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$SU(2)$ and $SL(2,\mathbb{C})$ Representations of a Class of Torus Knots

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Abstract: Let $K_{m,2}$ be the torus knot of type $(m,2)$. With the help of the explicit description of the $SL(2,\mathbb{C})$ character variety of this class of torus knots given by the author in a previous work, we study the relationship between the representations over $SU(2)$ and over $SL(2,\mathbb{C})$ of the fundamental group of $S^3 \setminus K_{m,2}$. In particular it is shown that the map from the moduli space of irreducible $SU(2)$-representations to the moduli space of $SL(2,\mathbb{C})$-representations is injective.

Key words: Character variety, Representation variety, Torus knot, $SU(2)$, $SL(2,\mathbb{C})$.

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1. Introduction

Let $K_{m,2}$ be the torus knot of type $(m,2)$ with odd m (see [8]) and denote by

$$H_m = \langle x, y \mid \underbrace{xyxy\cdots yx}_{\text{length } m} = \overbrace{yxyx\cdots xy}^{\text{lenght } m} \rangle$$

the fundamental group of its complement in S^3. In [2] the variety of representations $R(H_m)$ and the character variety $X(H_m)$ of H_m over $SL(2,\mathbb{C})$ were defined, together with the projection $t : R(H_m) \longrightarrow X(H_m)$ which, in our case, is given by $t(\rho) = (\mathrm{tr}\rho(x), \mathrm{tr}\rho(xy))$. In [7] an explicit geometric description of $X(H_m)$ was given via a refinement of the description obtained using the techniques in [3]. In [5] and [6] the analysis was extended to describe the $SL(2,\mathbb{C})$-character varieties of general torus knots. Finally, in [1] or [4] the space of representations over $SU(2)$, which is clearly a real algebraic subvariety of $R(H_m)$, was studied.

In the previous situation it is natural to be interested in the image under t of the representations over $SU(2)$. In particular this paper is devoted to compute explicitly (using the main result in [7]) $t(\tilde{R}(H_m))$, where $\tilde{R}(H_m)$ is the set of non-abelian representations over $SU(2)$. This computation will then be used to study the relationship between representations of H_m over $SL(2,\mathbb{C})$ and $SU(2)$.

2. Preliminaries

Let G be any group and $H \leq GL(2,\mathbb{C})$ be a subgroup of matrices. A representation $\rho : G \longrightarrow H$ is just a group homomorphism. We say that two representations ρ and ρ' are equivalent if there exists $P \in GL(2,\mathbb{C})$ such that $\rho'(g) = P^{-1}\rho(g)P$ for every $g \in G$. A representation ρ is reducible if the elements of $\rho(G)$ share a common eigenvector, otherwise it is irreducible. A representation ρ is abelian if $\rho(G)$ is an abelian subgroup of H. Note that abelian representations are always reducible.

Let $K_{m,2}$ be the torus knot of type $(m,2)$. The fundamental group of its complement admits a presentation [8]:

$$H_m = \langle x, y \mid \underbrace{xyxy\cdots yx}_{\text{length } m} = \overbrace{yxyx\cdots xy}^{\text{lenght } m} \rangle.$$

Writing $L_x = (xy)^{\frac{m-1}{2}}$, we see that $L_x x = y L_x$, so x and y are conjugate.

2.1. $SU(2)$-representation spaces.

In this section we will follow the notation in [1]. We also refer to [4] for a good account on this topic. The following well-known lemma will be useful in the sequel.

Lemma 1. *$SU(2) \cong S^3$ where, if we see S^3 as the set of unit quaternions, the isomorphism is given by:*

$$a_0 + a_1 i + a_2 j + a_3 k \longleftrightarrow \begin{pmatrix} a_0 + a_1 i & a_2 + a_3 i \\ a_2 - a_3 i & a_0 - a_1 i \end{pmatrix}.$$

In particular, $trA \in \mathbb{R}$ for every $A \in SU(2)$.

