)t^k}{(k!)^{3/2}}\right)^2 dt$$
$$= 2\int_0^\infty \exp[-2\beta(A)t]\left(\sum_{j,k=0}^\infty \frac{g_I^{k+j}(A)t^{k+j}}{(j!k!)^{3/2}}\right) dt$$
$$= \sum_{j,k=0}^\infty \frac{2(k+j)!g_I^{j+k}(A)}{(2\beta(A))^{j+k+1}(j!\ k!)^{3/2}},$$

as claimed. ∎

If A is normal, then $g_I(A) = 0$ and with $0^0 = 1$ we have $\hat{J}(A) = \frac{1}{\beta(A)}$. The latter lemma and Theorem 2.1 imply

Corollary 3.2. *Let $A, \tilde{A} \in \mathcal{B}(\mathcal{H})$ and let the conditions* (1.1), (3.1) *and $q\hat{J}(A) < 1$ hold. Then*

$$\cos\tau(\tilde{A}, X) \ge \frac{(1 - q\hat{J}(A))}{(1 + q\hat{J}(A))}\cos\tau(A, X).$$

4. Integral operators

As usually $L^2 = L^2(0,1)$ is the space of scalar-valued functions h defined on $[0,1]$ and equipped with the norm

$$\|h\| = \left[\int_0^1 |h(x)|^2 dx\right]^{1/2}.$$

Consider in $L^2(0,1)$ the operator $\tilde{A}$ defined by

$$(\tilde{A}h)(x) = a(x)h(x) + \int_0^1 k(x,s)h(s)ds \quad (h \in L^2, x \in [0,1]), \tag{4.1}$$

where $a(x)$ is a real bounded measurable function with

$$a_0 := \inf a(x) > 0, \tag{4.2}$$

and $k(x,s)$ is a scalar kernel defined on $0 \le x, s \le 1$, and

$$\int_0^1 \int_0^1 |k(x,s)|^2 ds\, dx < \infty. \tag{4.3}$$

So the Volterra operator V defined by

$$(Vh)(x) = \int_x^1 k(x,s)h(s)ds \quad (h \in L^2, x \in [0,1]),$$

is a Hilbert-Schmidt one. Define operator A by

$$(Ah)(x) = a(x)h(x) + \int_x^1 k(x,s)h(s)ds \quad (h \in L^2, x \in [0,1]).$$

Then $A = D + V$, where D is defined by $(Dh)(x) = a(x)h(x)$. Due to Lemma 7.1 and Corollary 3.5 from [3] we have $\sigma(A) = \sigma(D)$. So $\sigma(A)$ is real and $\beta(A) = a_0$. Moreover,

$$N_2(A_I) = N_2(V_I) \le N_2(V) = \left[\int_0^1 \int_x^1 |k(x,s)|^2 ds\, dx\right]^{1/2}.$$

Here $V_I = (V - V^*)/2i$. Thus,

$$g_I(A) \le g_V := \sqrt{2} N_2(V)$$

and

$$\|A - \tilde{A}\| \le q_0 := \left[\int_0^1 \int_0^x |k(x,s)|^2 ds\, dx\right]^{1/2}.$$

Simple calculations show that under consideration

$$\hat{J}(A) \le \hat{J}_0 := \sum_{j,k=0}^{\infty} \frac{g_V^{j+k}(k+j)!}{2^{j+k} a_0^{j+k+1} (j!\ k!)^{3/2}}.$$

Making use of Corollary 3.2 and taking into account that in the considered case $\cos\tau(A,X) = 1$, we arrive at the following result.

Corollary 4.1. *Let $\tilde{A}$ be defined by* (4.1) *and the conditions* (4.2) *and* (4.3) *hold. If, in addition, $q_0\hat{J}_0 < 1$, then $\sigma(\tilde{A})$ lies in the angular sector*

$$\left\{ z \in \mathbb{C} \,:\, |\arg z| \le \arccos \frac{(1 - q_0\hat{J}_0)}{(1 + q_0\hat{J}_0)} \right\}.$$

Example 4.2. To estimate the sharpness of our results consider in $L^2(0,1)$ the operators

$$(Ah)(x) = 2h(x) \quad \text{and} \quad (\tilde{A}h)(x) = (2+i)h(x) \quad (h \in L^2, x \in [0,1]).$$

$\sigma(A)$ consists of the unique point $\lambda = 2$ and so $\cos(A, X) = \cos\arg\lambda = 1$. We have

$$J(A) = 2\int_0^\infty e^{-4t}dt = 1/2 \quad \text{and} \quad q = 1.$$

By Corollary 3.2

$$\cos\tau(\tilde{A}, X) \geq \frac{1 - 1/2}{1 + 1/2} = 1/3.$$

Compare this inequality with the sharp result: $\sigma(\tilde{A})$ consists of the unique point $\tilde{\lambda} = 2 + i$. So $\tan(\arg\tilde{\lambda}) = 1/2$, and therefore $\cos(\arg\tilde{\lambda}) = 2/(\sqrt{5})$.

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