EXTRACTA MATHEMATICAE Vol. **27**, Núm. 1, 31 – 58 (2012)

Cyclicity Results for Some Antianalytic Toeplitz Operators Acting on H^p

Gilles Cassier, Réda Choukrallah

Institut Camille Jordan, CNRS UMR 5208*, Université de Lyon, Université Lyon* 1*,*
43 *boulevard du* 11 *Novembre* 1918*,* 69622 *Villeurbanne, France*
cassier@math.univ-lyon1.fr

College St-Jean-sur-Richelieu, 30 *boulevard du Séminaire Nord,*
C.P. 1018 *St-Jean-sur-Richelieu, Québec* J3B 5J4*, Canada*
reda.choukrallah@cstjean.qc.ca

Presented by Alfonso Montes Received October 27, 2010

Abstract: This article deals with some cyclic families of functions for antianalytic Toeplitz operators whose symbol is a finite Blaschke product in the spaces H^p where $1 < p < \infty$. We give a description of the invariant subspaces for this type of operators generated by special decompositions and by lacunary decompositions of functions. To that end, we study some particular decomposition properties of a function in H^p associated with an inner function which are valid in general.

Key words: Cyclicity, Hardy spaces, Toepliz operators, Blaschke products.

AMS *Subject Class.* (2010): 47A16, 47B35, 47B38, 30H10, 47A15.

Introduction

The Hardy spaces H^p, $1 \le p < \infty$ are the spaces of functions f with values in $\mathbb{C}$ and which are analytic in the open unit disk $\mathbb{D} = \{\zeta : \zeta \in \mathbb{C}\,, |\zeta| < 1\}$ and such that

$$\|f\|_p^p := \sup_{0 \le r < 1} \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta) < \infty\,,$$

where m is the normalized Lebesgue measure on the unit circle $\mathbb{T} = \{\zeta : |\zeta| = 1\}$. As usual, H^∞ is the space of bounded analytic functions on $\mathbb{D}$. For $1 < p < +\infty$, the dual of H^p will be identified by means of the standard isomorphism with H^q $(1/p + 1/q = 1)$ under the usual integral paring

$$\langle f \mid g \rangle = \int_0^{2\pi} f(e^{i\theta})\overline{g\,(e^{i\theta})} dm(\theta), \quad (f,\, g) \in H^p \times H^q.$$

A function $u \in H^p$ is an inner function if $u \in H^\infty$ and if $|u| = 1$ a.e. on $\mathbb{T}$. For every inner function, the subspace uH^p is a closed complemented subspace of

H^p. A canonical supplementary subspace of uH^p is $K_u^p = \{f - uP_+(\overline{u}f); f \in H^p\}$ (see [18]) where P_+ is the Riesz projection of L^p on H^p $(p > 1)$ and the bar stands for the complex conjugate. In the case $p = 2$, this direct sum is an orthogonal direct sum. Moreover, we have $K_u^p = H^p \cap (\overline{uH_0^p})$ when $1 < p < \infty$, where H_0^p is the subspace of functions in H^p which vanish on 0. For this reason, we define the subspace K_u^∞ of H^∞ by setting $K_u^\infty = H^\infty \cap (\overline{uH_0^\infty})$.

A function of the form

$$b = \prod_{k \geq 1} b_{\lambda_k}, \text{ where } b_{\lambda_k} = \frac{\overline{\lambda_k}}{|\lambda_k|} \frac{\lambda_k - z}{1 - \overline{\lambda_k} z}$$

and the $\lambda_k \in \mathbb{D}$ satisfy the Blaschke condition $\sum_{k \geq 1}(1 - |\lambda_k|) < \infty$, is called a Blaschke product. The set $\{\lambda_k\}$ may be finite (b is a finite Blaschke product). It is understood that $b_{\lambda_k}(z) = z$ when $\lambda_k = 0$.

For $u \in L^\infty(\mathbb{T})$, the Toepliz operator T_u with symbol u is defined via the multiplication operator u, followed by the Riesz projection, that is $T_u f = P_+(uf)$ for $f \in H^p$. The Toeplitz operator is called anti-analytic (or co-analytic) when the symbol is anti-analytic. In what follows, when the inner function u is fixed, we write sometimes only T instead of $T_{\overline{u}}$ for shortness. Given a function $f \in H^p$, we consider the smallest closed subspace invariant under T_u and containing f which is $E_f^p \stackrel{def}{=} span(T_u^n f : n \geq 0)$. A function $f \in H^p$ is called cyclic for T if $E_f^p = H^p$.

A sequence of positive integers n_k is said to be Hadamard-lacunary, or simply lacunary, if there exists a constant $d > 1$ such that $n_{k+1} > dn_k$ for all k. A power series $\sum a_k z^{n_k}$ is a lacunary power series if the sequence n_k is lacunary. An interesting arithmetic property satisfied by these sequences, which will be used later, is that if $(n_k)_{k \geq 1}$ is a lacunary sequence, then we have

$$\sup_{m \geq 1} \text{card} \left\{(j, k) \in \mathbb{N}^* : N = n_j - n_k\right\} < \infty,$$

which means in others words that there is a number M such that no positive integer N has more than M representations of type $N = n_j - n_k$ (see [8] p. 52-53). A lacunary decomposition in H^p $(1 \leq p < \infty)$ associated with an inner function u is a function $f \in H^p$ which can be written in the form

$$f(z) = \sum_{k \geq 0} f_k(z) u^{n_k}(z) \text{ such that } \frac{n_{k+1}}{n_k} \geq d > 1 \text{ for all } k \geq 1,$$

(where the series is norm convergent in H^p, $f_k \in K_u^p$ and d is a constant independent of k).

Let u be an inner function, recall that φ is a u-p-inner function if $\|\varphi\|_p = 1$ and

$$\int_0^{2\pi} \left|\varphi\left(e^{i\theta}\right)\right|^p u\left(e^{i\theta}\right)^k dm(\theta) = 0$$

for any positive integer k (see [13]). Observe that, in case $u(z) = z$, u-p-inner functions are classical inner functions. When b is a finite Blaschke product, we have the following result describing the subspaces which are invariant under the multiplication by b in H^p ([13]).

THEOREM 1. (T. L. Lance and M. I. Stessin) *If b is a finite Blaschke product of order n and $p \geq 1$ then any b invariant subspace M is of the form*

$$M = \sum_{i=1}^{k} h_{i,p}(z) H^p \circ b = \Big\{ \sum_{i=1}^{k} h_{i,p}(z) \varphi \circ b : \varphi \in H^p \Big\}$$

where the function $h_{i,p}$ are b-p-inner, $i = 1, \dots, k$ and $k \leq n$.

The cyclicity of lacunary series was studied for the Backward Shift by many authors, R. G. Douglas, H. S. Shapiro, A. L. Shields in their famous article on the cyclicity of the Backward Shift, E. Abakumov in his paper on the cyclicity of lacunary series on the spaces ℓ_p (see [1]) and A. B. Aleksandrov who gives some very interesting results of cyclicity. For example series with frequency spectrum in $\Lambda(1)$ sets which are more general sets including lacunary or Sidon sets (see [2], [3]).

One of the principal ideas of this work is to study the cyclicity of some anti-analytic Toeplitz operator on Hardy spaces H^p $(p > 1)$ in regards with some decompositions related with an inner function. We start by studying two kinds of such decompositions of a function f in H^p. Firstly, we decompose f with respect to the powers of an inner function u with coefficients in K_u^p (in the case $p = 2$, this decomposition can be obtained by using the Wold decomposition), and secondly we decompose f with respect to some particular unconditional basis of K_u^2 with coefficients in $H^p \circ u$. In both cases, the decomposition are valid for every inner function u. These decompositions can be used in other contexts and are of independent interest. This study comes out from the works of G. Cassier (cf. [6]) and R. Choukrallah (see [7]). We will give in what follows the construction in the spaces H^p to obtain a class of cyclic functions for the anti-analytic Toeplitz operators associated to a finite Blaschke product and we will describe in a precise way some invariant subspaces for these operators.

In what follows, this article is structured in three parts. In the first one, we study how to decompose a function in H^p according to a given inner function and we introduce some useful results for our approach. The second part is devoted to the study of cyclicity for some anti-analytic Toeplitz operators of some special decompositions and of some lacunary decompositions in H^p spaces when $2 < p < \infty$. The third part deals with an analogue study for some lacunary decompositions in H^p spaces when $1 < p < 2$. The main results in this paper are Theorems 4 and 6 in Subsection 2.2.

1. Decompositions of functions in H^p spaces according to a given inner function

The next Lemma will be very useful in what follows. The proof is based on the Riesz brothers Theorem (cf. [10]).

Lemma 1. *Let u an inner function. If $f \in K_u^p$, then the function $\tilde{f}$ given by $\tilde{f}(e^{i\theta}) = u(e^{i\theta})e^{-i\theta}\overline{f(e^{i\theta})}$ (where the bar stands for the complex conjugate) still belongs to the space K_u^p. Moreover, when $p = 2$ we have $K_u^2 = H^2 \ominus uH^2$.*

If $\lambda \in \mathbb{D}$ and $l \in \mathbb{N}$, we set

$$e_{\lambda,l} = \frac{l!z^l}{\left(1 - \overline{\lambda}z\right)^{l+1}}$$

For any function $f \in H^2$, we have $\langle f \mid e_{\lambda,l}\rangle = f^{(l)}(\lambda)$. That is why the function $e_\lambda = e_{\lambda,0}$ is called the reproducing kernel of H^2.

Lemma 2. *Let b a Blaschke product and let $(\lambda_k)_{k\geq 1}$ be the sequence of distinct zeros of b where each λ_k has an order of multiplicity d_k, then we have*

$$K_b^p = span\big(e_{\lambda_k,l} : k \geq 1, 0 \leq l \leq d_k - 1\big)$$

for all $p \geq 1$.

Notice that if b is a finite Blaschke product, then this subspace is finite dimensional and does not depend on p (Notation: $K_b^p = K_b$, for the proof see for example [17]).