Recall that any element of S^3 can be written in the form $(P,\varphi) = \cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}P$ with $P^2 = -1$ being a pure quaternion. Since the group H_m is generated by x and y, a representation ρ over $SU(2)$ will be determined by the images of its generators $\rho(x) = (P, \varphi)$ and $\rho(y) = (Q, \varphi)$, where the angle $\varphi \in [0, 2\pi]$ is the same for $\rho(x)$ and $\rho(y)$ because x and y are conjugate. Moreover, this representation will be non-abelian if $P \neq \pm Q$ and $\varphi \neq 2k\pi$.

Now, if ψ denotes the unoriented angle between the (oriented) axes P and Q and putting $\tau = \cos\psi$, $\gamma = \cot\frac{\varphi}{2}$ it can be seen that there exists a bijection between the set

$$\mathcal{R}(H_m) = \{(\tau, \gamma) \in \mathbb{R}^2 \mid \tau = \cos\psi,\ \gamma = \cot\frac{\varphi}{2}\}$$

and the set of $SU(2)$-equivalence classes of non-abelian representations of H_m over $SU(2)$ [1, p. 104]. Note that $\psi \in (0, \pi)$ hence $\tau \in (-1, 1)$, $\gamma \in \mathbb{R}$.

Let us define a family of polynomials in the following recursive way ([1, Lemma 1.2.]):

$$z_0(\tau, \gamma) = 1,$$
$$z_1(\tau, \gamma) = \gamma^2 - 2\tau - 1,$$
$$z_2(\tau, \gamma) = \gamma^4 + \gamma^2(-6\tau - 4) + 4\tau^2 + 2\tau - 1,$$
$$z_n(\tau, \gamma) = (\gamma^2 - 1 - 2\tau)(z_{n-1} + (\gamma^2 + 1)z_{n-2}) - (\gamma^2 + 1)^3 z_{n-3},\ n \geq 3.$$

Given the group H_m with odd m, let $\mathcal{C}(H_m)$ be the plane algebraic curve

$$\mathcal{C}(H_m) = \{(\tau, \gamma) \in \mathbb{R}^2 \mid z_{\frac{m-1}{2}}(\tau, \gamma) = 0\}.$$

If we consider $D = \{(\tau, \gamma) \in \mathbb{R}^2 \mid -1 < \tau < 1\}$, it can be seen [1, Theorem 1.3.] that $\mathcal{R}(H_m) = \mathcal{C}(H_m) \cap D$.

2.2. $SL(2, \mathbb{C})$-REPRESENTATION AND CHARACTER VARIETIES. Like in the previous section, consider the group H_m. Then the set

$$R(H_m) = \{(\rho(x), \rho(y)) \mid \rho \text{ is a representation of } H_m \text{ over } SL(2, \mathbb{C})\}$$

is (see [2]) a well-defined affine algebraic set, up to canonical isomorphism.

Recall that given a representation $\rho : H_m \longrightarrow SL(2, \mathbb{C})$ its character $\chi_\rho : H_m \longrightarrow \mathbb{C}$ is defined by $\chi_\rho(g) = \mathrm{tr}\rho(g)$. Note that two equivalent representations ρ and ρ' have the same character, and the converse is also true if either ρ or ρ' is irreducible [2, Prop. 1.5.2.]. Now, choose any $g \in H_m$ and define $t_g : R(H_m) \longrightarrow \mathbb{C}$ by $t_g(\rho) = \chi_\rho(g)$. Let T denote the ring generated by $\{t_g \mid g \in H_m\}$, then ([2, Prop. 1.4.1.]) T is a finitely generated ring and using the well-known identities (A, $B \in SL(2, \mathbb{C})$):

$$\mathrm{tr}A = \mathrm{tr}A^{-1}, \quad \mathrm{tr}AB = \mathrm{tr}BA, \quad \mathrm{tr}AB = \mathrm{tr}A\mathrm{tr}B - \mathrm{tr}AB^{-1},$$

it can be shown [3, Cor. 4.1.2.] that T is generated by the set $\{t_x, t_{xy}\}$. Note that x and y being conjugate, $t_x = t_y$.

