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About a Characterization of Dodds-Fremlin Regarding Positive Compact Operators

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Abstract: We give sufficient and necessary conditions, different from that of Dodds-Fremlin, which characterize compact operators between Banach lattices, relying on semi-compact and AM-compact operators.

Key words: compact operator, AM-compact operator, semi-compact operator, order continuous norm, discrete Banach lattice.

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This note extends some of the results in [2]. Throughout this note, E and F will be Banach lattices and X will be a Banach space. We refer the reader to [1, 3] for definitions and notations related to Banach lattices. Recall that an operator $T : E \to X$ is *AM-compact* if it maps order intervals into relatively compact sets; an operator $T : X \to F$ is *semi-compact* if $T(B_X)$ is almost order bounded, i.e., for every $\varepsilon > 0$ there exists $u \in F^+$ such that $T(B_X) \subset [-u, u] + \varepsilon B_F$. It is clear that the both properties are weaker than compactness.

Remark 1. Suppose that F is order continuous and $0 \leq T : E \to F$. It is easy to see that if T is semi-compact and T' is AM-compact then T is compact. Indeed, T is L-weakly compact by [3, Proposition 3.6.2], so that T' is M-weakly compact by [3, Proposition 3.6.11], hence T' (and, therefore, T) is compact by [3, Proposition 3.7.4] (see [2, Theorem 1(i)] for another proof).

Theorem 2. *If every positive operator $T : E \to F$ is compact whenever it is semi-compact and T' is AM-compact then one of the following assertions holds:*

1. *F is order continuous, or*
2. *E' is order continuous and discrete.*

Proof. Suppose that F is not order continuous. It suffices to show that if E' is not order continuous or not discrete then there exists a positive non-compact operator $T : E \to F$ such that T is semi-compact and T' is AM-compact.

Suppose that E' is not order continuous. By [3, Theorem 2.4.14], E contains a lattice copy of ℓ_1, i.e., there is a closed sublattice $Y \subseteq E$ and a surjective lattice isomorphism $V : Y \to \ell_1$. Put $u_n = V^{-1}(e_n)$. By [3, Proposition 2.3.11], there exists a positive projection $P : E \to E$ with Range $P = Y$. Let $j : \ell_1 \to c_0$ be the normal inclusion. [3, Theorem 2.4.2] guarantees that there exist $y \in F^+$ and a disjoint normalized sequence (y_n) in $[0, y]$. Define $S : c_0 \to F$ via $S(e_n) = y_n$. Since for every $x = \sum_{i=1}^{\infty} \alpha_i e_i \in c_0$ we have

$$\left| S\left(\sum_{i=1}^{n} \alpha_i e_i\right) \right| = \sum_{i=1}^{n} |\alpha_i| \, y_i \le (\max_{i=1,\dots,n} |\alpha_i|) y \le \|x\|_\infty \, y$$

for all $n \ge 1$, it follows that $\|S(x)\| \le \|y\| \, \|x\|_\infty$, so that S is indeed a bounded operator from c_0 to F.

Put $T = SjVP$. Then T is not compact as $T(u_n) = y_n$ which has no convergent subsequences. To see that T is semi-compact, observe that jVP maps B_E into $M \cdot B_{c_0}$ where $M = \|jVP\|$, so that for each $x \in B_E$ we have $jVP(x) = \sum_{i=1}^{\infty} \alpha_i e_i$ with $|\alpha_i| \le M$ for all i. It follows that

$$|T(x)| = \left| \sum_{i=1}^{\infty} \alpha_i y_i \right| \le \sum_{i=1}^{\infty} |\alpha_i| \, y_i \le My.$$

Then $T(B_E) \subseteq M[-y, y]$ and so T is semi-compact. Finally, T' is AM-compact because $S' : F' \to \ell_1$ is positive and order intervals in ℓ_1 are compact. This proves that E' is order continuous.

We will now show that E' is discrete. Suppose not. Then it follows from [4, Theorem 1] that there exist two operators $0 \le S \le T : E \to F$ such that T is compact while S is not. However, S is semi-compact by [1, Theorem 5.72(b)] and S' is AM-compact by [3, Proposition 3.7.2].

Example 3. The converse is false. Indeed, let $T : \ell_2 \to \ell_\infty$ be the natural embedding. Note that $\ell_2' = \ell_2$ is discrete and order continuous; T is semi-compact and T' is AM-compact, but, nevertheless, T is not compact.

Theorem 4. *The following assertions are equivalent:*

1. *Every positive operator $T : E \to F$ is compact whenever it is AM-compact and T' is semi-compact;*

2. *F is finite-dimensional or E' is order continuous.*

Proof. (1) $\Rightarrow$ (2) Suppose that E' is not order continuous and $\dim F = \infty$. Let P, V, and (u_n) be as in the proof of Theorem 2. Since $\dim F = \infty$, it follows from $B_F \subseteq (B_F)^+ - (B_F)^+$ that $(B_F)^+$ is not compact; fix a sequence (y_n) in $(B_F)^+$ with no convergent subsequences. Define $S : \ell_1 \to F$ via $S(e_n) = y_n$; S is bounded because $\|S(\sum_{i=1}^{\infty} \alpha_i e_i)\| \leq \sum_{i=1}^{\infty} |\alpha_i|$. Put $T = SVP$. Since order intervals in ℓ_1 are compact, the operator VP is AM-compact, so that T is AM-compact. Being an operator on ℓ_∞, $(VP)'$ is semi-compact, so that T' is semi-compact. However, $T(u_n) = y_n$, so that T is not compact.

(2) $\Rightarrow$ (1) If F is finite-dimensional, (1) holds trivially, while if E' is order continuous then the proof is analogous to Remark 1.

Remark 5. A quick glance at the proof reveals that we can replace a Banach lattice F with a Banach space X in Theorem 4 as long as we remove the word “positive” from (1).

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