Remarks on Gurariĭ Spaces

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Abstract: We present selected known results and some new observations, involving Gurariĭ spaces. A Banach space is *Gurariĭ* if it has certain natural extension property for almost isometric embeddings of finite-dimensional spaces. Deleting the word "almost", we get the notion of a *strong Gurariĭ* space. There exists a unique (up to isometry) separable Gurariĭ space, however strong Gurariĭ spaces cannot be separable. The structure of the class of non-separable Gurariĭ spaces seems to be not very well understood. We discuss some of their properties and state some open questions. In particular, we characterize non-separable Gurariĭ spaces in terms of skeletons of separable subspaces, we construct a non-separable Gurariĭ space with a projectional resolution of the identity and we show that no strong Gurariĭ space can be weakly Lindelöf determined.

Key words: Gurariĭ space, (almost) linear isometry, universal disposition, projection, rotund renorming, complementation.

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Introduction

The *Gurariĭ space*, constructed by Gurariĭ [7] in 1965, is the unique separable Banach space $\mathbb{G}$ satisfying the following condition: Given finite-dimensional Banach spaces $X \subseteq Y$, given $\varepsilon > 0$, given an isometric linear embedding $f\colon X \to \mathbb{G}$ there exists an injective linear operator $g\colon Y \to \mathbb{G}$ extending f and satisfying $\|g\| \cdot \|g^{-1}\| \leqslant 1 + \varepsilon$. Almost straight from this definition, it is not hard to prove that such a space is unique up to isomorphism of norm arbitrarily close to one. Surprisingly, it has been unknown for some time whether the Gurariĭ space is unique up to isometry; it was answered affirmatively by

Lusky [17] in 1976. His proof used the method of *representing matrices*, explored earlier by Lazar and Lindenstrauss [14]. Very recently, Solecki and the second author [12] have found a simple and elementary proof of the uniqueness of the Gurariĭ space. We sketch the arguments in Section 2 below.

The defining condition of a Gurariĭ space can clearly be applied to non-separable spaces, obtaining the notion of a *Gurariĭ space*. Removing ε from the definition, one gets the notion of a *strong Gurariĭ space*. Besides their existence, not much is known about the structure of strong Gurariĭ spaces. Few years ago, the second author found, assuming the continuum hypothesis, a unique Banach space V of density continuum and satisfying the following stronger property: every isometric embedding $f \colon S \to V$ from a subspace of an arbitrary fixed separable space T can be extended to an isometric embedding $g \colon T \to V$. In fact, this is a special case of a general theory of *Fraïssé-Jónsson limits*. Recently, the authors of [3] developed the idea of "generating" Banach spaces by using pushouts, finding strong Gurariĭ spaces of arbitrarily large density above the continuum.

In this note we survey the basic properties of the separable Gurariĭ space, we explain the pushout constructions, and we characterize Gurariĭ spaces in terms of skeletons of separable spaces. We also show that Banach spaces constructed by pushout iterations from finite-dimensional spaces are not universal for spaces of density $\aleph_1$. More specifically, we show that every copy of c_0 is complemented in such spaces. Finally, we state some questions regarding the structure of Gurariĭ spaces.

The paper is organized as follows.

Contents

Section 1 contains the basic definitions and an overview of the Pushout Lemma, crucial for the existence of Gurariĭ spaces. Section 2 has a survey character. We introduce Gurariĭ spaces, describe two natural constructions, and sketch the proof of their isometric uniqueness. We also provide a proof of the result of Wojtaszczyk [20] on 1-complemented subspaces of the Gurariĭ space. Section 3 studies non-separable Gurariĭ spaces. We characterize them in terms of skeletons of separable subspaces. As an application, we observe that no Gurariĭ space is complemented in a $C(K)$ space and we prove that every Banach space embeds isometrically into a Gurariĭ space of the same density. We also show that there exists a Gurariĭ space of density $\aleph_1$ and with a projectional resolution of the identity. Section 4 deals with a natural generalization of the notion of a strong Gurariĭ space, when the class of finite-dimensional spaces is replaced by a larger class $\mathcal{K}$. The property is then called "universal disposition for $\mathcal{K}$". We review the "pushout construction" which is the main tool in [3] for constructing spaces of universal disposition for various classes. Section 5 addresses the structure of strong Gurariĭ spaces. Using the fact that the Gurariĭ space is not 1-injective for finite-dimensional spaces, we observe that strong Gurariĭ spaces cannot contain skeletons of 1-complemented separable subspaces; in particular no weakly compactly generated space can be a strong Gurariĭ space. We finally show, using some arguments from [2] that strong Gurariĭ spaces constructed by pushout iterations in [3] have the property that every copy of c_0 is complemented. Section 7 contains some concluding remarks and some open questions.

1. Preliminaries

We shall use standard notions concerning Banach spaces and linear operators (all linear operator are, by default, bounded). We shall consider real Banach spaces, although the result are valid for the complex case, without any significant changes.

The following well-known notion will be used throughout this paper. Let X, Y be Banach spaces, $\varepsilon > 0$. A linear operator $f\colon X \to Y$ is an *ε-isometric embedding* if

$$(1 + \varepsilon)^{-1} \cdot \|x\| \leqslant \|f(x)\| \leqslant (1 + \varepsilon) \cdot \|x\|.$$

holds for every $x \in X \setminus \{0\}$. If the above condition holds with strict inequalities, we shall say that f is a *strict ε-isometric embedding*. An operator f is an *isometric embedding* iff it is an ε-isometric embedding with $\varepsilon = 0$. A bijective (ε-)isometric embedding is called an (ε-)*isometry*. (The word "isometry"

always means "linear isometry".) Two Banach spaces are *linearly isometric* if there exists a linear isometry between. Two Banach spaces are *almost linearly isometric* if for every $\varepsilon > 0$ there exists a linear ε-isometry between them. Two norms on the same Banach space are *ε-equivalent* if the identity is an ε-isometry.

We shall need the following simple and standard fact on extending equivalent norms.

LEMMA 1.1. *Let $E \subseteq F$ be Banach spaces, $\varepsilon > 0$ and let $|\cdot|_E$ be a norm on E that is ε-equivalent to the original norm of E (inherited from F). Then there exists a norm $|\cdot|_F$ that extends $|\cdot|_E$ and is ε-equivalent to the original norm of F.*

Proof. Let $\|\cdot\|$ be the original norm of F and let $S = \{\varphi \in E^* \colon |\varphi|_E = 1\}$ be the dual sphere in E^* with respect to $|\cdot|_E$. Then $\|\varphi\| \leqslant 1 + \varepsilon$ for every $\varphi \in S$. Given $y \in F$, define

$$|y|_F = \sup\{\psi(y) \colon \psi \upharpoonright E \in S \text{ and } \|\psi\| \leqslant 1 + \varepsilon\}.$$

It is clear that $|\cdot|_F$ extends $|\cdot|_E$ and is ε-equivalent $\|\cdot\|$.

We finish this section with the rather well-known, important category-theoretic property of Banach spaces, crucial for the existence of Gurariĭ spaces.

LEMMA 1.2. (The Pushout Lemma) *Let Z, X, Y be Banach spaces, let $i \colon Z \to X$ be an isometric embedding and let $f \colon Z \to Y$ be an ε-isometric embedding, where $\varepsilon > 0$. Then there exist a Banach space W, an isometric embedding $j \colon Y \to W$ and an ε-isometric embedding $g \colon X \to W$ for which the diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\ j\ } & W \\ {\scriptstyle f}\big\uparrow & & \big\uparrow{\scriptstyle g} \\ Z & \xrightarrow[\ i\ ]{} & X \end{array}$$

commutes. Furthermore, if X, Y are finite-dimensional then so is W.

Proof. For simplicity, let us assume that i is the inclusion $Z \subseteq X$. Define $W = (X \oplus Y)/\Delta$, where $X \oplus Y$ is endowed with the ℓ_1 norm and

$$\Delta = \{\langle z, -f(z)\rangle \colon z \in Z\}.$$

Let g and j be the quotients of the canonical embeddings, i.e. $g(x) = \langle x, 0 \rangle + \Delta$ and $j(y) = \langle 0, y \rangle + \Delta$ for $x \in X$, $y \in Y$. Observe that

$$\|g(x)\| = \inf_{z \in Z} \Big(\|x + z\|_X + \| - f(z)\|_Y \Big) \leqslant \|x\|_X.$$

Similarly,

$$\|j(y)\| = \inf_{z \in Z} \Big(\|z\|_X + \|y - f(z)\|_Y \Big) \leqslant \|y\|_Y.$$

It remains to estimate $\|g(x)\|$ and $\|j(y)\|$ from below.