Now define the map $t : R(H_m) \longrightarrow \mathbb{C}^2$ by $t(\rho) = (t_x(\rho), t_{xy}(\rho))$. Put $X(H_m) = t(R(H_m))$, then $X(H_m)$ is an algebraic variety which is well-defined up to canonical isomorphism [2, Cor 1.4.5.] and which is called the character variety of the group H_m in $SL(2, \mathbb{C})$. Note that $X(H_m)$ is the set of all characters χ_ρ of representations $\rho \in R(H_m)$.

We are now interested in giving a more explicit description of $X(H_m)$ (see [3, 7] for details). Let us start by recursively defining a family of polynomials $\{q_n\}_{n\geq 1}$:

$$q_1(T) = T - 2,$$
$$q_2(T) = T + 2,$$
$$\prod_{1\neq d|n} q_d\left(X + \frac{1}{X}\right) = \frac{X^{n-1} + X^{n-2} + \cdots + X + 1}{X^{\frac{n-1}{2}}} \text{ if } n \geq 3 \text{ is odd,}$$
$$\prod_{1,2\neq d|n} q_d\left(X + \frac{1}{X}\right) = \frac{X^{n-2} + X^{n-4} + \cdots + X^2 + 1}{X^{\frac{n-2}{2}}} \text{ if } n \geq 4 \text{ is even.}$$

Observe that if we denote by $\{c_n\}_{n\geq 1}$ the family of cyclotomic polynomials, then for $n \geq 3$ it holds that

$$c_n(X) = X^{\frac{\varphi(n)}{2}} q_n\left(X + \frac{1}{X}\right).$$

With this we have the following description.

Proposition 1. ([7], Cor. 4.3.)

$$X(H_m) \cong \{(X, Z) \in \mathbb{C}^2 \mid (X^2 - Z - 2) \prod_{1\neq d|n} q_d^*(Z) = 0\},$$

where $q_d^*(Z) = (-1)^{\deg q_d} q_d(-Z)$.

3. Computing $t(\tilde{R}(H_m))$

Let $\rho : H_m \longrightarrow SU(2)$ be a non-abelian representation given by

$$\begin{cases} \rho(x) = (P, \varphi) = \cos\frac{\varphi}{2} + P\sin\frac{\varphi}{2} \\ \rho(y) = (Q, \varphi) = \cos\frac{\varphi}{2} + Q\sin\frac{\varphi}{2} \end{cases}$$

where $P = ai + bj + ck$ and $Q = a'i + b'j + c'k$ are pure unit quaternions such that $P^2 = Q^2 = -1$, $P \neq \pm Q$ and $\varphi \in (0, 2\pi)$. By some straightforward computations we obtain that

$$\rho(xy) = \rho(x)\rho(y) = \cos^2\frac{\varphi}{2} - \langle P, Q\rangle \sin^2\frac{\varphi}{2} + (P + Q)\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}.$$

And making use of Lemma 1 we get $\mathrm{tr}\rho(x) = \mathrm{tr}\rho(y) = 2\cos\frac{\varphi}{2}$ and $\mathrm{tr}\rho(xy) = 2\cos^2\frac{\varphi}{2} - 2\langle P, Q\rangle \sin^2\frac{\varphi}{2}$.