Remark 1. Let $(\lambda_k)_{k\in I}$ $(I \subset \mathbb{N})$ be the sequence of zeros of a Blaschke product b with simple zeros, then $(e_{\lambda_k})_{k\in I}$ admits a biorthogonal sequence given by

$$e^*_{\lambda_k} = \frac{b_k(z)}{b_k(\lambda_k)}(1 - |\lambda_k|^2)e_{\lambda_k} \text{ where } b_k(z) = \prod_{j\in I,\, j\neq k} b_{\lambda_j}.$$

It is obvious that $e^*_{\lambda_k} \in K_b$ because for every $h \in H^p$,

$$\langle e^*_{\lambda_k} \mid bh\rangle = \frac{1-|\lambda_k|^2}{b_k(\lambda_k)}\Big\langle e_{\lambda_k} \mid \frac{\lambda_k - z}{1-\overline{\lambda}_k z}h\Big\rangle = 0.$$

Moreover, since we can apply Lemma 1, then we have

$$\tilde{e}_{\lambda_k} = be^{-i\theta}\overline{e}_{\lambda_k}(e^{-i\theta}) = -\frac{\lambda_k}{|\lambda_k|}b_k(e^{-i\theta})e_{\lambda_k}.$$

Therefore,

$$\tilde{e}_{\lambda_k} = -\frac{\lambda_k}{|\lambda_k|}\frac{b_k(\lambda_k)}{1-|\lambda_k|^2}e^*_{\lambda_k}.$$

And so $\mathbb{C}\tilde{e}_{\lambda_k} = \mathbb{C}e^*_{\lambda_k}$.

Proposition 1. *Let $p \geq 1$, $u \in H^p$ be an inner function and*

$$\mathcal{E}_u = span\left(\sum f_k u^k : \text{ the sums are finite and } f_k \in K_u^\infty\right).$$

Then $\mathcal{E}_u$ is dense in H^p.

Proof. Let $p > 1$, q be the conjugate of p and $\varphi \in H^q$ such that φ is orthogonal to $\mathcal{E}_u$. Then for every $\alpha \in \mathbb{D}$, we have

$$\int_0^{2\pi} \varphi(e^{it})\frac{1-u(\alpha)\overline{u}(e^{it})}{1-\alpha e^{-it}}\, dm(t) = 0.$$

For $p > 1$, we can write that $\varphi = \varphi_0 + u\varphi_1$ where $\varphi_0 \in K_u^q$ and then,

$$\varphi_0(\alpha) + \int_0^{2\pi} \varphi_1(e^{it})\frac{u(e^{it})-u(\alpha)}{1-\alpha e^{-it}}\, dm(t) = 0.$$

But, $\int_0^{2\pi}\varphi_1(e^{it})\frac{u(e^{it})-u(\alpha)}{1-\alpha e^{-it}}\, dm(t) = 0$, then $\varphi_0(\alpha) = 0$ for every $\alpha \in \mathbb{D}$ and therefore $\varphi_0 = 0$ and $\varphi = u\varphi_1$. By iterating the process, we prove that for every $n \geq 0$, there exists $\varphi_n \in H^q$ such that $\varphi = u^n\varphi_n$.
Let $z \in \mathbb{D}$ be fixed, and notice that $\varphi_n(z) = \frac{1}{2\pi}\int_0^{2\pi}\varphi_n(e^{it})\frac{1}{1-ze^{-it}}\, dm(t)$ and that $\|\varphi\|_q = \|\varphi_n\|_q$ for every $n \geq 0$. Then,

$$|\varphi(z)| \leq \|\varphi_n\|_q\|e_z\|_p|u(z)|^n \leq \|\varphi\|_q\|e_z\|_p|u(z)|^n.$$

And since $\lim_{n\to\infty}|u(z)|^n = 0$ then $\varphi = 0$. And so, $\overline{\mathcal{E}_u} = H^p$ for every $p > 1$.

If $p = 1$, let $f \in H^2 \subset H^1$, there exists a sequence $(g_n)_{n\geq 0} \subset \mathcal{E}_u$ such that $\lim_{n\to\infty}\|f-g_n\|_2 = 0$. By using the density of H^2 in H^1 and the fact that $\|f-g_n\|_1 \leq \|f-g_n\|_2$, we can approximate functions in H^1 by functions in $\mathcal{E}_u$. ∎

The following result shows that we can decompose any function $f \in H^p$ $(1 < p < \infty)$ relatively to the subspaces $K_u^p u^k$. In addition, we give explicit integral formulas for the component functions.

THEOREM 2. *Let u be an inner function and let f be in H^p $(p > 1)$. Then f can be uniquely decomposed with respect the powers of u under the form*

$$f = \sum_{k=0}^{+\infty} f_k u^k$$

where $f_k \in K_u^p$ and the series is norm convergent in H^p. Moreover, for any $z \in \mathbb{D}$ we have

$$u(z)^k f_k(z) = u(z)^k \int_0^{2\pi} \overline{u(e^{i\theta})}^k f(e^{i\theta}) \frac{1 - u(z)\overline{u(e^{i\theta})}}{1 - ze^{-i\theta}}\, dm(\theta) =: P_k(f)(z) \quad (1)$$

and

$$r_m(z) = u(z)^{m+1} \int_0^{2\pi} \frac{f(e^{i\theta})\overline{u(e^{i\theta})}^{m+1}}{1 - ze^{-i\theta}}\, dm(\theta). \quad (2)$$

Remark 2.

1) In the case $p = 2$, observe that the decomposition relatively to the subspaces $K_u^2 u^k$ is related to the Wold decomposition of the Toeplitz operators whose symbol is an inner function (see [16] for more precisions).
2) For $f \in H^1$, we still denote by $P_k(f)$ the function given by the first integral formula of Theorem 2.

Recall that T is defined in H^p in the following way $T(f) = P_+(\overline{u} f)$ and T has a right inverse R $(TR = I)$ defined by $Rf = uf$. Observe that R is an isometry. Let us denote by P_k the operator $R^k T^k - R^{k+1} T^{k+1}$ for any integer k. Then we have the following useful lemma.

LEMMA 3. *The operator P_k is a projection whose range is the closed subspace $K_u^p u^k$. Moreover, we have $P_k \circ P_l = 0$ if $k \neq l$.*

Proof. Let $(k, l) \in \mathbb{N}^2$. We have

$$P_k \circ P_l = R^k T^k R^l T^l - R^{k+1} T^{k+1} R^l T^l - R^k T^k R^{l+1} T^{l+1} + R^{k+1} T^{k+1} R^{l+1} T^{l+1}.$$

We distinguish three different cases:

- If $k > l$, since $T^i R^i = Id$ for any positive integer, we get

$$P_k \circ P_l = R^k T^{k-l} T^l - R^{k+1} T^{k+1-l} T^l - R^k T^{k-l-1} T^{l+1} + R^{k+1} T^{k-l} T^{l+1} = 0.$$

- If $k < l$, a similar computation also gives $P_k \circ P_l = 0$.
- If $k = l$, we have

$$\begin{aligned} P_k \circ P_l &= R^k T^k R^k T^k - R^{k+1} T^{k+1} R^k T^k - R^k T^k R^{k+1} T^{k+1} \\ &\quad + R^{k+1} T^{k+1} R^{k+1} T^{k+1} \\ &= R^k T^k - R^{k+1} T T^k - R^k R T^{k+1} + R^{k+1} T^{k+1} \\ &= P_k. \end{aligned}$$

Let $f \in K_u^p u^k$, we can write $f = u^k \varphi$ where $\varphi \in K_u^p$. From Lemma 1, we know that $\varphi = u e^{-i\theta} \overline{\widetilde{\varphi}}$ where $\widetilde{\varphi} \in K_u^p$. Therefore, we get

$$\begin{aligned} P_k(f) &= R^k T^k f - R^{k+1} T^{k+1} f = u^k P_+(\overline{u}^k f) - u^{k+1} P_+(\overline{u}^{k+1} f) \\ &= u^k \varphi - u^{k+1} P_+(\overline{u}\varphi) = u^k \varphi - u^{k+1} P_+(e^{-i\theta} \overline{\widetilde{\varphi}}) = u^k \varphi = f \end{aligned}$$

and hence the subspace $K_u^p u^k$ is contained in the range $R(P_k)$ of P_k. Conversely, let $f \in R(P_k)$, then we have

$$f = P_k f = u^k P_+(\overline{u}^k f) - u^{k+1} P_+(\overline{u}^{k+1} f),$$

which yields to

$$P_+\left(\overline{u}^k f\right) = P_+\left(P_+(\overline{u}^k f) - u P_+(\overline{u}^{k+1} f)\right) = P_+(\overline{u}^k f) - u P_+(\overline{u}^{k+1} f).$$

Thus, $P_+(\overline{u}^{k+1} f) = 0$. On the one hand, we can write $f = u^{k+1} e^{-i\theta} \overline{g}$ where $g \in H^p$. On the other hand, we derive that $f = u^k P_+(\overline{u}^k f)$ and hence $\overline{u}^k f = u e^{-i\theta} \overline{g} \in H^p$. Combining these two last facts, we see that g belongs to the subspace K_u^p. It implies that $f \in K_u^p u^k$. This finishes the proof of Lemma 3. ∎

Now, we turn to the proof of Theorem 2 For any $f \in H^p$, we set $f_k = \overline{u}^k P_k(f)$. Let $\varepsilon > 0$ and consider a fixed function $f \in H^p$. Using Proposition 1, we see that there exists a finite sum $g = \sum_{k=0}^m g_k u^k$ in $\mathcal{E}_u$ such that $\|f - g\| \leq$

$\varepsilon / \|P_+\|$. For any $n > m$, we observe that

$$\begin{aligned}
\left\| f - \sum_{k=0}^{n} f_k u^k \right\| &= \left\| f - \sum_{k=0}^{n} R^k T^k f - R^{k+1} T^{k+1} f \right\| \\
&= \left\| R^{n+1} T^{n+1} f \right\| \\
&\le \left\| R^{n+1} T^{n+1} (f-g) \right\| + \left\| R^{n+1} T^{n+1} g \right\| \\
&= \left\| R^{n+1} T^{n+1} (f-g) \right\| \\
&\le \sup_{n \ge 0} \|T^n\| \, \|f-g\| \\
&\le \|P_+\| \, \|f-g\| \le \varepsilon.
\end{aligned}$$

Thus the series $\sum_{k=0}^{+\infty} f_k u^k$ is norm convergent to f in H^p.