Fix $x \in X$. Given $z \in Z$, we have

$$\|x + z\|_X + \| - f(z)\|_Y \geqslant (1 + \varepsilon)^{-1} \big(\|x + z\|_X + \| - z\|_X \big) \geqslant (1 + \varepsilon)^{-1} \|x\|_X.$$

It follows that $\|g(x)\| \geqslant (1 + \varepsilon)^{-1} \|x\|_X$.

Now fix $y \in Y$. Given $z \in Z$, we have

$$\|z\|_X + \|y - f(z)\|_Y \geqslant (1 + \varepsilon)^{-1} \big(\|f(z)\|_Y + \|y - f(z)\|_Y \big) \geqslant \|y\|_Y.$$

Thus $\|j(y)\| \geqslant \|y\|_Y$. This completes the proof.

Note that Lemma 1.1 can be viewed as a special case of the Pushout Lemma.

We shall use several times the "isometric" version of the Pushout Lemma: Namely, if f in the statement above is an isometric embedding then so is g. Note also that the lemma above is valid when "ε-isometric embedding" is replaced by "linear operator of norm $\leqslant 1 + \varepsilon$". The proof is the same (see [3] for more details).

A word of explanation on the name "Pushout Lemma" is in place. Namely, the commutative square from the lemma is usually called an *amalgamation* of X and Y or, more precisely, of i and f. It turns out however that the amalgamation constructed in the proof is the pushout of i and f in the category of Banach spaces with bounded linear operators. Specifically, given arbitrary bounded linear operators $T\colon X \to V$, $S\colon Y \to V$ such that $T \circ i = S \circ f$, there exists a unique linear operator $h\colon W \to V$ satisfying $h \circ g = T$ and $h \circ j = S$. Finally, the norm of h does not exceed $\max(\|T\|, \|S\|)$.

Recall that a space $Y \subseteq X$ is *complemented* (more precisely: *k-complemented*) in X if there exists a projection $P\colon X \to X$ of norm $\leqslant k$ and such that $Y = \operatorname{im} P$. Officially, P is a *projection* if $P^2 = P$, however we shall say that a linear operator $P\colon X \to Y$ is a *projection* if $Y \subseteq X$ and $P \upharpoonright Y = \operatorname{id}_Y$. It is clear that both definitions lead to the same concept.

Coming back to the previous remarks, the following property of a pushout deserves some attention:

LEMMA 1.3. (cf. [3]) *Under the assumptions of the Pushout Lemma, if f is a linear operator of norm $\leqslant 1$ and $i[Z]$ is k-complemented in X then $j[Y]$ is k-complemented in W.*

Furthermore, if i and f are inclusions then every bounded projection from X onto Z extends to a projection from W onto Y, preserving the norm.

Proof. Let $P\colon X \to Z$ be such that $P \circ i = \mathrm{id}_Z$ and $\|P\| \leqslant k$. Define $T = f \circ P$ and $S = \mathrm{id}_Y$. Then $\|T\| \leqslant k\|f\| \leqslant k$, $T \circ i = S \circ f$, therefore by the property of the pushout, there exists a unique operator $h\colon W \to Z$ of norm $\leqslant k$, such that $h \circ g = T$ and $h \circ j = S = \mathrm{id}_Y$. In particular, $j \circ h$ gives a projection onto $j[Y] \subseteq W$. Finally, if i and f are inclusions then $h \circ g = T$ translates to $h \restriction X = P$. ∎

Recall that a finite-dimensional Banach space X is *polyhedral* if its unit ball is a polyhedron. In other words, there exist functionals $\varphi_0, \varphi_1, \ldots, \varphi_{m-1} \in X^*$ such that

$$\|x\| = \max_{i<m} |\varphi_i(x)|$$

for every $x \in X$. An infinite-dimensional Banach space is called *polyhedral* if each of its finite-dimensional subspaces is polyhedral. Typical examples of polyhedral Banach spaces are $\ell_1(n)$ and $\ell_\infty(n)$, the n-dimensional variants of ℓ_1 and ℓ_∞, respectively. The spaces $\ell_\infty(n)$ play a special role, due to the following two facts.

PROPOSITION 1.4. *A finite-dimensional Banach space is polyhedral if and only if it embeds isometrically into $\ell_\infty(n)$ for some $n \in \mathbb{N}$.*

Proof. Let the norm of X be of the form

$$\|x\| = \max_{i<n} |\varphi_i(x)|,$$

where $\varphi_0, \ldots, \varphi_{n-1} \in X^*$. Define $e\colon X \to \ell_\infty(n)$ by $e(x)(i) = \varphi_i(x)$. It is clear that e is an isometric embedding.

Conversely, it is obvious that $\ell_\infty(n)$ is polyhedral, and every subspace of a polyhedral space is polyhedral. ∎

Recall that a Banach space X is 1-*injective* if, given a pair of Banach spaces $E \subseteq F$, every bounded linear operator $T\colon E \to X$ extends to a linear operator $\tilde{T}\colon F \to X$ so that $\|\tilde{T}\| = \|T\|$ holds.

PROPOSITION 1.5. *For every $n \in \mathbb{N}$ the space $\ell_\infty(n)$ is 1-injective.*

Proof. Fix T and define $T_i(x) = T(x)(i)$ for $x \in E$. By the Hahn-Banach Theorem, each T_i extends to a linear functional $\tilde{T}_i$, preserving the norm. Define $\tilde{T}(x)(i) = \tilde{T}_i(x)$ for $x \in F$. It is clear that $\tilde{T}$ extends T and $\|\tilde{T}\| = \|T\|$. ∎

The proof of the following fact is an easy exercise, noticing that the norm of the pushout space is the convex hull of two polyhedra.

LEMMA 1.6. *Let $i\colon Z \to X$, $j\colon Z \to Y$ be two isometric embeddings of finite-dimensional polyhedral spaces. Then there exist a polyhedral space W and isometric embeddings $i'\colon X \to W$ and $j'\colon Y \to W$ such that the square*

$$
\begin{array}{ccc}
Y & \xrightarrow{\ j'\ } & W \\
\big\uparrow{\scriptstyle j} & & \big\uparrow{\scriptstyle i'} \\
Z & \xrightarrow[\ i\ ]{} & X
\end{array}
$$

is commutative. Furthermore, W can be taken to be the space coming from the Pushout Lemma.

2. THE SEPARABLE GURARIĬ SPACE

This section has a survey character. We introduce the definition of a Gurariĭ space, show its existence, uniqueness and basic properties.

We start with some general definitions, originally due to Gurariĭ [7].

DEFINITION 2.1. Let $\mathfrak{K}$ be a class of Banach spaces. A Banach space X is of *(almost) universal disposition* for $\mathfrak{K}$ if for every pair of spaces $S \subseteq T$, both in $\mathfrak{K}$, for every isometric embedding $f\colon S \to X$ (and for every $\varepsilon > 0$), there exists an (ε-)isometric embedding $g\colon T \to X$ such that $g \restriction S = f$. If this holds, we shall write briefly "X is (almost) UD($\mathfrak{K}$)". We shall write UD(fin-dim) and UD(sep) for "universal disposition for finite-dimensional spaces" and "universal disposition for separable spaces", respectively.

DEFINITION 2.2. A Banach space is *Gurariĭ* if it is of almost universal disposition for finite-dimensional spaces. A *strong Gurariĭ space* is a Banach space of universal disposition for finite-dimensional spaces.

The starting point of our study is the following result.

THEOREM 2.3. (Gurariĭ [7]) *There exists a separable Gurariĭ space.*

We shall present two constructions in Subsection 2.3.

2.1. ISOMETRIC UNIQUENESS. A standard back-and-forth argument shows that every two separable Gurariĭ spaces are almost isometric. Below we sketch the arguments showing isometric uniqueness.

The following lemmas come from [12]. The proof of the first one is a bit technical, yet completely elementary. The second lemma follows directly from the first one, applying the definition of a Gurariĭ space.