Recalling the notation from Section 2, the previous paragraph leads to:

$$\tau = \cos\psi = \langle P, Q\rangle,$$

$$\gamma = \cot\frac{\varphi}{2},$$

$$X = \mathrm{tr}\rho(x) = \mathrm{tr}\rho(y) = 2\cos\frac{\varphi}{2},$$

$$Z = \mathrm{tr}\rho(xy) = 2\cos^2\frac{\varphi}{2} - 2\langle P, Q\rangle \sin^2\frac{\varphi}{2}.$$

Now, some simple computations give:

$$\gamma^2 = \frac{X^2}{4 - X^2}, \quad \tau = \frac{X^2 - 2Z}{4 - X^2},$$

$$X^2 = \frac{4\gamma^2}{1 + \gamma^2}, \quad Z = \frac{2\gamma^2(1 - \tau) - 2\tau(1 + \gamma^2)}{1 + \gamma^2}.$$

If we put $\omega = \gamma^2$, a close look at its definition shows that we can write $z_n(\tau, \gamma)$ as a polynomial in the variables (τ, ω). As a consequence we can consider the rational function $\tilde{z}_n(X, Z) = z_n(\tau(X, Z), \omega(X, Z)) \in \mathbb{Z}(X, Z)$ which allows us to define the polynomial $\hat{z}_n(X, Z) = (4 - X^2)^n \tilde{z}_n(X, Z) \in \mathbb{Z}[X, Z]$. This polynomial defines a real plane algebraic curve $V_n = \{(X, Z) \in \mathbb{R}^2 \mid \hat{z}_n(X, Z) = 0\}$. In particular, given the group H_m we will define $V(H_m) = V_{\frac{m-1}{2}}$.

In Proposition 1 we defined the $SL(2, \mathbb{C})$ character variety of H_m as the complex algebraic curve

$$X(H_m) = \{(X, Z) \in \mathbb{C}^2 \mid (X^2 - Z - 2) \prod_{1 \neq d | m} q_d(-Z) = 0\} \subset \mathbb{C}^2.$$

Observe that we can see $X(H_m) \subset \mathbb{R}^4$ as a real affine algebraic set and consequently, if we define $X_{\mathbb{R}}(H_m) = X(H_m) \cap \{\mathrm{Im}X = \mathrm{Im}Z = 0\}$, then

$$X_{\mathbb{R}}(H_m) = \{(X,Z) \in \mathbb{R}^2 \mid (X^2 - Z - 2) \prod_{1 \neq d \mid m} q_d(-Z) = 0\} \subset \mathbb{R}^2$$

is a real plane algebraic curve.

Now we will se that $V(H_m) \subset X_{\mathbb{R}}(H_m)$. This is essentially proved in the following proposition.

Proposition 2. *For every $n \geq 1$,*

$$\hat{z}_n(X,Z) = 4^n \prod_{1 \neq d \mid 2n+1} q_d^*(Z).$$

Proof. We will proceed by induction on n. The cases $n = 1, 2, 3$ can be easily verified by direct computations. Now, if $n \geq 4$ we have that

$$z_n(\tau, \gamma) = (\gamma^2 - 1 - 2\tau)(z_{n-1} + (\gamma^2 + 1)z_{n-2}) - (\gamma^2 + 1)^3 z_{n-3},$$

and putting $\gamma^2 = \frac{X^2}{4-X^2}$ and $\tau = \frac{X^2-2Z}{4-X^2}$ this recurrence relation becomes:

$$\hat{z}_n(X,Z) = 4^n(Z-1)\Bigg(\prod_{1 \neq d \mid 2n-1} q_d^*(Z) + \prod_{1 \neq d \mid 2n-3} q_d^*(Z) \Bigg) - 4^n \prod_{1 \neq d \mid 2n-5} q_d^*(Z).$$

Finally, recalling the definition of the polynomials $\{q_d\}$ and setting $Z = W + \frac{1}{W}$ we get:

$$Z - 1 = \frac{W^2 - W + 1}{W},$$

$$\prod_{1 \neq d \mid 2n-1} q_d^*\left(W + \frac{1}{W}\right) = \frac{\sum_{i=0}^{2n-2}(-1)^i W^{2n-2-i}}{W^{n-1}},$$

$$\prod_{1 \neq d \mid 2n-3} q_d^*\left(W + \frac{1}{W}\right) = \frac{\sum_{i=0}^{2n-4}(-1)^i W^{2n-4-i}}{W^{n-2}},$$

$$\prod_{1 \neq d \mid 2n-5} q_d^*\left(W + \frac{1}{W}\right) = \frac{\sum_{i=0}^{2n-6}(-1)^i W^{2n-6-i}}{W^{n-3}}.$$

Now it is enough to substitute and operate in the previous relation to get the result. ∎

Remark 1. Observe that if two polynomials coincide over values of the form $a + \frac{1}{a}$, then they must be equal.