We now prove formula (1). Let $k \in \mathbb{N}$ be fixed, we have

$$\begin{aligned}
\left\langle \overline{u}^k f \mid (1 - \overline{u(z)} u) e_z \right\rangle = &\sum_{i=0}^{k-1} \left\langle \overline{u}^{k-i} f_i \mid (1 - \overline{u(z)} u) e_z \right\rangle + \left\langle f_k \mid (1 - \overline{u(z)} u) e_z \right\rangle \\
&+ \sum_{i \ge k+1} \left\langle u^{i-k} f_i \mid (1 - \overline{u(z)} u) e_z \right\rangle.
\end{aligned}$$

We distinguish three different cases:

If $0 \le i \le k-1$: it is sufficient to do it for $i = 0$ and then using the integral form of the inner product and applying Lemma 1 to the function f_0, we obtain

$$\begin{aligned}
\left\langle \overline{u}^k f_0 \mid (1 - \overline{u(z)} u) e_z \right\rangle = &\int_0^{2\pi} \overline{u(e^{i\theta})}^k e^{-i\theta} u(e^{i\theta}) \overline{\tilde{f}_0(e^{i\theta})} \overline{e_z(e^{i\theta})} \, dm(\theta) \\
&- u(z) \int_0^{2\pi} \overline{u(e^{i\theta})}^{k+1} e^{-i\theta} u(e^{i\theta}) \overline{\tilde{f}_0(e^{i\theta})} \overline{e_z(e^{i\theta})} \, dm(\theta) = 0
\end{aligned}$$

because the functions under the integrals are anti-analytic.

If $i > k$: set $m = i - k \in \mathbb{N}^*$, then

$$\left\langle u^m f_{k+m} \mid (1 - \overline{u(z)} u) e_z \right\rangle = u^m(z) f_{k+m}(z) - u(z) u(z)^{m-1} f_{k+m}(z) = 0.$$

If $i = k$: We develop the inner product and we apply Lemma 1 to the function f_k, then we get

$$\begin{aligned}\langle f_k \mid (1-\overline{u(z)}u)e_z\rangle &= f_k(z) - u(z)\int_0^{2\pi} f_k(e^{i\theta})\overline{u(e^{i\theta})e_z(e^{i\theta})}\,dm(\theta)\\ &= f_k(z) - u(z)\int_0^{2\pi} e^{-i\theta}u(e^{i\theta})\overline{\tilde{f}_k(e^{i\theta})}\,\overline{u(e^{i\theta})e_z(e^{i\theta})}\,dm(\theta)\\ &= f_k(z).\end{aligned}$$

Thus, formula (1) is proved.

To prove formula (2), since $T^m f = f_m + r_m$ where

$$r_m(z) = u(z)^{m+1}\sum_{k=m+1}^{\infty} f_k(z)u^{k-m-1}(z),$$

we obtain by using the integral representation of the f_k given by (1),

$$\begin{aligned}r_m(z) &= u(z)^{m+1}\sum_{k=m+1}^{\infty}\int_0^{2\pi}\overline{u(e^{i\theta})}^{k} f(e^{i\theta})u(z)^{k-m-1}\frac{1-u(z)\overline{u(e^{i\theta})}}{1-ze^{-i\theta}}\,dm(\theta)\\ &= u(z)^{m+1}\int_0^{2\pi} f(e^{i\theta})\overline{u(e^{i\theta})}^{m+1}\Big[\sum_{j=0}^{\infty}\big(\overline{u(e^{i\theta})}u(z)\big)^j\Big]\frac{1-u(z)\overline{u(e^{i\theta})}}{1-ze^{-i\theta}}\,dm(\theta)\end{aligned}$$

Computing the sum between the hooks and after simplification, it comes that

$$r_m(z) = u(z)^{m+1}\int_0^{2\pi}\frac{f(e^{i\theta})\overline{u(e^{i\theta})}^{m+1}}{1-ze^{-i\theta}}\,dm(\theta).$$

∎

Let us now give the second type of decomposition relatively to some particular unconditional basis of K_u^2 with coefficients in $H^p \circ u$.

THEOREM 3. *Let u be a non constant inner function and $(e_k)_{k\in J}$ ($J \subseteq \mathbb{N}$) be a normalized unconditional basis of the subspace K_u^2 of H^2. Then, any function $f \in H^p$ ($p \geq 2$) is uniquely represented as a sum*

$$f(z) = \sum_{k\in J}\widehat{f}_k(u(z))e_k(z) \qquad (z \in \mathbb{D})$$

with

$$\sum_{k\in J}\left\|\widehat{f}_k\right\|_2^2 < +\infty.$$

Moreover, if the sequence of coefficient functionals $(e_k^*)_{k\in J}$ associated to $(e_k)_{k\in J}$ is contained in H^∞, we have $\widehat{f_k} \in H^p$ and

$$\left\|\widehat{f_k}\right\|_p \le \|e_k^*\|_\infty \left[\frac{1+|u(0)|}{1-|u(0)|}\right]^{\frac{1}{q}} \|f\|_p .$$

Remark 3.

1) Observe that the series $\sum_{k\in J} \widehat{f_k}(u(z))e_k(z)$ is necessarily unconditionally convergent in H^2, but it is not the case in H^p $(1 < p \neq 2)$ as we can see by taking $u(z) = z$ and applying a well known result about Fourier series (see for instance [21]).

2) The above result extends, in a different way, results from [13] and gives estimates of the associated constants involving the basis $(e_k^*)_{k\in J}$ of K_u^2 and the inner function u.

Proof. Since $(e_k)_{k\in J}$ is a normalized unconditional basis of K_u^2, we know from Bari's Theorem (see for instance [21]) that there exists an orthonormal basis $(\varepsilon_k)_{k\in J}$ of K_u^2 and a positive invertible operator $A \in B(K_u^2)$ such that $A\varepsilon_k = e_k$. Let us denote by E the subspace of H^2 of the finite sums $\sum_{k\in J\cap\{1,\dots,m\}} \varphi_k(u(z))\varepsilon_k(z)$ where the functions φ_k belong to H^2 and m is an non negative integer. We define the operator B on E by setting

$$B\Bigg(\sum_{k\in J_m} \varphi_k(u(z))\varepsilon_k(z)\Bigg) = \sum_{k\in J_m} \varphi_k(u(z))e_k(z)$$

where $J_m = J \cap \{1,\dots,m\}$. We have

$$\begin{aligned}
&\Bigg\langle B\Bigg(\sum_{k\in J_m} \varphi_k(u(z))\varepsilon_k(z)\Bigg) \mid \sum_{k\in J_m} \varphi_k(u(z))\varepsilon_k(z)\Bigg\rangle \\
&= \sum_{k\in J_m}\sum_{l\in J_m} \langle \varphi_k \circ u\, e_k \mid \varphi_l \circ u\, \varepsilon_l\rangle \\
&= \sum_{k\in J_m}\sum_{l\in J_m} \langle \varphi_k \mid \varphi_l\rangle\langle e_k \mid \varepsilon_l\rangle = \sum_{k\in J_m}\sum_{l\in J_m}\left[\sum_{n=0}^{+\infty} \varphi_{k,n}\overline{\varphi_{l,n}}\right]\langle A\varepsilon_k \mid \varepsilon_l\rangle
\end{aligned}$$

$$= \sum_{n=0}^{+\infty} \Big\langle A\Big(\sum_{k\in J_m} \varphi_{k,n}\varepsilon_k \Big) \mid \sum_{k\in J_m} \varphi_{k,n}\varepsilon_k \Big\rangle \leq \|A\| \sum_{n=0}^{+\infty} \Big\| \sum_{k\in J_m} \varphi_{k,n}\varepsilon_k \Big\|^2$$
$$= \|A\| \sum_{n=0}^{+\infty} \sum_{k\in J_m} |\varphi_{k,n}|^2 = \|A\| \sum_{k\in J_m} \|\varphi_k\|^2 = \|A\| \Big\| \sum_{k\in J_m} \varphi_k(u(z))\varepsilon_k(z) \Big\|^2.$$

In the same way, we also obtain

$$\Big\langle B\Big(\sum_{k\in J_m} \varphi_k(u(z))\varepsilon_k(z) \Big) \mid \sum_{k\in J_m} \varphi_k(u(z))\varepsilon_k(z) \Big\rangle$$
$$\geq \left\|A^{-1}\right\|^{-1} \Big\| \sum_{k\in J_m} \varphi_k(u(z))\varepsilon_k(z) \Big\|^2.$$

Then, we use the following lemma whose proof is left to the reader.

Lemma 4. *Let E be a dense subspace of a Hilbert space H and T a linear application from E into H such that for any $x \in E$, we have*

$$a\,\|x\|^2 \leq \langle Tx \mid x\rangle \leq b\,\|x\|^2$$

where a, b are two positive real numbers. Then, the operator T admits a unique bounded extension to H which is a positive invertible operator.

Since E is dense in H^2, using Lemma 4 we see that B admits an extension to H^2 which is a positive invertible operator. Let $f \in H^2$, using the continuity of B^{-1} we easily see that for any $z \in \mathbb{D}$ we have

$$B^{-1}(f)(z) = \sum_{k\in J} \widehat{f}_k(u(z))\varepsilon_k(z)$$

and $\sum_{k\in J} \left\|\widehat{f}_k\right\|^2 = \left\|B^{-1}(f)\right\|^2 \leq \left\|B^{-1}\right\|^2 \|f\|^2 < +\infty$.