LEMMA 2.4. *Let $f\colon X \to Y$ be a strict ε-isometric embedding of Banach spaces, $\varepsilon > 0$. Then there exist a Banach space Z and isometric embeddings $g\colon Y \to Z$, $h\colon X \to Z$, such that $\|g \circ f - h\| < \varepsilon$.*

LEMMA 2.5. *Let G be a Gurariĭ space. Then for every pair X, Y of finite-dimensional Banach spaces such that $X \subseteq G$, for every $\varepsilon > 0$, for every $\delta > 0$, for every strict ε-isometric embedding $f\colon X \to Y$ there exists a δ-isometric embedding $j\colon Y \to G$ such that $\|jf(x) - x\| < \varepsilon\|x\|$ for every non-zero $x \in X$.*

THEOREM 2.6. (Lusky [17]) *Every two separable Gurariĭ spaces are linearly isometric.*

Proof. Let E and F be two separable Gurariĭ spaces. Define inductively two sequences of linear operators $f_n\colon X_n \to Y_n$ and $g_n\colon Y_n \to X_{n+1}$ satisfying the following conditions.

(i) $X_n \subseteq E$ and $Y_n \subseteq F$ are finite-dimensional spaces.

(ii) f_n and g_n are 2^{-n}-isometric embeddings.

(iii) $\|g_n f_n(x) - x\| < 2^{-n}\|x\|$ for every $x \in X_n \setminus \{0\}$.

(iv) $\|f_{n+1} g_n(y) - y\| < 2^{-n}\|y\|$ for every $y \in Y_n \setminus \{0\}$.

We start with $X_0 = 0$ and we take Y_0 to be any finite-dimensional subspace of F. We find g_0 by using Lemma 2.5. Having defined f_n and g_n, we use Lemma 2.5 both for E and F to find first f_{n+1} and next g_{n+1}. Note that we have some freedom to choose the subspaces X_{n+1} and Y_{n+1}. Thus, the inductive construction can be carried out so that $\bigcup_{n\in\omega} X_n$ is dense in E and $\bigcup_{n\in\omega} Y_n$ is dense in F.

Given $x \in X_n$, using (iv) and (ii), we have

$$\|f_n(x) - f_{n+1}g_n f_n(x)\| < 2^{-n}\|f_n(x)\| \leqslant 2^{-n+1}.$$

Similarly, using (ii) and (iii), we get

$$\|f_{n+1}(x) - f_{n+1}g_n f_n(x)\| \leqslant \|f_{n+1}\| \cdot \|x - g_n f_n(x)\| < 2^{-n+1}.$$

Thus $\|f_n(x) - f_{n+1}(x)\| < 2^{-n+2}$. It follows that the sequence $\{f_n\}_{n \in \omega}$ is pointwise convergent. Its limit extends uniquely to an isometry $f_\infty \colon E \to F$. The same arguments show that $\{g_n\}_{n \in \omega}$ pointwise converges to an isometry $g_\infty \colon F \to E$. Finally, (iii) and (iv) show that $g_\infty \circ f_\infty = \mathrm{id}_E$ and $f_\infty \circ g_\infty = \mathrm{id}_F$. ∎

From now on, we can speak about *the* Gurariĭ space, the unique separable space of almost universal disposition for finite-dimensional spaces. This space will always be denoted by $\mathbb{G}$.

The proof above is actually a simplified version of that in [12], where it is shown that for every strict ε-isometry f between finite-dimensional subspaces of $\mathbb{G}$ there exists a bijective isometry $h \colon \mathbb{G} \to \mathbb{G}$ such that $\|f - h\| < \varepsilon$.

2.2. A CRITERION FOR BEING GURARIĬ. Note that there are continuum many isometric types of finite-dimensional Banach spaces. Thus, to check that a given Banach space is Gurariĭ, one needs to show the existence of suitable extensions of continuum many isometric embeddings. Of course, this can be relaxed. One way to do it is to consider a natural countable subcategory of the category of all finite-dimensional Banach spaces.

We need to introduce some notation. Every finite-dimensional Banach space E is isometric to $\mathbb{R}^n$ with some norm $\|\cdot\|$. We shall say that E is *rational* if it is isometric to $\langle \mathbb{R}^n, \|\cdot\| \rangle$, such that the unit sphere is a polyhedron whose all vertices have rational coordinates. Equivalently, X is rational if, up to isometry, $X = \mathbb{R}^n$ with a "maximum norm" $\|\cdot\|$ induced by finitely many functionals $\varphi_0, \ldots, \varphi_{m-1}$ such that $\varphi_i[\mathbb{Q}^n] \subseteq \mathbb{Q}$ for every $i < m$. More precisely,

$$\|x\| = \max_{i < m} |\varphi_i(x)|$$

for $x \in \mathbb{R}^n$. Typical examples of rational Banach spaces are $\ell_1(n)$ and $\ell_\infty(n)$, the n-dimensional variants of ℓ_1 and ℓ_∞, respectively. On the other hand, for $1 < p < \infty$, $n > 1$, the spaces $\ell_p(n)$ are not rational. Of course, every rational Banach space is polyhedral.

It is clear that there are (up to isometry) only countably many rational Banach spaces and for every $\varepsilon > 0$, every finite-dimensional space has an ε-isometry onto some rational Banach space.

A pair of Banach spaces $\langle E, F \rangle$ will be called *rational* if $E \subseteq F$ and, up to isometry, $F = \mathbb{R}^n$ with a rational norm, and $E \cap \mathbb{Q}^n$ is dense in E. Note that if $\langle E, F \rangle$ is a rational pair then both E and F are rational Banach spaces. It is clear that there are, up to isometry, only countably many rational pairs of Banach spaces.

THEOREM 2.7. *Let X be a Banach space. Then X is Gurariĭ if and only if it satisfies the following condition.*

(G) *Given $\varepsilon > 0$, given a rational pair of spaces $\langle E, F \rangle$, for every strict ε-isometric embedding $f \colon E \to X$ there exists an ε-isometric embedding $g \colon F \to X$ such that*
$$\| g \restriction E - f \| \leqslant \varepsilon.$$

Furthermore, in condition (G) *it suffices to consider ε from a given set $T \subseteq (0, +\infty)$ with $\inf T = 0$.*

Proof. Every Gurariĭ space satisfies (G), almost by definition. Assume X satisfies (G). Fix two finite-dimensional spaces $E \subseteq F$ and fix an isometric embedding $f \colon E \to X$. Fix $\varepsilon > 0$. Fix a linear basis $\mathcal{B} = \{e_0, \ldots, e_{m-1}\}$ in F so that $\mathcal{B} \cap E = \{e_0, \ldots, e_{k-1}\}$ is a basis of E (so E is k-dimensional and F is m-dimensional). Choose $\delta > 0$ small enough. In particular, δ should have the property that for every linear operators $h, g \colon F \to X$, if $\max_{i<m} \| h(e_i) - g(e_i) \| < \delta$ then $\| h - g \| < \varepsilon/3$. In fact, δ depends on the norm of F only; a good estimation is $\varepsilon/(3M)$, where
$$M = \sup \left\{ \sum_{i<m} |\lambda_i| \colon \left\| \sum_{i<m} \lambda_i e_i \right\| = 1 \right\}.$$

Now choose a δ-equivalent norm $\| \cdot \|'$ on F such that $E \subseteq F$ becomes a rational pair (in particular, the basis $\mathcal{B}$ gives a natural coordinate system in which all e_is have rational coordinates). The operator f becomes a δ-isometric embedding, therefore by (G) there exists a δ-isometric embedding $g \colon F \to X$ such that $\| f - g \restriction E \|' < \delta$.

Now let $h \colon F \to X$ be the unique linear operator satisfying $h(e_i) = f(e_i)$ for $i < k$ and $h(e_i) = g(e_i)$ for $k \leqslant i < m$. Then $h \restriction \mathcal{B}$ is δ-close to $g \restriction \mathcal{B}$ with respect to the original norm, therefore $\| h - g \| < \varepsilon/3$. Clearly, $h \restriction E = f$. If

δ is small enough, we can be sure that g is an $\varepsilon/3$-isometric embedding with respect to the original norm of F. Finally, assuming that $\varepsilon < 1$, a standard calculation shows that h is an ε-isometric embedding, being $(\varepsilon/3)$-close to g.

The "furthermore" part obviously follows from the arguments above. ∎

Note that, for a given separable Banach space X, the criterion stated above can be applied by "testing" countably many almost isometric embeddings, namely, only those that map rational vectors to a fixed countable dense subset of X. More precisely, given a dense set $D \subseteq X$, every strict ε-isometric embedding $f\colon \mathbb{R}^n \to X$ (where $\mathbb{R}^n$ is endowed with some rational norm) can be approximated by strict ε-isometric embeddings $g\colon \mathbb{R}^n \to X$ satisfying $g[\mathbb{Q}^n] \subseteq D$.

Theorem 2.7 together with Lemma 2.4 provide another natural criterion for being Gurariĭ.