Corollary 1. *If $m \geq 1$ is odd,*

$$V(H_m) = \{(X, Z) \in \mathbb{R}^2 \mid \hat{z}_{\frac{m-1}{2}}(X, Z) = 0\} =$$
$$= \{(X, Z) \in \mathbb{R}^2 \mid \prod_{1 \neq d | m} q_d(-Z) = 0\} \subset X_{\mathbb{R}}(H_m).$$

is an algebraic subvariety. In particular, it consists of $\frac{m-1}{2}$ straight lines.

Proof. It is a clear consequence of the previous proposition. For the last assertion note that each q_d has $\frac{\varphi(d)}{2}$ distinct real roots. ∎

Let us now define the following subsets of $\mathbb{R}^2$:

$$E^+ = \left\{(a, b) \in \mathbb{R}^2 \mid 0 \leq a < 2,\ -2 < b < a^2 + 2\right\},$$

$$E^- = \left\{(a, b) \in \mathbb{R}^2 \mid -2 < a \leq 0,\ -2 < b < a^2 + 2\right\}$$

and put $E = E^+ \cup E^-$. The following lemma is easy to prove.

Lemma 2. *Recall that $D = \{(\tau, \gamma) \mid \tau \in (-1, 1),\ \gamma \in \mathbb{R}\}$. The applications $f^+ : D \longrightarrow E^+$ and $f^- : D \longrightarrow E^-$ given by:*

$$f^+(\tau, \gamma) = \left(+\sqrt{\frac{4\gamma^2}{1+\gamma^2}}, \frac{2\gamma^2(1-\tau) - 2\tau(1+\gamma^2)}{1+\gamma^2}\right),$$

$$f^-(\tau, \gamma) = \left(-\sqrt{\frac{4\gamma^2}{1+\gamma^2}}, \frac{2\gamma^2(1-\tau) - 2\tau(1+\gamma^2)}{1+\gamma^2}\right),$$

are well-defined and surjective.

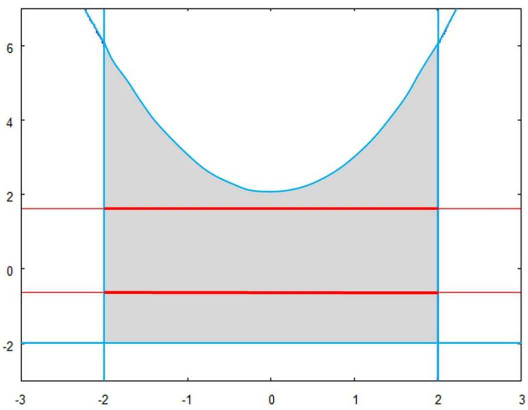

Figure 1: $V(H_5) \cap E$

EXAMPLE 1. In Figure 1 we can see the intersection between the region E, limited by the lines $x = -2$, $x = 2$, $y = -2$ and $y = x^2 + 2$, and the curve $V(H_5)$ which is given (due to Corollary 1) by $y^2 - y - 1 = 0$ and consists of 2 horizontal straight lines.

Recall that the set of non-abelian representations of H_m over $SU(2)$ was bijective to $\mathcal{R}(H_m) = \mathcal{C}(H_m) \cap D$. Thus, it makes sense to study the restrictions $f^{\pm}|_{\mathcal{R}(H_m)}$.