From now on, we assume that the sequence of coefficient functionals $(e_k^*)_{k\in J}$ associated to $(e_k)_{k\in J}$ is contained in H^∞. From Theorem 3.5 of [6], we have

$$\widehat{f}_k(z) = \int_0^{2\pi} \frac{1-|z|^2}{\left|1 - z\overline{u(e^{i\theta})}\right|^2} f(e^{i\theta})\overline{e_k^*(e^{i\theta})}\, dm(\theta).$$

To shorten formulas, we set $\Phi(r,t,z) = (1-r^2)\left|1-re^{it}\overline{u(z)}\right|^{-2}$. Then, we obtain

$$\begin{aligned}
&\int_0^{2\pi}\left|\widehat{f}_k(re^{it})\right|^p dm(t)\\
&\le \int_0^{2\pi}\left[\int_0^{2\pi}\Phi(r,t,e^{i\theta})\left|f(e^{i\theta})\right|^p dm(\theta)\right]\\
&\qquad\qquad\qquad\left[\int_0^{2\pi}\Phi(r,t,e^{i\theta})\left|e_k^*(e^{i\theta})\right|^q dm(\theta)\right]^{\frac{p}{q}} dm(t)\\
&\le \|e_k^*\|_\infty^p\int_0^{2\pi}\left[\int_0^{2\pi}\Phi(r,t,e^{i\theta})\left|f(e^{i\theta})\right|^p dm(\theta)\right]\\
&\qquad\qquad\qquad\left[\int_0^{2\pi}\Phi(r,t,e^{i\theta}) dm(\theta)\right]^{\frac{p}{q}} dm(t)\\
&= \|e_k^*\|_\infty^p\int_0^{2\pi}\left[\int_0^{2\pi}\Phi(r,t,e^{i\theta})\left|f(e^{i\theta})\right|^p dm(\theta)\right]\left[\int_0^{2\pi}\Phi(r,t,0) dm(\theta)\right]^{\frac{p}{q}} dm(t)\\
&\le \|e_k^*\|_\infty^p\left[\frac{1+|u(0)|}{1-|u(0)|}\right]^{\frac{p}{q}}\int_0^{2\pi}\int_0^{2\pi}\frac{1-r^2}{\left|1-re^{it}\overline{u(e^{i\theta})}\right|^2}\left|f(e^{i\theta})\right|^p dm(\theta)dm(t)\\
&= \|e_k^*\|_\infty^p\left[\frac{1+|u(0)|}{1-|u(0)|}\right]^{\frac{p}{q}}\int_0^{2\pi}\left[\int_0^{2\pi}\frac{1-r^2}{\left|1-re^{it}\overline{u(e^{i\theta})}\right|^2}dm(t)\right]\left|f(e^{i\theta})\right|^p dm(\theta)\\
&= \|e_k^*\|_\infty^p\left[\frac{1+|u(0)|}{1-|u(0)|}\right]^{\frac{p}{q}}\|f\|_p^p .
\end{aligned}$$

We get the result by letting $r\to 1^-$. ∎

Lemma 5. *If b is a finite Blaschke product, $P_k : H^1 \to K_b^1$ is continuous and* $\sup_{k\ge 0}\|P_k\|_1 < \infty$.

Proof. Let b be a finite Blaschke product, it suffices to do the computations for P_0. Let $f\in H^2$, then

$$\begin{aligned}
P_0(f) &= \int_0^{2\pi} f(e^{it})\frac{1-b(z)\overline{b(e^{it})}}{1-ze^{-it}}\,dm(t)\\
&= \int_0^{2\pi}\frac{e^{it}f(e^{it})}{b(e^{it})}\frac{b(e^{it})-b(z)}{e^{it}-z}\,dm(t).
\end{aligned}$$

And if we take $z = re^{i\theta}$, then for a fixed r,

$$\begin{aligned}\|\big(P_0(f)\big)_r\|_1 &= \int_0^{2\pi}\left|\int_0^{2\pi}\frac{e^{it}f(e^{it})}{b(e^{it})}\frac{b(e^{it})-b(re^{i\theta})}{e^{it}-re^{i\theta}}\,dm(t)\right|dm(\theta)\\ &\leq \int_0^{2\pi}\int_0^{2\pi}\big|f(e^{it})\big|\left|\frac{b(e^{it})-b(re^{i\theta})}{e^{it}-re^{i\theta}}\right|dm(t)dm(\theta)\\ &\leq \sup_{r,t,\theta}\left|\frac{b(e^{it})-b(re^{i\theta})}{e^{it}-re^{i\theta}}\right|\|f\|_1.\end{aligned}$$

In case b is a finite Blaschke product, by taking $z_1 = e^{it}$ and $z_2 = re^{i\theta}$, we can write $b(z_1) = \frac{P(z_1)}{Q(z_1)}$ where P, Q are polynomials with Q bounded from below in the disc and so

$$\begin{aligned}\left|\frac{b(e^{it})-b(re^{i\theta})}{e^{it}-re^{i\theta}}\right| &= \left|\frac{1}{z_1-z_2}\left(\frac{P(z_1)}{Q(z_1)}-\frac{P(z_1)}{Q(z_1)}\right)\right|\\ &= \left|\frac{P(z_1)Q(z_2)-P(z_2)Q(z_1)}{(z_1-z_2)Q(z_1)Q(z_2)}\right|.\end{aligned}$$

We remark that it is possible to factorize $P(z_1)Q(z_2)-P(z_2)Q(z_1)$ by z_1-z_2, and then by writing $P(z_1)Q(z_2)-P(z_2)Q(z_1) = (z_1-z_2)L(z_1, z_2)$ (where L is a polynomial of two variables) then

$$\left|\frac{b(e^{it})-b(re^{i\theta})}{e^{it}-re^{i\theta}}\right| = \left|\frac{L(z_1, z_2)}{Q(z_1)Q(z_2)}\right|.$$

This quantity is bounded because b is a finite Blaschke product and finally $\sup_{k\geq 0}\|P_k\|_1 < \infty$. ∎

We use the previous lemma in order to prove the following result which is of independent interest.

Proposition 2. *Let b be a finite Blaschke product, $p \geq 2$, and q is the conjugate exponent of p and we define*

$$\begin{aligned}\Phi_q : H^q &\longrightarrow \ell^p(K_b^1)\\ f &\longmapsto \big(\bar{b}^k P_k(f)\big)_{k\geq 0}.\end{aligned}$$

Then Φ_q is continuous and there exists $C > 0$ such that for every $f \in H^q$,

$$\left(\sum_{k\geq 0}\|f_k\|_q^p\right)^{\frac{1}{p}} \leq C\|f\|_q.$$

Proof. We take

$$\begin{array}{rcl}\Phi_1 : H^1 & \longrightarrow & \ell^\infty(K_b^1) \\ f & \longmapsto & \big(\bar{b}^k P_k(f)\big)_{k\geq 0}.\end{array}$$

Then Φ_1 is a linear continuous application because $\|\Phi_1\| \leq \sup_{k\geq 0}\|P_k\|_1$ from the integral form of Lemma 5. In the same way, we define

$$\begin{array}{rcl}\Phi_2 : H^2 & \longrightarrow & \ell^2(K_b^1) \\ f & \longmapsto & \big(\bar{b}^k P_k(f)\big)_{k\geq 0}.\end{array}$$

Let us now recall a fundamental notion in geometry of Banach spaces. Given two compatible Banach spaces X and Y, an interpolation space W, $X \bigcap Y \subseteq W \subseteq X + Y$, is a Banach space with the following property: if L is a linear operator from $X + Y$ into itself, which is continuous from X into itself and from Y into itself, then it is also continuous from W into itself. Furthermore, $W = [X, Y]_\theta$ is said of exponent θ $(0 < \theta < 1)$ if there exists a constant C such that $\|L\|_{W;W} \leq C\|L\|_{X;X}^{1-\theta}\|L\|_{Y;Y}^{\theta}$ for such operators L (see [4] for more informations).

Then, by using from one side, the interpolation result of Jones to $[H^1, H^2]_\theta$ (see [11]) and from the other side the interpolation Theorem for the vector-valued ℓ^p spaces (cf. [4]) to $[\ell^\infty(K_b^1), \ell^2(K_b^1)]_\theta$, we obtain that

$$\Phi_q : H^q = [H^1, H^2]_\theta \longrightarrow \big[\ell^\infty(K_b^1), \ell^2(K_b^1)\big]_\theta = \ell^p(K_b^1)$$

is continuous and there exists $M > 0$ such that $\|\Phi_q\| \leq M\|\Phi_1\|^{1-\theta}\|\Phi_2\|^\theta$. ∎

PROPOSITION 3. *Let b be an infinite interpolating Blaschke product, then $T_{\overline{b}}$ does not define a continuous operator from H^1 into H^1.*

Proof. Let $(\lambda_k)_{k\geq 1}$ be the zero sequence of b enumerated in such a way that $|\lambda_k| \leq |\lambda_{k+1}|$. We define the following,

$$e^*_{\lambda_k}(z) = \frac{1-|\lambda_k|^2}{b_k(\lambda_k)}\frac{b_k(z)}{1-\overline{\lambda_k}z} \in K_b^1.$$

Let $h_k(z) = e^*_{\lambda_k}(z)^2 \in H^1$. Then $h_k(z) = e^*_{\lambda_k}(z) + \big(e^*_{\lambda_k}(z)^2 - e^*_{\lambda_k}(z)\big) = e^*_{\lambda_k}(z) + b(z)g_k(z)$ with $g_k \in H^1$ because $e^*_{\lambda_k}(z)^2 - e^*_{\lambda_k}(z) \in bH^1$. So, $P_0(h_k) = e^*_{\lambda_k}$. On the one hand,

$$\|h_k\|_1 = \|e^*_{\lambda_k}\|_2^2 = \frac{\big(1-|\lambda_k|^2\big)^2}{|b_k(\lambda_k)|^2}\|e_{\lambda_k}\|_2^2 = \frac{\big(1-|\lambda_k|^2\big)}{|b_k(\lambda_k)|^2}.$$

On the other hand,

$$\|P_0(h_k)\|_1 = \|e^*_{\lambda_k}\|_1 = \frac{1-|\lambda_k|^2}{|b_k(\lambda_k)|}\|e_{\lambda_k}\|_1.$$

It follows from Carleson's interpolation theorem [5], that the ratio $\frac{\|P_0(h_k)\|_1}{\|h_k\|_1} = \|e_{\lambda_k}\|_1 |b_k(\lambda_k)| \geq c\|e_{\lambda_k}\|_1$ for some constant $c > 0$ independent of k. The computation of $\|e_{\lambda_k}\|_1$ by taking $\lambda_k = |\lambda_k|e^{i\theta_k}$ gives

$$\begin{aligned}\|e_{\lambda_k}\|_1 = \int_0^{2\pi} \frac{dm(\theta)}{|1-\overline{\lambda}_k e^{i\theta}|} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{dm(\theta)}{(1+|\lambda_k|^2 - 2|\lambda_k|\cos(\theta-\theta_k))^{\frac{1}{2}}}. \\ &= \frac{1}{\pi}\int_0^{\pi} \frac{dt}{(1+|\lambda_k|^2-2|\lambda_k|\cos\, t)^{\frac{1}{2}}}.\end{aligned}$$