THEOREM 2.8. *A Banach space X is Gurariĭ if and only if it satisfies the following condition.*

(F) *Given $\varepsilon, \delta > 0$, given a rational pair of spaces $\langle E, F \rangle$, for every strict ε-isometric embedding $f\colon E \to X$ there exists a δ-isometric embedding $g\colon F \to X$ such that $\|f - g \restriction E\| < \varepsilon$.*

Proof. It is clear that (F) implies (G) and, by Theorem 2.7 this implies that X is Gurariĭ. It remains to show that every Gurariĭ space satisfies (F). For this aim, fix a rational pair $\langle E, F \rangle$ and a strict ε-isometric embedding $f\colon E \to X$. Let $Y = f[E]$. By Lemma 2.4, there are a finite-dimensional space Z and isometric embeddings $i\colon E \to Z$ and $j\colon Y \to Z$ such that $\|j \circ f - i\| < \varepsilon$. Using the Pushout Lemma, we can extend Z so that it also contains F. Since X is Gurariĭ, there exists a δ-isometric embedding $h\colon Z \to X$ extending j^{-1}. Finally, $g = h \restriction F$ is as required. ∎

2.3. TWO CONSTRUCTIONS.   There are several ways to see the existence of the Gurariĭ space $\mathbb{G}$. Actually, in Theorem 4.2 below, we shall show the existence of strong Gurariĭ spaces; in view of Theorem 3.4 below, such spaces contain many isometric copies of the Gurariĭ space. However, this is a rather indirect way of showing the existence of $\mathbb{G}$. A direct way is to construct a certain chain of finite-dimensional spaces. The crucial point is the Pushout Lemma.

THEOREM 2.9. (Gurariĭ [7], Gevorkjan [6]) *The Gurariĭ space exists and is isometrically universal for all separable Banach spaces.*

Proof. Fix a separable Banach space X and fix a countable dense set $D \subseteq X$. Fix a rational pair of Banach spaces $E \subseteq F$, fix a linear basis B in E consisting of vectors with rational coordinates, and fix $\varepsilon > 0$. Furthermore, fix a strict ε-isometric embedding $f \colon E \to X$ such that $f[B] \subseteq D$. Using the Pushout Lemma, we can find a separable Banach space $X' \supseteq X$ such that f extends to a strict ε-isometric embedding $g \colon F \to X'$. Note that there are only countably many pairs of rational Banach spaces and almost isometric embeddings as described above. Thus, there exists a separable Banach space $G(X) \supseteq X$ such that, given a rational pair $E \subseteq F$, for every ε-isometric embedding $f \colon E \to X$ there exists an ε-isometric embedding $g \colon F \to X$ such that $g \restriction E$ is arbitrarily close to f.

Repeat this construction infinitely many times. Namely, let

$$G = \mathrm{cl}\left(\bigcup_{n \in \omega} X_n \right),$$

where $X_0 = X$ and $X_{n+1} = G(X_n)$ for $n \in \omega$. Clearly, G is a separable Banach space. By Theorem 2.7, G is the Gurariĭ space.

Since the space X was chosen arbitrarily, this also shows that the Gurariĭ space contains an isometric copy of every separable Banach space. ∎

Next we show how to construct the Gurariĭ space as a "random" or "generic" Banach space. Uncountable variants, forcing the universe of set theory to be extended, have been recently studied by Lopez-Abad and Todorcevic [16]. Our idea is similar in spirit to that of Gurariĭ from [7], however it does not use any topological structure on spaces of norms.

Recall that c_{00} denotes the linear subspace of $\mathbb{R}^\omega$ consisting of all vectors with finite support. In other words, $x \in c_{00}$ iff $x \in \mathbb{R}^\omega$ and $x(n) = 0$ for all but finitely many $n \in \omega$. Given a finite set $S \subseteq \omega$, we shall identify the vector space $\mathbb{R}^S$ with the suitable subset of c_{00}, namely, $\mathbb{R}^S = \{x \in c_{00} \colon x(n) = 0 \text{ for every } n \in \omega \setminus S\}$.

Let $\mathbb{P}$ be the following partially ordered set. An element of $\mathbb{P}$ is a pair $p = \langle S_p, \|\cdot\|_p \rangle$, where $S_p \subseteq \omega$ is a finite set and $\|\cdot\|_p$ is a norm on $\mathbb{R}^{S_p} \subseteq c_{00}$. We define $p \leqslant q$ iff $S_p \subseteq S_q$ and $\|\cdot\|_q$ extends $\|\cdot\|_p$. Clearly, $\leqslant$ is a partial order. Suppose

$$p_0 < p_1 < p_2 < \cdots$$

is a sequence in $\mathbb{P}$ such that the chain of sets $\bigcup_{n\in\omega} S_{p_n} = \omega$. Then c_{00} naturally becomes a normed space. Let X be the completion of c_{00} endowed with this norm. We shall call it the *limit* of $\{p_n\}_{n\in\omega}$ and write $X = \lim_{n\to\infty} p_n$. It is rather clear that every separable Banach space is of the form $\lim_{n\to\infty} p_n$ for some sequence $\{p_n\}_{n\in\omega}$ in $\mathbb{P}$. We are going to show that for a "typical" sequence in $\mathbb{P}$, its limit is the Gurariĭ space.

Given a partially ordered set $\mathbb{P}$, recall that a subset $D \subseteq \mathbb{P}$ is *cofinal* if for every $p \in \mathbb{P}$ there exists $d \in D$ with $p \leqslant d$. Below is a variant of the well-known Rasiowa-Sikorski Lemma, which is actually an abstract version of the Baire Category Theorem.

LEMMA 2.10. *Let $\mathbb{P}$ be a partially ordered set and let $\mathcal{D}$ be a countable family of cofinal subsets of $\mathbb{P}$. Then there exists a sequence*

$$p_0 \leqslant p_1 \leqslant p_2 \leqslant \cdots$$

such that for each $D \in \mathcal{D}$ the set $\{n \in \omega : p_n \in D\}$ is infinite.

Proof. Let $\mathcal{D} = \{D_n : n \in \omega\}$ so that for each $D \in \mathcal{D}$ the set $\{n \in \omega : D_n = D\}$ is infinite. Using the fact that each D_n is cofinal, construct inductively $\{p_n\}_{n\in\omega}$ so that $p_n \in D_n$ for $n \in \omega$. ∎

A sequence $\{p_n\}_{n\in\omega}$ satisfying the assertion of the lemma above is often called $\mathcal{D}$-*generic*.

We now define a countable family of open cofinal sets which is good enough for producing the Gurariĭ space. Namely, fix a rational pair of spaces $\langle E, F\rangle$, fix a positive integer n and fix a rational embedding $f\colon E \to c_{00}$, that is, an injective linear operator mapping vectors with rational coordinates to $c_{00} \cap \mathbb{Q}^\omega$. The point is that there are only countably many possibilities for E and f. Define $D_{E,F,f,n}$ to be the set of all $p \in \mathbb{P}$ such that $n \in S_p$ and p satisfies the following implication:

> If f is a $(1/n)$-isometric embedding into $\langle \mathbb{R}^{S_p}, \|\cdot\|_p\rangle$, then there exists a $(1/n)$-isometric embedding $g\colon F \to \langle \mathbb{R}^{S_p}, \|\cdot\|_p\rangle$ such that $g \restriction E = f$.

CLAIM 2.11. *The set $D_{EF,f,n}$ is cofinal in $\mathbb{P}$.*

Proof. Fix $p \in \mathbb{P}$. Suppose that f is a $(1/n)$-isometric embedding into $\langle \mathbb{R}^{S_p}, \|\cdot\|_p\rangle$ (otherwise clearly $p \in D_{E,F,f,n}$). Using the Pushout Lemma, find a finite-dimensional Banach space W extending $\langle \mathbb{R}^{S_p}, \|\cdot\|_p\rangle$ and a $(1/n)$-isometric embedding $g\colon F \to W$ such that $g \restriction F = f$. We may assume that

$W = \langle \mathbb{R}^T, \| \cdot \|_W \rangle$ for some $T \supseteq S_p$, where the norm $\| \cdot \|_W$ extends $\| \cdot \|_p$. Let $q = \langle T, \| \cdot \|_W \rangle \in \mathbb{P}$. Clearly, $p \leqslant q$. Finally, $q \in D_{E,F,f,n}$ because f is a $(1/n)$-isometric embedding into $\langle \mathbb{R}^T, \| \cdot \| W \rangle$ and g extends f. ∎

Let $\mathcal{D}$ consist of all sets of the form $D_{E,F,f,n}$ as above. Then $\mathcal{D}$ is countable, therefore applying Lemma 2.10 we obtain a $\mathcal{D}$-generic sequence $\{p_n\}_{n \in \omega}$.