LEMMA 3. *$f^{\sigma}|_{\mathcal{R}(H_m)} : \mathcal{R}(H_m) \longrightarrow V(H_m) \cap E^{\sigma}$ is surjective for $\sigma = \pm$.*

Proof. We will focus on the case $\sigma = +$, the other one being analogous. Let us assume that $(\tau_0, \gamma_0) \in \mathcal{R}(H_m)$ and put $(X_0, Z_0) = f^+(\tau_0, \gamma_0)$. By the previous lemma, and since $(\tau_0, \gamma_0) \in \mathcal{R}(H_m) = \mathcal{C}(H_m) \cap D$, it is clear that $(X_0, Z_0) \in E^+$. Moreover,

$$0 = z_{\frac{m-1}{2}}(\tau_0, \gamma_0) = \tilde{z}_{\frac{m-1}{2}}(X_0, Z_0) = \frac{\hat{z}_{\frac{m-1}{2}}(X_0, Z_0)}{(4 - X_0^2)^{\frac{m-1}{2}}}$$

with $X_0 \neq 2$, so $(X_0, Z_0) \in V(H_m)$ and we have that $f^+(\mathcal{R}(H_m)) \subseteq V(H_m) \cap E^+$.

Now, let $(X_0, Z_0) \in V(H_m) \cap E^+$. Since f^+ is surjective we can choose $(\tau_0, \gamma_0) \in D$ such that $f^+(\tau_0, \gamma_0) = (X_0, Z_0)$. On the other hand,

$$0 = \hat{z}_{\frac{m-1}{2}}(X_0, Z_0) = (4 - X_0^2)^{\frac{m-1}{2}} \tilde{z}_{\frac{m-1}{2}}(X_0, Z_0) = z_{\frac{m-1}{2}}(\tau_0, \gamma_0)$$

so $(\tau_0, \gamma_0) \in \mathcal{C}(H_m)$ and surjectivity follows. ∎

Remark 2. Let $(\tau_0, \gamma_0) \in \mathcal{C}(H_m) \cap D$ be a point such that $f^\sigma(\tau_0, \gamma_0) = (X_0, Z_0)$. Then the point (τ_0, γ_0) determines the equivalence class of a non-abelian representation ρ of H_m over $SU(2)$. Clearly we have that $t(\rho) = (X_0, Z_0)$, where $t : R(H_m) \longrightarrow X(H_m)$ is the projection defined in Section 1.

We are now in the conditions to prove the main result of the paper.

THEOREM 1. $t(\tilde{R}(H_m)) = V(H_m) \cap E$.

Proof. If $\rho \in \tilde{R}(H_m)$, let $(\tau_0, \gamma_0) \in \mathcal{R}(H_m)$ be the point given by the identification between $\mathcal{R}(H_m)$ and the set of $SU(2)$-equivalence classes of elements of $\tilde{R}(H_m)$ (recall Section 2.1). If we put $(X_0, Z_0) = t(\rho)$ we have already seen that $(X_0, Z_0) = f^\sigma(\tau_0, \gamma_0)$ with $\sigma = \pm$ and it is enough to apply the previous lemma.

Conversely, if $(X_0, Z_0) \in V(H_m) \cap E$ we choose $(\tau_0, \gamma_0) \in (f^\sigma)^{-1}(X_0, Z_0)$ with $\sigma = \pm$ (recall that $E = E^+ \cup E^-$) and the result follows from the remark above. ∎

We can give an interpretation of the previous result in terms of representations.

COROLLARY 2. *Let $\rho : H_m \longrightarrow SL(2, \mathbb{C})$ be an irreducible representation such that $(tr\rho(x), tr\rho(xy)) \in X_{\mathbb{R}}(H_m) \cap E$. Then, there exists a representation $\rho' : H_m \longrightarrow SU(2)$ such that ρ and ρ' are equivalent.*

Proof. If ρ is irreducible, then $t(\rho) \in V(H_m)$. Thus, $t(\rho) \in V(H_m) \cap E = t(\tilde{R}(H_m))$ and there exists $\rho' \in \tilde{R}(H_m)$ such that $t(\rho) = t(\rho')$ and in these conditions ρ and ρ' are equivalent due to [2, Prop. 1.5.2.]. ∎

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