We proceed by absurdum and suppose that P_0 is bounded. Then, there exists $M > 0$ such that $M \geq \frac{\|P_0(h_k)\|_1}{\|h_k\|_1}$ and for every ε, $0 < \varepsilon \leq \pi$, we have

$$M \geq \frac{1}{\pi}\int_\varepsilon^{\pi} \frac{dt}{\big(1+|\lambda_k|^2-2|\lambda_k|\cos\, t\big)^{\frac{1}{2}}}.$$

By studying the derivative of the function $f(r) = (1+r^2-2r\cos t)^{-1}$, it is easy to prove that there exists an integer k_0 such that $k \geq k_0$, $f(\lambda_{k+1}) \leq f(\lambda_k)$ for every $t \in [0,\, \pi]$. We can then apply the classical Beppo Levi's theorem on monotone convergence (see for instance [15]) and obtain

$$M \geq \int_0^{\pi} \frac{dt}{\sqrt{1-\cos\, t}}.$$

This gives a contradiction with the fact that the convergence of the integral $\int_0^{\pi} \frac{dt}{\sqrt{1-\cos\, t}}$ is equivalent to the convergence of the integral $\int_0^{\pi} \frac{\sqrt{2}dt}{t} = +\infty$. Then P_0 is not continuous. Therefore the operator $T = T_{\overline{b}}$ is not continuous because otherwise $P_0 = I - RT$ will be continuous. ∎

2. Cyclicity in H^p spaces where $2 \leq p < \infty$

In this section, we will describe in H^p, $1 < p < \infty$ the $T_{\overline{b}}$-invariant subspaces generated by some lacunary decompositions of functions f which will also lead to have an explicit criteria of cyclicity for these functions. We will look at the smallest $T_{\overline{b}}$-invariant subspace containing f which is

$span(T_{\bar{b}}^{n} f : n \geq 0)$.

NOTATION: We define in H^p, $1 < p < \infty$ the following subspaces,

$$E_f = \overline{span}_{H^p}(T^n f : n \geq 0).$$

$$K_*(f) = \bigcap_{n \geq 0} \overline{span}_{H^p}(f_k : k \geq n)$$

where the functions f_k are the component functions of f given by Theorem 2. Afterward, when there is no ambiguity on the considered function f, we will write K_* for $K_*(f)$ instead of shortness. We also define the tensor product of two linear subspaces of function F and G, denoted by $F \otimes G$, as the closure in H^p of the vector space generated by all the products fg where $f \in F$ and $g \in G$.

2.1. PREPARATORY LEMMAS. Firstly, we recall the following well known lemma.

LEMMA 6. ([12]) *Let $(b_n)_{n \geq 1}$ be a sequence of nonnegative real numbers such that $\sum_{n \geq 1} b_n < \infty$. Then for every real $\gamma \geq 1$,*

$$\sum_{k \geq 1} \frac{b_k}{\left(\sum_{j > k} b_j\right)^{\gamma}} = \infty.$$

The next Lemma will be useful in the sequel. Notice that Lemma 7 comes from general properties of T-invariant subspaces and does not depend on the "lacunarity" of the decomposition of the considered function.

LEMMA 7. *Let b be a finite Blaschke product and f, g be any two functions of H^p with $p > 1$. Then,*

(1) $g \in E_f^p \Rightarrow g_i \in span(f_j : j \geq 0) \quad \forall\, i \geq 0.$

(2) *Let $F \subset K_b^p$ be such $F \otimes [H^p \circ b] \subset E_f^p$. Then, $F \subset K_*(f)$.*

Proof. From Theorem 2, we can write

$$f = \sum_{i=0}^{\infty} f_i b^i, \quad g = \sum_{i=0}^{\infty} g_i b^i.$$

(1) For any fixed integer i_0,

$$T^{i_0} g = g_{i_0} + \sum_{i>i_0} g_i b^{i-i_0}.$$

Since $g \in E_f^p$ is stable by T then $T^{i_0} g \in E_f^p$ and there exists a sequence of complex polynomials $p_n = \sum_{j=0}^{m_n} a_j(n) z^j$ such that,

$$T^{i_0} g = \lim_{n\to\infty} p_n(T) f = g_{i_0} + \sum_{i>i_0} g_i b^{i-i_0}.$$

Denote by $\lambda_1, \ldots, \lambda_d$ the zeros of b and by d_k the order of multiplicity of λ_k. Then we have

$$\begin{aligned}
g_{i_0} &= \sum_{k=1}^{d} \sum_{l=1}^{d_k} \langle g_{i_0} \mid e_{\lambda_k,l} \rangle e^*_{\lambda_k,l} = \lim_{n\to\infty} \sum_{k=1}^{d} \sum_{l=1}^{d_k} \langle p_n(T)(f) \mid e_{\lambda_k,l} \rangle e^*_{\lambda_k,l} \\
&= \lim_{n\to\infty} \sum_{j=0}^{m_n} a_j(n) \sum_{k=1}^{d} \sum_{l=1}^{d_k} f_j{}^{(l)}(\lambda_k) e^*_{\lambda_k,l} \\
&= \lim_{n\to\infty} \sum_{k=1}^{d} \sum_{l=1}^{d_k} \Big\langle \sum_{j=0}^{m_n} a_j(n) f_j \mid e_{\lambda_k,l} \Big\rangle e^*_{\lambda_k,l}.
\end{aligned}$$

This implies that $g_{i_0} = \lim_{n\to\infty} \sum_{j=0}^{m_n} a_j(n) f_j$. Thus $g_{i_0} \in span(f_j : j \geq 0)$.
(2) Let $g \in F \otimes [H^p \circ b]$. If we consider $T^k f$ instead of f in (1), then

$$g \in E_{T^k f} \Rightarrow g_i \in span(f_j : j \geq k) \quad \forall\, i \geq 0.$$

And since this is true for any integer k, we obtain the inclusion. ∎

Let us state the following useful lemma.

LEMMA 8. *Let E be a closed subspace in H^p, $1 < p < +\infty$, and u be an inner function. Then we have*

$$E \in Lat(T_{\overline{u}}) \cap Lat(T_u) \Leftrightarrow E = (K_u^p \cap E) \otimes [H^p \circ u].$$

Proof. Assume that $E \in Lat(T_{\overline{u}}) \cap Lat(T_u)$ and let $f = \sum_{k\geq 0} f_k u^k \in E$, where $f_k \in K_u^p$ for each k. Then $T_{\overline{u}}(f) = \sum_{k\geq 1} f_k u^{k-1}$. Since $T_u \circ T_{\overline{u}}(f) = \sum_{k\geq 1} f_k u^k \in E$, we have $f_0 = f - T_u \circ T_{\overline{u}}(f) \in E$. In a similar way, we can prove that $f_k \in E$ for any nonnegative integer k. The reverse inclusion and the implication in the other direction are obvious. ∎

2.2. MAIN RESULTS ABOUT CYCLICITY. Let $h \in H^p$ ($p \geq 2$) and b be a finite Blaschke product, we consider the decomposition $h = \sum_{k=0}^{+\infty} h_k u^k$ given by Theorem 2 and we set $R_k(h) = \sum_{l>k} \|h_l\|_p{}^q \in (\mathbb{R}_+ \cup \{+\infty\})$. We have the following result which is concerned with a more general class of functions than classical lacunary functions.

THEOREM 4. *Let b be a finite Blaschke product and $f = \sum_{k\geq 0} f_k b^{n_k} \in H^p$ ($p \geq 2$) which admits an infinite decomposition with respect to the powers of the finite Blaschke product b (i.e. card $\{k \geq 0; f_k \neq 0\} = +\infty$). Assume that we have $\lim_{k\to+\infty}(n_{k+1} - n_k) = +\infty$, $\sum \|f_k\|_p^q < +\infty$ and*

$$\underline{\lim} \frac{R_{k-1}(f)}{R_k(f)} > 1.$$

Then, we have $E_f^p = K_*^p(f) \otimes H^p \circ b \oplus E_\varphi^p$, where φ is a function finitely represented with respect to the powers of b (E_φ^p is a finite dimensional subspace of H^p).

Remark 4. For every $p \geq 2$, observe that we always have

$$\underline{\lim} \frac{R_{k-1}(f)}{R_k(f)} \geq 1.$$

Note also that when $p = 2$ we always have $\sum \|f_k\|_2^2 < +\infty$.

Proof. By assumption, we know that there exist $\rho > 1$ and k_0 such that

$$\frac{R_{k-1}(f)}{R_k(f)} \geq \rho$$

for any $k \geq k_0$.

Let E be a maximal subspace in $Lat(T_{\overline{b}}) \cap Lat(T_b)$ (for the inclusion) which is contained in E_f^p. By Lemma 8, we know that $E = F \otimes H^p \circ b$ where F the subspace $K_b^p \cap E$. Choose $(e_k)_{k=1,\ldots,m}$ a basis of K_b^p(K_b^p is a finite dimensional subspace) such that $(e_k)_{k=1,\ldots,d}$ is a basis of F. The associated coefficient functionals $(e_k^*)_{k=1,\ldots,m}$ belongs to $K_b^q = K_b^\infty \subseteq H^\infty$ because b is a finite Blaschke product. Applying Theorem 3, we see that we can decompose f under the form $f = \sum_{k=1}^m \widehat{f_k} \circ u e_k$, where the functions $\widehat{f_k}$ are in H^p. Using the Littlewood Subordination Principle (see for instance [20]), we get that the functions $\widehat{f_k} \circ u$ belong to H^p and hence the functions $g = \sum_{k=1}^d \widehat{f_k} \circ u e_k$ and $\varphi = \sum_{k=d+1}^m \widehat{f_k} \circ u e_k$ are both in H^p. We easily see that $g \in E$, therefore we derive that $\varphi \in E_f^p$. Using the unicity of the decomposition of f with respect to the powers of b, we easily see that $\varphi = \sum_{k\geq 0} \varphi_k b^{n_k}$. Assume that this decomposition is infinite, that is $card\,\{k \geq 0; \varphi_k \neq 0\} = +\infty$. Firstly, we note that for any $l \in \mathbb{N}$, we have

$$\sum_{k=l}^{+\infty} \|\varphi_k\|_p^q \leq \left\| \sum_{i=d+1}^{m} e_i \otimes e_i^* \right\|^q \sum_{k=l}^{+\infty} \|f_k\|_p^q < +\infty \tag{3}$$