THEOREM 2.12. *Let $\mathcal{D}$ be as above and let $\{p_n\}_{n \in \omega}$ be a $\mathcal{D}$-generic sequence. Then the space $\lim_{n \to \infty} p_n$ is Gurariĭ.*

Proof. Let $X = \lim_{n \to \infty} p_n$. Notice that $\bigcup_{n \in \omega} S_{p_n} = \omega$. Fix a positive integer k, fix a rational pair of spaces $\langle E, F \rangle$ and fix a $1/(k+1)$-isometric embedding $f \colon E \to X$. We can modify f in such a way that it remains to be a $(1/k)$-isometric embedding, while at the same time $f[E] \subseteq c_{00}$, and it maps rational vectors into $c_{00} \cap \mathbb{Q}^\omega$. Now $D_{E,F,f,k} \in \mathcal{D}$ therefore there exists $n \in \omega$ such that $p_n \in D_{E,F,f,k}$ and $\mathbb{R}^{S_{p_n}}$ contains the range of f. By the definition of $D_{E,F,f,k}$, f extends to a $(1/k)$-isometric embedding $g \colon F \to X$. By Theorem 2.7, this shows that X is Gurariĭ. ∎

Let us remark that some modifications of the poset $\mathbb{P}$ still give the Gurariĭ space. For instance, we can consider only polyhedral norms for $\| \cdot \|_p$, because the Pushout Lemma holds for this class. We shall use this observation later.

2.4. SCHAUDER BASES AND LINDENSTRAUSS SPACES. We now present the proof that the Gurariĭ space has a monotone Schauder basis. This fact has already been noticed by Gurariĭ in [7].

Recall that a *Schauder basis* in a separable Banach space X is a sequence $\{e_n\}_{n \in \omega}$ of non-zero vectors of X, such that for every $x \in X$ there exist uniquely determined scalars $\{\lambda_n\}_{n \in \omega}$ satisfying

$$x = \sum_{n \in \mathbb{N}} \lambda_n e_n.$$

The series above is supposed to converge in the norm. Given a Schauder basis $\{e_n\}_{n \in \omega}$, one always has the associated *canonical projections*

$$P_N\Big(\sum_{n \in \mathbb{N}} \lambda_n e_n\Big) = \sum_{n < N} \lambda_n e_n.$$

Note that each P_N is a projection and $P_N P_M = P_{\min(N,M)}$ for every $N, M \in \mathbb{N}$. By the Banach-Steinhaus principle, $\sup_{N \in \mathbb{N}} \|P_N\| < +\infty$. The basis is

monotone if $\|P_N\| \leqslant 1$ for each $N \in \mathbb{N}$. We shall consider monotone Schauder bases only. It turns out that the existence of a monotone Schauder basis can be deduced from the canonical projections:

PROPOSITION 2.13. (Mazur) *Let X be a Banach space and let $\{P_n\}_{n\in\omega}$ be a sequence of norm one projections such that $P_0 = 0$, $\dim(P_{n+1}X/P_nX) \leqslant 1$, $\bigcup_{n\in\mathbb{N}} P_nX$ is dense in X, and $P_nP_m = P_{\min(n,m)}$ for every $n, m \in \mathbb{N}$. Then there exists a monotone Schauder basis $\{e_n\}_{n\in\omega}$ in X such that $\{P_n\}_{n\in\omega}$ is the sequence of canonical projections associated to $\{e_n\}_{n\in\omega}$.*

Proof. Let us first prove that $\lim_{n\to\infty} P_n x = x$ for every $x \in X$. For this aim, fix $x \in S_X$ and $\varepsilon > 0$. Find n_0 such that $\|x - y\| < \varepsilon/2$ for some $y \in P_{n_0}X$. Given $n \geqslant n_0$, we have

$$\|P_n x - x\| \leqslant \|P_n x - y\| + \|y - x\| < \|P_n(x - y)\| + \varepsilon/2 < \varepsilon.$$

Now let φ_n be such that $P_{n+1}x - P_n x = \varphi_n(x)e_n$ for some $e_n \in S_X$. Here we use the fact that $P_{n+1}X = P_nX \oplus \mathbb{R}e_n$ for some $e_n \in \ker P_n \cap S_X$. Finally, given $x \in X$, we have

$$x = \lim_{n\to\infty} P_n x = \lim_{N\to\infty} \sum_{n<N} (P_{n+1} - P_n)x = \lim_{N\to\infty} \sum_{n<N} \varphi_n(x)e_n.$$

Finally, if $0 = \sum_{n\in\mathbb{N}} \lambda_n e_n$ then, by easy induction, we show that $\lambda_n = 0$ for every $n \in \mathbb{N}$. This shows that $\{e_n\}_{n\in\omega}$ is a Schauder basis. Clearly, P_ns are the canonical projections, therefore the basis is monotone. ∎

We now recall an important class of Banach spaces, containing the Gurariĭ space:

DEFINITION 2.14. A Banach space X is called a *Lindenstrauss space* if X^* is linearly isometric to $L_1(\mu)$ for some measure μ.

It turns out that among separable Banach spaces the class of Lindenstrauss spaces coincides with π_1^∞ spaces of Michael & Pełczyński [18]: A Banach space X is π_1^∞ if it contains a directed family $\mathcal{F}$ such that $\bigcup\mathcal{F}$ is dense in X and each $F \in \mathcal{F}$ is linearly isometric to some $\ell_\infty(n)$. Recall that a family $\mathcal{F}$ is *directed* if for every $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ such that $A \cup B \subseteq C$. The following characterization combines results of Michael & Pełczyński [18] and Lazar & Lindenstrauss [15].

THEOREM 2.15. *For a separable Banach space X, the following conditions are equivalent:*

(a) *X is Lindenstrauss.*

(b) *X is π_1^∞.*

(c) *X is the completion of the union of a chain $E_1 \subseteq E_2 \subseteq \cdots$, where each E_n is linearly isometric to $\ell_\infty(n)$. (The chain is finite in case X is finite-dimensional.)*

The implication (b) $\implies$ (c), due to Michael & Pełczyński, follows from an interesting geometric property of $\ell_\infty(n)$ spaces: Given $E \subseteq \ell_\infty(l)$ isometric to $\ell_\infty(k)$ for some $k < l$, there exists a space F isometric to $\ell_\infty(k+1)$ and such that $E \subseteq F \subseteq \ell_\infty(l)$ (see [18, Lemma 3.2]).

The basic infinite-dimensional example of a Lindenstrauss space is c_0; other examples are $C(K)$ spaces with K compact metric.

Theorem 2.15 combined with Proposition 2.13 gives the following

COROLLARY 2.16. (Gurariĭ [8], Michael & Pełczyński [18]) *Every separable Lindenstrauss space has a monotone Schauder basis.*

THEOREM 2.17. (Gurariĭ [7]) *The Gurariĭ space is Lindenstrauss.*

Proof. Let $\mathbb{P}$ be the partially ordered set defined before Theorem 2.12. Define $\mathbb{P}_0$ to be the set of all $p \in \mathbb{P}$ such that the norm $\|\cdot\|_p$ is polyhedral. It is easy to verify that, with the same family $\mathcal{D}$ of cofinal sets, the limit of a $\mathcal{D}$-generic sequence is Gurariĭ. In fact, the only difference is in using the polyhedral variant of the Pushout Lemma, namely, Lemma 1.6. Now add to the family $\mathcal{D}$ the following set:

$$E = \big\{ p \in \mathbb{P}_0 : \langle \mathbb{R}^{S_p}, \|\cdot\|_p \rangle \text{ is linearly isometric to some } \ell_\infty(n) \big\}.$$

Since all the norms $\|\cdot\|_p$ are polyhedral, the set E is cofinal in $\mathbb{P}_0$. The limit of a $(\mathcal{D} \cup \{E\})$-generic sequence is necessarily a π_1^∞ space; since such a sequence is also $\mathcal{D}$-generic, its limit is the Gurariĭ space. ∎

It has been proved by Lazar & Lindenstrauss [14] that if X is a separable space such that X^* is isometric to a non-separable $L_1(\mu)$ space then X contains an isometric copy of $C(2^{\mathbb{N}})$, where $2^{\mathbb{N}}$ is the Cantor set. In particular, such a space X contains an isometric copy of every separable Banach space. This gives another (rather indirect) proof of isometric universality of the Gurariĭ space.