(where $u \otimes v$ is the rank one operator: $f \to \langle f \mid v\rangle\, u$) and $R_l(\varphi) > 0$ for any $l \in \mathbb{N}$. Let N and j be two fixed nonnegative integers, there exists k_0 such that $n_k \geq N + j$ for each $k \geq k_0$. Then we have

$$\frac{1}{\|\varphi_k\|} T^{n_k - N - j} T^N \varphi = \frac{\varphi_k}{\|\varphi_k\|} b^j + b^j \sum_{l>k} \frac{\varphi_l}{\|\varphi_k\|} b^{n_l - n_k}$$

for each $\varphi_k \neq 0$, and in this case we set $r_k = \sum_{l>k} \frac{\varphi_l}{\|\varphi_k\|} b^{n_l - n_k}$. We consider neighborhoods, for the weak topology, of the following form

$$V = \left\{ \varphi \in H^p; |\langle \varphi \mid h_i \rangle| < 1, 1 \leq i \leq n \right\},$$

where each function h_i belongs to H^q $(\frac{1}{p} + \frac{1}{q} = 1)$. Using Proposition 2 and Holder inequality, we obtain

$$\begin{aligned}
|\langle r_k \mid h_i \rangle| &\leq \frac{1}{\|\varphi_k\|} \left| \sum_{l>k} \langle \varphi_l \mid h_{i,n_l - n_k} \rangle \right| \\
&\leq \frac{1}{\|\varphi_k\|} \sum_{l>k} \|\varphi_l\|_p \, \|h_{i,n_l - n_k}\|_q \\
&\leq \frac{1}{\|\varphi_k\|} \left[\sum_{l>k} \|\varphi_l\|_p^q \right]^{\frac{1}{q}} \left[\sum_{l>k} \|h_{i,n_l - n_k}\|_q^p \right]^{\frac{1}{p}}.
\end{aligned}$$

Suppose that none of the r_k belongs to V, then for every $k \geq k_0$, we have

$$1 \leq \max_{i=1,\ldots,n} |\langle r_k \mid h_i \rangle|^p \leq \frac{1}{\|\varphi_k\|^p} \left[\sum_{l>k} \|\varphi_l\|_p^q \right]^{\frac{p}{q}} \sum_{i=1}^{n} \sum_{l>k} \|h_{i,n_l - n_k}\|_q^p .$$

Therefore, we get

$$\left[\frac{R_{k-1}(\varphi) - R_k(\varphi)}{R_k(\varphi)} \right]^{\frac{p}{q}} \leq \sum_{i=1}^{n} \sum_{l > n_{k+1} - n_k} \|h_{i,l}\|_q^p .$$

From Proposition 2, we know that the series $\sum_{l\geq 0}\|h_{i,l}\|_q^p$ are convergent. Since $\lim\,(n_{k+1}-n_k)=+\infty$, we derive that

$$\lim_{k\to+\infty}\frac{R_{k-1}(\varphi)}{R_k(\varphi)}=1.$$

Thus, there exists a positive integer k_1 such that for any $k\geq k_1$

$$\frac{R_{k-1}(\varphi)}{R_k(\varphi)}\leq\frac{1+\rho}{2}$$

Set $\alpha_k=R_k(\varphi)/R_k(f)$ and notice that $\alpha_k\in\left]0,\left\|\sum_{i=d+1}^m e_i\otimes e_i^*\right\|^q\right]$ (see (3)). From the previous inequalities, we deduce that

$$\rho\frac{\alpha_{k-1}}{\alpha_k}\leq\frac{\alpha_{k-1}}{\alpha_k}\frac{R_{k-1}(f)}{R_k(f)}=\frac{R_{k-1}(\varphi)}{R_k(\varphi)}\leq\frac{1+\rho}{2}$$

which implies that (α_k) is a increasing sequence. Thus (α_k) converges to a strictly positive number α. Hence, we obtain

$$\lim\frac{R_{k-1}(f)}{R_k(f)}=\lim\frac{\alpha_k}{\alpha_{k-1}}\frac{R_{k-1}(\varphi)}{R_k(\varphi)}=1,$$

which gives a contradiction. Consequently, we see that 0 is in the weak closure of $\{r_k; k\in\{l;\varphi_l\neq 0\}\}$. Since the sequence $\varphi_k/\left\|\varphi_k\right\|_p$ is contained in the finite dimensional subspace K_b^p, we derive that there exists a subsequence $\varphi_{m_k}/\left\|\varphi_{m_k}\right\|_p$ which is convergent to a nonzero function ψ. Observe that the sequence (r_k) does not depend on N nor on j. It implies that $\psi\otimes H^p\circ b\subseteq E^p_{T^Nf}$ for any N. By construction, each functions φ_{m_k} belongs to $G=span\{e_{d+1},\ldots,e_m\}$, so ψ is in G. It implies that $E\subsetneqq E+\psi\otimes H^p\circ b\in Lat(T_{\bar{b}})\cap Lat(T_b)$, but this fact contradicts the maximality of E. Consequently, the function φ is finitely represented with respect to the powers of b and we can write $\varphi=\sum_{k=0}^n\varphi_k b^k$ where $\varphi_k\in K_b^p$ for some positive integer n. On the one hand, it follows that $T^kf=T^kg$ for any $k>m$, hence we necessarily have $K_*^p(f)=K_*^p(g)\subseteq F$. On the other hand, from Lemma 7, we know that the converse inclusion is also true. Finally, we get $E_f^p\subseteq E_g^p+E_\varphi^p\subseteq E+E_\varphi^p=K_*^p(f)\otimes H^p\circ b+E_\varphi^p\subseteq E_f^p$ $(\varphi\in E_f^p)$. Observe that we have $(K_*^p(f)\otimes H^p\circ b)\cap E_\varphi^p=\{0\}$ by construction of g and φ. Therefore E_f^p is the direct vectorial sum of $K_*^p(f)\otimes H^p\circ b$ and of the finite dimensional subspace E_φ^p. ∎

COROLLARY 1. *Let f and b be as in Theorem 4. In addition, assume that $K_*^p(f) = K_b^p$. Then f is cyclic for T in H^p.*

Remark 5. Let $p \geq 2$, $\alpha > 1$ and $(a_k)_{k\geq 0}$ be a sequence of complex numbers such that $\sum_{k=0}^{+\infty}|a_k|^q < +\infty$ (q is the conjugate exponent of p). Assume that the sequence $(\beta_k)_{k\geq 0}$ defined by setting $\beta_k = \sum_{l>k}|a_l|^q$ satisfies the condition $\underline{\lim}\beta_{k-1}/\beta_k > 1$. Recall that the Rademacher functions $(r_k(t))_{k\geq 0}$ are given by $r_k(t) = sgn(\sin(2^k\pi t))$ for any $t \in [0,1]$, where $sgn(x) = 1$ when $x > 0$, $sgn(0) = 0$ and $sgn(x) = -1$ when $x < 0$. Let b be the Blaschke product given by $b(z) = zb_{\lambda_1}(z)\cdots b_{\lambda_m}(z)$ where $\lambda_1, \ldots, \lambda_m$ are m distinct points contained in $\mathbb{D} \setminus \{0\}$. We set $u_0(z) = 1$ and $u_j(z) = b_{\lambda_1}(z)\cdots b_{\lambda_l}(z)$ for any $j \in \{1, \ldots, m\}$. Let $s \in \{1, \ldots, m\}$, for every $t \in [0,1]$ we consider the infinite linear combination f_t of Blaschke products given by the series $f_t(z) = \sum_{k=0}^{\infty} a_k r_k(t) u_{\gamma(k)}(z) b(z)^{[k^\alpha]}$ ($z \in \mathbb{D}$) where $\gamma(k)$ is the rest of rest of the euclidean division of k by $s+1$ and $[x]$ is the integer part of the real number x. Let us observe that $[k^\alpha]$ is not a lacunary sequence. Using Khinchin's inequality (see for instance [14]), Fubini's theorem and Theorem 4, we see that there exists a Borelian subset $E \in [0,1]$ of Lebesgue measure 1 such that for all $p \geq 2$ we have $f_t \in H^p$ and $E_{f_t}^p = K_{zu_s}^p \otimes H^p \circ b$ for any $t \in E$. In particular, when $s = m$ we see that for almost every choice of signs $\{\varepsilon_k\}$, the series $\sum_{k=0}^{+\infty} \varepsilon_k u_{\gamma(k)}(z) b(z)^{[k^\alpha]}$ is a $T_{\overline{b}}$ cyclic vector in all H^p for $p \geq 2$.

PROPOSITION 4. *Let $f(z) = \sum_{k\geq 0} f_k(z) b^k(z) \in H^2$, be the decomposition of f with respect to the sequence of the powers of a Blaschke product b and where none of the functions f_k are zero. If we suppose that the sequence in K_b of the f_k is orthogonal, then f is not cyclic for T in H^2.*

Proof. Indeed, suppose that f is cyclic and let $p_n(z) = \sum_{i=0}^{d_n} a_i(n) z^i$ be a sequence of complex polynomials such that

$$f_0 = \lim_{n\to\infty} p_n(T)f.$$

Then, $c = \|f_0\|^2 = \lim_{n\to+\infty}\langle f_0 \mid p_n(T)f\rangle = \lim_{n\to+\infty} p_n(0) = \lim_{n\to+\infty} a_0(n)$. And in a similar way, we can prove that $\lim_{n\to+\infty} a_i(n) = 0$ for any $i = 1, \ldots d_n$. Consider,

$$p_n(T)f = \sum_{i=0}^{d_n} a_i(n) T^i f = \sum_{i=0}^{d_n} a_i(n) f_i + b\left(\sum_{i=0}^{d_n} a_i(n) f_{i+1}\right) + \sum_{j=2}^{\infty} b^j \left(\sum_{i=0}^{d_n} a_i(n) f_{i+j}\right).$$

We take $\Psi_{1,n} = \sum_{i=0}^{d_n} a_i(n) f_{i+1}$. Then, $\|\Psi_{1,n}\|^2 \leq \|p_n(T)f\|^2 \leq M$ for some constant $M > 0$. Therefore there exists a subsequence Ψ_{1,n_k} which converges weakly to a function Ψ. Or, from one hand, we have

$$\lim_{k\to\infty} \langle p_{n_k}(T)f \mid bf_1\rangle = \langle f_0 \mid bf_1\rangle = 0.$$

And from the other hand this same sequence converges to a non null limit because

$$\langle p_{n_k}(T)f \mid bf_1\rangle = \langle \Psi_{1,n_k} \mid f_1\rangle = a_0(n_k)\|f_1\|^2 \xrightarrow[k\to+\infty]{} c\|f_1\|^2 \neq 0.$$

Then we have a contradiction. And f is not cyclic. ∎

The following result will be useful later and is of independent interest. In the scalar case, observe that the criterion of pseudo-continuation of Douglas-Shapiro-Shields (cf. [8]) gives this property in a different way.