THEOREM 2.18. (Wojtaszczyk [20]) *Every separable Lindenstrauss space is isometric to a 1-complemented subspace of* $\mathbb{G}$.

Proof. Fix a separable Lindenstrauss space X and let $\{X_n\}_{n\in\omega}$ be a chain of spaces such that $X_0 = \{0\}$, $\bigcup_{n\in\mathbb{N}} X_n$ is dense in X, and each X_n is linearly isometric to $\ell_\infty(n)$ (see Theorem 2.15). In case X is finite-dimensional, we put $X_n = X = \ell_\infty(\dim X)$.

Let us look back at the simple proof of Lemma 2.10, where a $\mathcal{D}$-generic sequence was constructed in the poset $\mathbb{P}$ defined just before Theorem 2.12 and $\mathcal{D}$ is the same countable collection of cofinal sets. For convenience, we shall write $U(p)$ for the Banach space $\langle \mathbb{R}^{S_p}, \|\cdot\|_p \rangle$, where $p \in \mathbb{P}$.

We claim that there exists a $\mathcal{D}$-cofinal sequence $\{p_n\}_{n\in\omega}$ together with isometric embeddings $i_n \colon X_n \to U(p_n)$ and norm one operators $P_n \colon U(p_n) \to X_n$ such that $P_n \circ i_n = \mathrm{id}_{X_n}$, i_{n+1} extends i_n and P_{n+1} extends P_n for each $n \in \mathbb{N}$. Recall that $\mathcal{D}$ was enumerated as $\{D_n\}_{n\in\omega}$, so that each $D \in \mathcal{D}$ occurs infinitely many times. Suppose p_n, i_n and P_n have been defined. Using the Pushout Lemma, find $q \geqslant p_n$ and an isometric embedding $j \colon X_{n+1} \to U(q)$ extending i_n. The property of the pushout gives a norm one projection $Q \colon U(q) \to X_{n+1}$ extending P_n (see Lemma 1.3).

Now, using the fact that D_{n+1} is cofinal, find $p_{n+1} \in D_{n+1}$ so that $p_{n+1} \geqslant q$. Finally, $i_{n+1} = j$, treated as an embedding into $U(p_{n+1})$ and P_{n+1} is any extension of Q preserving the norm, which exists because X_{n+1} is linearly isometric to some $\ell_\infty(m)$.

This finishes the inductive construction. By Theorem 2.12, we know that $\lim_{n\to\infty} p_n = \mathbb{G}$ and taking the pointwise limits of i_n and P_n we obtain an isometric embedding $i \colon X \to \mathbb{G}$ and a norm one operator $P \colon \mathbb{G} \to X$ such that $P \circ i = \mathrm{id}_X$. This shows that $i[X]$ is 1-complemented in $\mathbb{G}$. ■

3. NON-SEPARABLE GURARIĬ SPACES

In this section we give a characterization of Gurariĭ spaces in terms of skeletons.

Let X be a Banach space. A family $\mathcal{F}$ of closed linear subspaces of X will be called a *skeleton* in X if the following conditions are satisfied.

(1) Each $F \in \mathcal{F}$ is separable.

(2) $\bigcup \mathcal{F} = X$.

(3) $\mathcal{F}$ is directed, i.e. for every $F_0, F_1 \in \mathcal{F}$ there is $G \in \mathcal{F}$ such that $F_0 \cup F_1 \subseteq G$.

(4) $\mathrm{cl}(\bigcup_{n\in\omega} F_n) \in \mathcal{F}$, whenever $\{F_n\}_{n\in\omega}$ is a countable chain in $\mathcal{F}$.

The notion of a skeleton makes sense for non-separable Banach spaces, since $\mathcal{F} = \{X\}$ is a skeleton if X is separable. Actually, notice that if $\mathcal{F}$ is a skeleton in X then for every separable subset $S \subseteq X$ there exists $F \in \mathcal{F}$ satisfying $S \subseteq F$. The significance of skeletons lies in the following well-known property.

PROPOSITION 3.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be skeletons in a fixed Banach space X. Then $\mathcal{F} \cap \mathcal{G}$ is again a skeleton in X.*

Proof. It is clear that $\mathcal{F} \cap \mathcal{G}$ satisfies (1) and (4). In order to prove (2) and (3) it suffices to show that for every separable subspace $S \subseteq X$ there exists $H \in \mathcal{F} \cap \mathcal{G}$ such that $S \subseteq H$.

Fix a separable set $S \subseteq X$. By the remark above, there exists $F_0 \in \mathcal{F}$ such that $S \subseteq F_0$. Similarly, there exists $G_0 \in \mathcal{G}$ such that $F_0 \subseteq G_0$. By induction, we construct two increasing sequences $\{F_n\}_{n\in\omega}$ and $\{G_n\}_{n\in\omega}$ in $\mathcal{F}$ and $\mathcal{G}$ respectively, so that $F_n \subseteq G_n \subseteq F_{n+1}$ holds for every $n \in \omega$. Finally, notice that $H = \mathrm{cl}(\bigcup_{n\in\omega} F_n) = \mathrm{cl}(\bigcup_{n\in\omega} G_n)$ belongs to both $\mathcal{F}$ and $\mathcal{G}$. ∎

We now turn to the announced characterization of Gurariǐ spaces in terms of skeletons.

LEMMA 3.2. *Let X be a Gurariǐ space and let $S \subseteq X$ be a countable set. Then there exists a subspace $Y \subseteq X$ linearly isometric to $\mathbb{G}$ and such that $S \subseteq Y$.*

Proof. This is a standard closing-off argument. The criterion for being Gurariǐ (Theorem 2.7) actually requires checking countably many almost isometric embeddings. The first step is to show that given a separable subspace $Z \subseteq X$ there exists a separable space $E(Z)$ (not uniquely determined) such that $Z \subseteq E(Z) \subseteq X$ and the following condition is satisfied.

(†) For every rational pair of spaces $\langle E, F \rangle$, for every $\varepsilon > 0$, for every strict ε-isometric embedding $f \colon E \to Z$ there exists a strict ε-isometric embedding $g \colon F \to E(Z)$ such that $\|f - g \upharpoonright E\| < \varepsilon$.

Once we have proved this, we construct a chain of separable spaces $Z_0 \subseteq Z_1 \subseteq \cdots \subseteq X$ such that $S \subseteq Z_0$, and $Z_{n+1} = E(Z_n)$ for every $n \in \mathbb{N}$. Then, using Theorem 2.7, we conclude that the space $Y = \mathrm{cl}(\bigcup_{n\in\omega} Z_n)$ is Gurariǐ, because of condition (†). It remains to show the existence of $E(Z)$ satisfying (†).

Fix Z and fix a countable dense subset D of Z. Let $\mathcal{A}$ consist of all quadruples of the form $\langle E, F, f, \varepsilon \rangle$, where $E \subseteq F$ is a rational pair of (finite-dimensional) spaces, $\varepsilon > 0$ is a rational number, and $f \colon E \to Z$ is a strict ε-isometric embedding such that $f[B] \subseteq D$, where B is a fixed linear basis of E consisting of vectors with rational coordinates. These assumptions ensure us that $\mathcal{A}$ is countable.

Using the fact that X is Gurariĭ, given $q = \langle E, F, f, \varepsilon \rangle$, we know that there exists a strict ε-isometric embedding $g \colon F \to X$ such that $\|f - g \restriction E\| < \varepsilon$. Denote by R_q the range of g. Finally, take $E(Z)$ to be the closure of the union $Z \cup \bigcup_{q \in \mathcal{A}} R_q$. It is clear that (†) is satisfied. ∎

LEMMA 3.3. *Let $\{X_n\}_{n \in \omega}$ be a chain of subspaces of a Banach space X such that $X = \mathrm{cl}(\bigcup_{n \in \omega} X_n)$ and each X_n is linearly isometric to $\mathbb{G}$. Then X is linearly isometric to the Gurariĭ space $\mathbb{G}$.*

Proof. Fix finite-dimensional spaces $E \subseteq F$ and an isometric embedding $f \colon E \to X$. Fix $\varepsilon > 0$. Choose a linear map $g \colon E \to X$ that is ε-close to f so that it is a strict ε-isometric embedding and $g[E] \subseteq X_n$ for some $n \in \omega$. Now, using the property of the Gurariĭ space X_n, there exists an extension $h \colon F \to X_n$ of g, that is also a strict ε-isometric embedding. Finally, $h \restriction E$ is ε-close to f. By Theorem 2.7, this shows that X is Gurariĭ. ∎

THEOREM 3.4. *Let X be a Banach space. The following properties are equivalent.*

(a) *X is a Gurariĭ space.*

(b) *X has a skeleton consisting of subspaces isometric to the Gurariĭ space $\mathbb{G}$.*

(c) *There exists a directed family $\mathcal{G}$ of spaces isometric to $\mathbb{G}$, such that $\bigcup \mathcal{G} = X$.*

Proof. (a) $\implies$ (c) Let $\mathcal{G}$ be the family of all subspaces of X that are isometric to $\mathbb{G}$. By Lemma 3.2, $\bigcup \mathcal{G} = X$ and $\mathcal{G}$ is directed. In fact, this follows from a stronger property of $\mathcal{G}$: every separable subset is covered by an element of $\mathcal{G}$.