Theorem 5. *Let b be a finite Blaschke product and let $f \in H^p$, $p \geq 2$. Then f is cyclic for $T = T_{\bar{b}}$ in H^p if and only if f is cyclic for T in H^2.*

Proof. Let us denote $E^r(f) = \overline{span}_{H^r}\{T^n f; n \geq 0\}$. Assume that f is not cyclic in H^p ($p \geq 2$), that is $E^p(f) \neq H^P$. Then $E^p(f)^\perp \in Lat(R)$ in H^q where q is the conjugate exponent of p ($1/p + 1/q = 1$). Applying Theorem 1, we see that there exist b-q-inner functions $h_1, \ldots, h_m$ such that $E^p(f)^\perp = h_1 H^q \circ b + \cdots + h_m H^q \circ b$. Since $E^p(f)^\perp \neq \{0\}$, we may suppose that each h_i is non zero. As b is a finite Blaschke product, we deduce easily from Corollary 2 of [13] that any h_i is in H^∞, hence in H^2. Thus $E^2(f)^\perp \neq \{0\}$ and f is not cyclic in H^2. The converse implication follows from the inequality $\|h\|_2 \leq \|h\|_p$ ($h \in H^p$). ∎

We have this following Theorem concerning the cyclicity of lacunary decomposition of functions.

Theorem 6. *Let b be a finite Blaschke product of degree d and f in H^p which admits a lacunary decomposition with respect to the powers of b ($f = \sum_{k\geq 0} f_k b^{n_k}$). Then the following statements are equivalent.*

(1) *f is cyclic for $T_{\bar{b}}$ in H^p.*

(2) *$K_*(f) = K_b$.*

(3) *Let* $\lambda_1, \ldots, \lambda_k$ *be the zeros of* b *and* $d_1, \ldots, d_k$ *respectively the orders of multiplicity of these zeros, then for every* $m \geq 0$, *there exists integers* m_i, $m_i \geq m$, $i = 1, \ldots, d$ *such that*

$$\begin{vmatrix} f_{m_1}(\lambda_1) & f_{m_2}(\lambda_1) & \cdots & \cdots & f_{m_d}(\lambda_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{m_1}^{(d_1-1)}(\lambda_1) & f_{m_2}^{(d_1-1)}(\lambda_1) & \cdots & \cdots & f_{m_d}^{(d_1-1)}(\lambda_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{m_1}(\lambda_k) & f_{m_2}(\lambda_k) & \cdots & \cdots & f_{m_d}(\lambda_k) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{m_1}^{(d_k-1)}(\lambda_k) & f_{m_2}^{(d_k-1)}(\lambda_k) & \cdots & \cdots & f_{m_d}^{(d_k-1)}(\lambda_k) \end{vmatrix} \neq 0.$$

Proof. We first prove that (1) $\Rightarrow$ (2). Suppose that $E_f^p = H^p$. Then, for every $g \in H^p$ and for every $N \geq 1$, there exists a sequence of complex polynomials $(p_n)_{n\geq 1}$ such that $\lim_{n\to\infty}\|p_n(T)f - b^N g\|_p = 0$. Then $\lim_{n\to\infty}p_n(T)T^N f = T^N \lim_{n\to\infty}p_n(T) = T^N b^N g = g$. Therefore, $E_{T^N f} = H^p$. From Lemma 7 applied to $T^{n_k} f$ instead of f for every $k \geq 1$, we obtain $K_b \subset span(f_j : j \geq k)$, so $K_b \subset K_*(f)$, and $K_b = K_*(f)$.

We turn now to the assertion (2) $\Rightarrow$ (1). Let $f \in H^p$ be such $K_*(f) = K_b$. Using Theorem 5, it is enough to prove that f is cyclic in H^2. Suppose the contrary, assume that f is not cyclic in H^2 and consider a maximal subspace $E \in Lat(T_{\bar{b}}) \cap Lat(T_b)$ which is contained in $E_f^2 \neq H^2$. From Lemma 7, we know that $E = F \otimes H^2 \circ b$ where $F = E \cap K_b$. We easily see that $g = \sum_{k=0}^{\infty} P_F(f_k) b^{n_k}$ (resp. $\varphi = \sum_{k=0}^{\infty} P_{F^\perp}(f_k) b^{n_k}$) is the orthogonal projection of the function f onto E (resp. $E^\perp$). Assume that φ is not finitely represented, it implies that there exists an infinite subsequence (m_k) of (n_k) such that $\varphi = \sum_{k=0}^{\infty} \varphi_k b^{m_k}$ where each φ_k is nonzero. Observe that (m_k) is necessarily a lacunary sequence. For any fixed integer N, $j \geq 0$, there exists an integer k_0 such that for every $k \geq k_0$, $m_k - m_{k-1} \geq j + N$ (which is possible because φ is a lacunary decomposition). We consider

$$\frac{1}{\|\varphi_k\|} T^{m_k - N - j} T^N \varphi = \frac{\varphi_k}{\|\varphi_k\|} b^j + b^j \sum_{l>k} \frac{\varphi_l}{\|\varphi_k\|} b^{m_l - m_k}.$$

We take $r_k = \sum_{l>k} (\varphi_l / \|\varphi_k\|)\, b^{m_l - m_k}$. To prove that 0 is in the weak adherence of the sequence (r_k), it suffices to prove that every neighborhood of 0 for the weak topology contains one of the r_k. We consider neighborhoods of the following form

$$V = \left\{h \in H^2 \, : \, |\langle h \mid h_i\rangle| < 1, \, 1 \leq i \leq n\right\}.$$

The functions h_i $i = 1, \dots n$ are some given functions of H^2. We obtain that

$$\begin{aligned} |\langle r_k \mid h_i\rangle| &= \frac{1}{\|\varphi_k\|_2}\left|\sum_{l>k}\langle \varphi_l \mid h_{i,\, m_l - m_k}\rangle\right| \\ &\leq \frac{1}{\|\varphi_k\|_2}\sum_{l>k}\|\varphi_l\|_2\big\|h_{i,\, m_l-m_k}\big\|_2 \\ &\leq \frac{1}{\|\varphi_k\|_2}\Big(\sum_{l>k}\|\varphi_l\|_2^2\Big)^{\frac{1}{2}}\Big(\sum_{l>k}\big\|h_{i,\, m_l-m_k}\big\|_2^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Suppose none of the r_k belongs to V. Then for every $k \geq k_0$

$$1 \leq \max_{1\leq i\leq n}|\langle r_k \mid h_i\rangle|^2 \leq \frac{1}{\|\varphi_k\|_2^2}\Big(\sum_{l>k}\|\varphi_l\|_2^2\Big)\Big(\sum_{i=1}^{n}\sum_{l>k}\big\|h_{i,\, m_l-m_k}\big\|_2^2\Big).$$

Since the sequence $(n_l)_{l\geq 0}$ is a lacunary sequence, it is well-known (see [8]) that there exists an integer number $M > 0$ such that for every non null integer N, there is no more than M distinct representations $N = n_r - n_s$ where $r, \, s \in \mathbb{N}$. Therefore, summing over k gives

$$\sum_{k\geq k_0}\frac{\|\varphi_k\|_2^2}{\Big(\sum_{l>k}\|\varphi_l\|_2^2\Big)} \leq \sum_{1\leq i\leq n} M\sum_{k\geq 0}\|h_{i,\, k}\|_2^2.$$

It comes that

$$\sum_{k\geq k_0}\left(\frac{\|\varphi_k\|_2^2}{\sum_{l>k}\|\varphi_l\|_2^2}\right) \leq M\sum_{1\leq i\leq n} C^p\|h_{i,\, k}\|_2^2.$$

On the one hand, the left side is a divergent sum by Lemma 6, on the other hand the right side is a convergent sum, this gives a contradiction. Then, by considering an adequate subsequence, as in the proof of Theorem 4, we derive that there exists a function ψ such that E is strictly contained in the subspace $(F + \psi) \otimes (H^2 \circ b)$ of E_f^2 which is a contradiction. Thus, the function φ is finitely represented. Similar analysis to that in the proof of Theorem 4 shows that $K_*(f) = K_*(g) = F$. Since $K_*(f) = K_b^2$ by assumption, we immediately deduce that $E = K_b^2 \otimes H^2 \circ b = H^2$, therefore the subspace E_f^2 is whole H^2 and f is cyclic. So we obtain (1).