(c) $\implies$ (b) Let $\mathcal{G}$ be as in (c) and let $\mathcal{F}$ be the family of all subspaces of X that are isometric to $\mathbb{G}$. We claim that $\mathcal{F}$ is a skeleton. Condition (1) is obvious, (2) follows from the property of $\mathcal{G}$ and (4) follows from Lemma 3.3. In order to show (3), it suffices to prove that every countable subset of X is

covered by an element of $\mathcal{F}$. Fix $D = \{d_n \colon n \in \omega\} \subseteq X$ and, using direct-edness, construct inductively $G_0 \subseteq G_1 \subseteq \cdots$ in $\mathcal{G}$ so that $d_n \in G_n$. Then $F = \mathrm{cl}(\bigcup_{n \in \omega} G_n)$ is an element of $\mathcal{F}$ and $D \subseteq F$.

(b) $\implies$ (a) Fix two finite-dimensional spaces $A \subseteq B$ and an isometric embedding $f \colon A \to X$. Then $f[A]$ is finite-dimensional, therefore there exists $F \in \mathcal{F}$ such that $f[A] \subseteq F$. Since F is the Gurariĭ space, given any $\varepsilon > 0$, f can be extended to an ε-isometry $g \colon B \to F$. ∎

The following corollary improves [3, Thm. 6.1], where the same was shown for Banach spaces of universal disposition for separable spaces.

COROLLARY 3.5. *No complemented subspace of a $C(K)$ space (or, more generally, an M-space) can be Gurariĭ.*

Proof. Suppose $X \subseteq C(K)$ is a Gurariĭ space and $P \colon C(K) \to X$ is a projection. Let $\mathcal{F}$ be a skeleton in $C(K)$ consisting of spaces of continuous functions over some metric compacta. By Theorem 3.4, X has a skeleton $\mathcal{G}$ such that each $G \in \mathcal{G}$ is isometric to the Gurariĭ space $\mathbb{G}$. A standard closing-off argument (see the proof of [3, Thm. 6.1]) shows that there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $PF = G$. The final contradiction comes from [3, Cor. 5.4], saying that the Gurariĭ space is not complemented in any C(K) space.

The arguments above can be repeated when C(K) spaces are replaced by M-spaces (see the comments in Sections 5,6 of [3]). ∎

It should be noted that Corollary 3.5 can actually be derived from [3, Thm. 6.1], using another result from [3] saying that ultraproducts of Gurariĭ spaces are UD(sep), while ultraproducts of $C(K)$ spaces are again $C(K)$ spaces. However, our argument using skeletons is elementary and perhaps more illustrative.

THEOREM 3.6. *Every Banach space embeds isometrically into a Gurariĭ space of the same density.*

Proof. We use induction on the density of the space. The statement is true for separable spaces, so fix a cardinal $\kappa > \aleph_0$ and suppose the statement holds for Banach spaces of density $< \kappa$.

Fix a Banach space X of density κ. Then X is the completion of the union of a chain $\{X_\alpha\}_{\alpha < \kappa}$ starting from a separable space X_0 and such that $\mathrm{dens}\,(X_\alpha) < \kappa$ for every $\alpha < \kappa$. We may assume that this chain is continuous, i.e., X_δ is the closure of $\bigcup_{\xi < \delta} X_\xi$, whenever δ is a limit ordinal. We construct

a sequence of isometric embeddings $f_\alpha \colon X_\alpha \to G_\alpha$, where each G_α is a Gurariĭ space of density $< \kappa$, and $G_\alpha \subseteq G_\beta$, $f_\beta \restriction G_\alpha = f_\alpha$ whenever $\alpha < \beta$.

Suppose G_α and f_α have been constructed for $\alpha < \eta$. If η is a limit ordinal, we take G_η to be the completion of $\bigcup_{\xi < \eta} G_\xi$. By Theorem 3.4, we know that G_η is a Gurariĭ space. The embedding f_η is uniquely determined.

Now suppose $\eta = \beta + 1$. Using the Pushout Lemma, we find a space $W \supseteq G_\beta$ so that f_β extends to an isometric embedding $j \colon X_{\beta+1} \to W$. Note that $\mathrm{dens}(W) < \kappa$. Using the inductive hypothesis, there exists a Gurariĭ space $G_{\beta+1} \supseteq W$ such that $\mathrm{dens}(G_{\beta+1}) = \mathrm{dens}(W)$. We define $f_{\beta+1} = j$.

Finally, the sequence $\{f_\alpha\}_{\alpha < \kappa}$ determines an isometric embedding of X into $G = \mathrm{cl}(\bigcup_{\alpha < \kappa} G_\alpha)$. Clearly, $\mathrm{dens}(G) = \kappa$ and G is Gurariĭ by Theorem 3.4. ∎

It seems that there are many non-isomorphic Gurariĭ spaces of density $\aleph_1$. We show that some of them have many projections. Recall that a *projectional resolution of the identity* (briefly: PRI) in a Banach space is a transfinite sequence of norm one projections $\{P_\alpha\}_{\alpha < \omega_1}$ whose images are separable, form a continuous chain covering the space, and $P_\alpha P_\beta = P_{\min\{\alpha,\beta\}}$ holds for every $\alpha, \beta < \omega_1$. The notion of a PRI is usually defined for arbitrary non-separable Banach spaces, see [4] and [5] for more information. It seems that PRI is the main tool for proving certain properties of a non-separable Banach space by transfinite induction. For example, every Banach space of density $\aleph_1$ with a PRI admits a bounded one-to-one linear operator into $c_0(\omega_1)$ (see, e.g., [10, Cor. 17.5]).

THEOREM 3.7. *There exists a Gurariĭ space E of density $\aleph_1$ that has a projectional resolution of the identity.*

Proof. First of all, there exists a norm one projection $Q \colon \mathbb{G} \to \mathbb{G}$ such that $\ker Q$ is non-trivial. This follows immediately from the proof of Theorem 2.18, where we can at the first step ensure that the embedding of $\mathbb{G}$ into $\mathbb{G}$ is not the identity.

We now construct a continuous chain of separable spaces $\{G_\alpha\}_{\alpha < \omega_1}$ with the following properties.

(i) Each G_α is linearly isometric to $\mathbb{G}$.

(ii) For each $\alpha < \omega_1$ there exists a projection $Q_\alpha^{\alpha+1} \colon G_{\alpha+1} \to G_\alpha$, isometric to Q.

Property (ii) ensures us that the chain is strictly increasing and its union $G_{\omega_1} = \bigcup_{\alpha < \omega_1} G_\alpha$ is indeed of density $\aleph_1$.

By Theorem 3.4, G_{ω_1} is a Gurariĭ space and by [13] (see also [10, Thm. 17.5]) it has a projectional resolution of the identity. ▮

Note that there are Banach spaces of density $\aleph_1$, not embeddable into any Banach space with a PRI (e.g., spaces with uncomplemented copies of c_0, see Section 6 below). Thus, by Theorems 3.6 and 3.7, there are at least two non-isomorphic Gurariĭ spaces of density $\aleph_1$.

4. Spaces of universal disposition for larger classes

In this section we discuss spaces of $\mathrm{UD}(\mathfrak{D}_{<\kappa})$, where $\mathfrak{D}_{<\kappa}$ is the class of Banach spaces of density $< \kappa$. If $\kappa = \aleph_0$, let $\mathfrak{D}_{<\kappa}$ be the class of all finite-dimensional Banach spaces.

Recall that a Banach space is *isometrically universal* for a class $\mathfrak{K}$ of spaces, if it contains an isometric copy of every space from $\mathfrak{K}$. The following general fact is well-known, we state it for the sake of completeness. A special case (for $\kappa = \aleph_0$) is contained in [7].