We have (2) $\Leftrightarrow$ (3) because K_b is of finite dimension and from Theorem 2, we know that a base is given by the functions $\big(e_{\lambda_k,l} : k \geq 1,\, 0 \leq l \leq d_k - 1\big)$. Since we have $\langle f \mid e_{\lambda,l}\rangle = f^{(l)}(\lambda)$, we see that the statement (3) gives the existence of $\sum_{i=1}^{k} d_k$ independent functions f_{m_i} in $K_*(f) \subset K_b$ therefore the two statements are equivalent. ∎

3. Cyclicity in H^p spaces where $1 < p < 2$

In this part, we will study the cyclicity of lacunary decompositions of functions in the spaces H^p where $1 < p < 2$. For that, we make the following observation,

Lemma 9. *Let u be an inner function and $f, g \in K_u^4$. Then $fg \in K_u^2 \oplus uK_u^2$.*

Proof. Let $f,\, g \in K_u^4$. The functions f and g can be respectively written in the form $f = ue^{-i\theta}\overline{\tilde{f}}$ and $g = ue^{-i\theta}\overline{\tilde{g}}$ with $\tilde{f}$ and $\tilde{g}$ in H^4. The products fg and $\tilde{f}\tilde{g}$ belong to H^2 and for every $h \in H^2$, we have

$$\begin{aligned}
\langle u^2 h \mid fg\rangle &= \int_0^{2\pi} u(e^{i\theta})^2 h(e^{i\theta}) \overline{f\,(e^{i\theta}) g\,(e^{i\theta})}\, dm(\theta) \\
&= \int_0^{2\pi} u(e^{i\theta})^2 h(e^{i\theta}) \overline{\Big[u(e^{i\theta}) e^{-i\theta} \overline{\tilde{f}(e^{i\theta})}\Big] \Big[u(e^{i\theta}) e^{-i\theta} \overline{\tilde{g}(e^{i\theta})}\Big]}\, dm(\theta) \\
&= \int_0^{2\pi} e^{2i\theta} h(e^{i\theta}) \tilde{f}\big(e^{i\theta}\big) \tilde{g}\big(e^{i\theta}\big)\, dm\,(\theta) = 0.
\end{aligned}$$

An immediate consequence of Theorem 2 is that $H^2 = K_u^2 \oplus uK_u^2 \oplus u^2 H^2$, consequently we necessarily have $fg \in K_u^2 \oplus uK_u^2$. ∎

We give here an extended version of a result which exists in the scalar case (see [19], p. 115). If Λ is a set of integers, we call by $\mathcal{E}_{b,\Lambda}$ the vectorial space formed by the sums with finite support of type $\sum_{k\in\Lambda, k\leq p} f_k b^k$ where $f_k \in K_b$. We denote by $E_{b,\Lambda}$ the closure in $L^1(dm)$ of the vectorial subspace $\mathcal{E}_{b,\Lambda}$.

Theorem 7. *Let $M \in \mathbb{N}^*$ and Λ be an infinite set in $\mathbb{Z}$ such that no integer has more than M representations as the sum of two elements of Λ. Then the space $E_{b,\Lambda}$ is a closed subset of $L^4(dm)$.*

Proof. Let $f = \sum_{k\in\Lambda} f_k b^k \in \mathcal{E}_{b,\Lambda}$, we have

$$\|f\|_4^4 = \|f^2\|_2^2 = \left\| \sum_{\substack{(k,l)\in\Lambda^2 \\ k+l\in 2\mathbb{Z}}} f_k f_l b^{k+l} + \sum_{\substack{(k,l)\in\Lambda^2 \\ k+l\in 2\mathbb{Z}+1}} f_k f_l b^{k+l} \right\|_2^2$$

$$\leq 2 \left\| \sum_{k\in\Lambda} f_k^2 b^{2k} + 2 \sum_{\substack{(k,l)\in\Lambda^2 \\ k+l\in 2\mathbb{Z},\, k<l}} f_k f_l b^{k+l} \right\|_2^2 + 8 \left\| \sum_{\substack{(k,l)\in\Lambda^2 \\ k+l\in 2\mathbb{Z}+1,\, k<l}} f_k f_l b^{k+l} \right\|_2^2 .$$

Considering the fact that an integer has not more than M representations as the sum of two elements of Λ and according to Lemma 9, we see that the terms occurring in the sums above can be rewritten as the sum of orthogonal blocks with at most M terms. For each of these functions we take the square of the norm in $L^2(dm)$ and we use the inequality $\|\varphi_1 + \cdots + \varphi_M\|_2^2 \leq M\big[\|\varphi_1\|_2^2 + \cdots + \|\varphi_M\|_2^2\big]$. We obtain,

$$\|f\|_4^4 \leq 2M \left[\sum_{k\in\Lambda} \|f_k^2\|_2^2 + 4 \sum_{\substack{(k,l)\in\Lambda^2 \\ k<l}} \|f_k f_l\|_2^2 \right]$$

$$\leq 2M \left[\sum_{k\in\Lambda} \|f_k\|_4^4 + 4 \sum_{\substack{(k,l)\in\Lambda^2 \\ k<l}} \|f_k\|_4^2 \, \|f_l\|_4^2 \right]$$

$$\leq 4M \left[\sum_{k\in\Lambda} \|f_k\|_4^2 \right]^2 \leq 4Mc^4 \left[\sum_{k\in\Lambda} \|f_k\|_2^2 \right]^2 = 4Mc^4 \, \|f\|_2^4 .$$

where c is the constant coming from the equivalence of the norms. $\|.\|_2$ and $\|.\|_4$ on the finite dimensional space K_b. Then we have $\|f\|_4 \leq 2^{\frac{1}{2}} M^{\frac{1}{4}} c \|f\|_2$. Using Hölder inequality with the conjugate exponents 3 and $\frac{3}{2}$, it comes that,

$$\|f\|_2^2 = \int_{\mathbb{T}} |f|^{\frac{4}{3}} |f|^{\frac{2}{3}} dm \leq \|f\|_4^{\frac{4}{3}} \, \|f\|_1^{\frac{2}{3}} \leq 2^{\frac{2}{3}} M^{\frac{1}{3}} c^{\frac{4}{3}} \|f\|_2^{\frac{4}{3}} \, \|f\|_1^{\frac{2}{3}} .$$

And finally, $\|f\|_4 \leq 2\sqrt{M} c^2 \|f\|_1$. The desired result is now a direct consequence of this inequality. ∎

THEOREM 8. *Let b be a finite Blaschke product and $T = T_{\overline{b}}$ be the associated antianalytic Toeplitz operator. We consider in H^p, where $1 < p \leq 2$, a* *function f which admits a lacunary decomposition with respect to the powers of b ($f = \sum_{k=1}^{\infty} f_k b^{n_k}$). Then f is cyclic if and only if one of the two last statements of Theorem 6 is satisfied.*

Proof. Let f be the lacunary decomposition of the statement. Since $f \in H^p$, $1 < p < 2$, and according to Theorem 2, the partial sums of the series $\sum_{k=1}^{\infty} f_k(z) b^{n_k}(z)$ converge to f in H^p, and also converge to f in H^1 and therefore $f \in E_{b,\Lambda}$ ($\Lambda = \{n_k\}$). Applying Theorem 7 we see that $f \in H^2$. According to Theorem 6 under the same hypothesis which are here fulfilled by f, f is cyclic and there exists a sequence of complex polynomials such that for every $h \in H^2$, $\lim_{n\to\infty} \|h - p_n(T)f\|_2 = 0$. Since $\|h - p_n(T)f\|_p \leq \|h - p_n(T)f\|_2$, we have

$$H^2 \subset \overline{span}_{H^p}\{T^n f : n \geq 0\}.$$

As H^2 is dense in H^p for $1 < p < 2$, we have $H^p = E_f^p$. ∎

References

[1] E.V. Abakumov, Cyclicity and approximation by lacunary power series, *Michigan. Math. J.* **42** (2) (1995), 277–299.

[2] A.B. Aleksandrov, Gap series and pseudocontinuations. An arithmetic approach, (Russian), *Algebra i Analiz* **9** (1) (1997), 3–31, translation in *St. Petersburg Math. J.* **9** (1) (1998), 1–20

[3] A.B. Aleksandrov, Lacunary series and pseudocontinuations, (Russian), *Investigations on linear operators and function theory. Part* 24, Zap. Nauchn. Sem. POMI, 232, POMI, St. Petersburg, 1996, 16–32, translation in *J. Math. Sci. (New York)* **92** (1) (1998), 3550–3559.

[4] J. Bergh, J. Löfström, "Interpolation Spaces, An Introduction", Grundlehren der Mathematischen Wissenschaften, 223, Springer-Verlag, Berlin-New York, 1976.

[5] L. Carleson, An interpolation problem for bounded analytic functions, *Amer. J. Math.* **80** (1958), 921–930.

[6] G. Cassier, Autour de quelques intéractions récentes entre l'analyse complexe et la théorie des opérateurs, in "Operator Theory and Banach Algebras", Proceeding of the International Conference in Analysis Rabat (Morocco), April 12-14, 1999, Theta, Bucharest 2003, 51–71.

[7] R. Choukrallah, "Lacunarité et Vecteurs Cycliques pour les Semi-Groupes de Translations à Gauche", Thèse de Doctorat, Université Bordeaux 1, 2006.

[8] R.G. Douglas, H.S. Shapiro, A.L. Shields, Cyclic vectors and invariant subspaces for the backward shift operator, *Ann. Inst. Fourier(Grenoble)* **20** (1) (1970), 37–76.

[9] C. Foias, B.Sz. Nagy, "Analyse Harmonique des Opérateurs de l'Espace de Hilbert", Masson et Cie, Paris, Akadémiai Kiadó, Budapest, 1967.

[10] K. Hoffman, "Banach Spaces of Analytic Functions", Prentice-Hall Series in Modern Analysis, Prentice-Hall, Englewood Cliffs, N.J., 1962.

[11] P. Jones, Interpolation between Hardy spaces, in "Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vol. II", (edited by W. Beckner, A. Calderòn, R. Fefferman and P. Jones), Wadsworth Inc., Belmont, CA, 1983, 437–451.

[12] K. Knopp, "Infinite Sequences and Series", Dover Publications, Inc., New York, 1956.

[13] T.L. Lance, M.I. Stessin, Multiplication invariant subspaces of Hardy spaces, *Canad. J. Math.* **49** (1) (1997), 100-118.

[14] J. Lindenstrauss, L. Tzafriri, "Classical Banach Spaces I", Springer Verlag, Berlin-New York, 1977.

[15] F. Riesz, B. Sz. Nagy, "Functional Analysis", Dover Publications, Inc., New York, 1990.

[16] M. Rosenblum, J. Rovnyak, "Hardy Classes and Operator Theory", Oxford University Press, New York, 1985.

[17] N.K. Nikolskii, "Operators, Functions and Systems: An Easy Reading, Volume 1: Hardy, Hankel and Toeplitz", Mathematical Surveys and Monographs, 92, American Mathematical Society, Providence, RI, 2002.

[18] N.K. Nikolskii, "Treatise on the Shift Operator", Springer-Verlag, Berlin, 1986.

[19] W. Rudin, "Analyse Fonctionnelle", Ediscience internationale, Paris, 1995.

[20] J.H. Shapiro, "Composition Operators and Classical Function Theory", Springer-Verlag, New York, 1993.

[21] I. Singer, "Bases in Banach Spaces I", Springer-Verlag, New York-Berlin, 1970.