PROPOSITION 4.1. *Assume κ is an infinite regular cardinal.*

(0) *Let U be a Banach space of $\mathrm{UD}(\mathfrak{D}_{<\kappa})$. Then for every pair of spaces $X \subseteq Y$ such that $\mathrm{dens}(X) < \kappa$ and $\mathrm{dens}(Y) \leqslant \kappa$, every isometric embedding $f \colon X \to U$ extends to an isometric embedding $g \colon Y \to U$.*

(1) *Every Banach space of $\mathrm{UD}(\mathfrak{D}_{<\kappa})$ is isometrically universal for the class of Banach spaces of density $\leqslant \kappa$.*

(2) *Let U, V be two Banach spaces of $\mathrm{UD}(\mathfrak{D}_{<\kappa})$ and of density κ. Then every linear isometry $f \colon X \to Y$ such that $X \subseteq U$, $Y \subseteq V$ and $X, Y \in \mathfrak{D}_{<\kappa}$, extends to a bijective linear isometry $h \colon U \to V$. In particular, U and V are linearly isometric.*

Proof. Let U be a Banach space of $\mathrm{UD}(\mathfrak{D}_{<\kappa})$ and fix Banach spaces $X \subseteq Y$ as in (0). Fix an isometric embedding $f \colon X \to U$. Choose a continuous chain $\{X_\alpha\}_{\alpha < \kappa}$ of closed subspaces of Y so that $X_0 = X$, $X_\alpha \in \mathfrak{D}_{<\kappa}$ and $\bigcup_{\alpha < \kappa} X_\alpha$ is dense in Y. Recall that a "continuous chain" means that X_δ is the closure of $\bigcup_{\xi < \delta} X_\xi$ for every limit ordinal $\delta < \kappa$. Using the definition of universal disposition, construct inductively a sequence of linear isometric embeddings $f_\alpha \colon X_\alpha \to U$ so that $f_0 = f$ and $f_\beta \upharpoonright X_\alpha = f_\alpha$ whenever $\alpha < \beta$. At limit steps

we use the continuity of the chain. The unique map $f_\kappa \colon X \to U$ satisfying $f_\kappa \restriction X_\alpha = f_\alpha$ for $\alpha < \kappa$ is an isometric embedding extending f. This shows both (0) and (1), since we may take $X = 0$.

The proof of (2) is a standard back-and-forth argument. Namely, let $\{U_\alpha\}_{\alpha<\kappa}$ and $\{V_\alpha\}_{\alpha<\kappa}$ be continuous chains of closed subspaces of U and V respectively, such that U_α, V_α are of density $< \kappa$ for $\alpha < \kappa$ and $U = \mathrm{cl}(\bigcup_{\alpha<\kappa} U_\alpha)$, $V = \mathrm{cl}(\bigcup_{\alpha<\kappa} V_\alpha)$ (note that the closure is irrelevant if $\kappa > \aleph_0$). Furthermore, we assume that $U_0 = X$ and $V_0 = Y$. Construct inductively isometric embeddings $f_\xi \colon U_{\alpha(\xi)} \to V_{\beta(\xi)}$ and $g_\xi \colon V_{\beta(\xi)} \to U_{\alpha(\xi+1)}$ so that $f_0 = f$, $g_\xi \circ f_\xi$ is the inclusion $U_{\alpha(\xi)} \subseteq U_{\alpha(\xi+1)}$, and $f_{\xi+1} \circ g_\xi$ is the inclusion $V_{\beta(\xi)} \subseteq V_{\beta(\xi+1)}$ for each $\xi < \kappa$. The limit steps make no trouble because of the continuity of both chains. The regularity of κ is used for the fact that every subspace of U (or V, respectively) of density $< \kappa$ is contained in some U_α (or V_β, respectively). The "limit" operators $f_\kappa \colon U \to V$ and $g_\kappa \colon V \to U$ are bijective linear isometries because $f_\kappa \circ g_\kappa = \mathrm{id}_V$ and $g_\kappa \circ f_\kappa = \mathrm{id}_U$. Finally, note that f_κ extends f, which completes the proof of (2). ∎

Given cardinal numbers μ, κ, by $\mu^{<\kappa}$ we denote the supremum of all cardinals μ^λ where $\lambda < \kappa$. The next result is a special case of more general constructions, known in model theory (see, e.g., Jónsson [9]). For Banach spaces this can be found in [3] and [11].

THEOREM 4.2. *Let μ be a cardinal and let κ be an uncountable cardinal. Let X be a Banach space of density $\leqslant \mu$. Then there exists a Banach space $Y \supseteq X$ of density $\mu^{<\kappa}$ that is of universal disposition for spaces of density $< \kappa$.*

Proof. The space Y will be constructed by using The Pushout Lemma. So, we need to compute first, how many "possibilities" we have. The idea is that we first want to extend X to a bigger Banach space $Z(X)$ such that every isometric embedding $f \colon E \to F$ with $E \subseteq X$ and F of density $< \kappa$ is *realized* in $Z(X)$, that is, there exists an isometric embedding $g \colon F \to Z(X)$ such that $g(f(x)) = x$ for $x \in E$.

Given an isometric embedding $f \colon E \to F$ such that $E \subseteq X$, let $P(X, f)$ be the resulting Banach space of the pushout of f and the inclusion $E \subseteq X$. Clearly, the density of $P(X, f)$ is the maximum of $\mathrm{dens}\,(X)$ and $\mathrm{dens}\,(F)$.

Observe that there are at most $\mu^{<\kappa}$ closed subspaces of X of density $< \kappa$. This follows from the fact that the cardinality of X is $\leqslant \mu^{\aleph_0}$. Now, given two spaces E and F of density $\lambda < \kappa$, the cardinality of the set of all isometric

embeddings of E into F cannot exceed $\lambda^\lambda = 2^\lambda \leqslant \mu^\lambda$. Finally, note that there are at most $2^{<\kappa}$ isometric types of Banach spaces of density $< \kappa$. Here we use the fact that κ is uncountable and therefore $2^{<\kappa} \geqslant \mathfrak{c}$.

It follows that there is a family $\mathcal{F}$ of cardinality $\leqslant \mu^{<\kappa}$ consisting of isometric embeddings $f\colon E \to F$ with $E \subseteq X$, the density of F is $< \kappa$ and every isometric embedding $g\colon G \to H$ satisfying these conditions is isometric to some element of $\mathcal{F}$. Write $\mathcal{F} = \{f_\xi\}_{\xi < \lambda}$, where $\lambda = |\mathcal{F}|$. Construct inductively a continuous chain of Banach spaces $\{X_\xi\}_{\xi < \lambda}$, starting with $X_0 = X$ and setting $X_{\xi+1} = P(X_\xi, f_\xi)$. Let $Z(X) = X_\lambda$, the completion of the union of $\{X_\alpha\}_{\alpha < \lambda}$.

Note that every isometry from a subspace of X of density $< \kappa$ into a space of density $< \kappa$ is realized in $Z(X)$, because we have taken care of all possibilities. Furthermore, observe that for $\mu_1 = \mathrm{dens}\,(Z(X))$ we have that $\mu_1^{<\kappa} = \mu^{<\kappa}$. This follows from the fact that $\mu_1 \leqslant \mu^{<\kappa}$ and $(\mu^{<\kappa})^{<\kappa} = \mu^{<\kappa}$.

By the remark above, we can repeat this procedure up to $\mu^{<\kappa}$ many times, not enlarging the density. That is, we construct a continuous chain of Banach spaces $\{Z_\alpha\}_{\alpha < \theta}$, where $\theta = \mu^{<\kappa}$, $Z_0 = X$ and $Z_{\alpha+1} = Z(Z_\alpha)$ for $\alpha < \theta$. We claim that the resulting Banach space $Y = \bigcup_{\alpha < \theta} Z_\alpha$ is of universal disposition for spaces of density $< \kappa$. Its density is exactly $\mu^{<\kappa}$. The only thing is to check that the cofinality of θ is $\geqslant \kappa$. In fact, a well known fact from cardinal arithmetic says that $\theta^{\mathrm{cf}(\theta)} > \theta$. On the other hand, $\theta^\lambda = \theta$ for every $\theta < \kappa$. Thus, indeed, the cofinality of θ is $\geqslant \kappa$ and therefore every subspace of Y that is of density $< \kappa$ is actually contained in some Z_α. This completes the proof. ∎

Since $\mathfrak{c}^{<\aleph_1} = \mathfrak{c}^{\aleph_0} = \mathfrak{c}$, we obtain the following corollary, without extra assumptions on cardinal arithmetic.

COROLLARY 4.3. ([3]) *There exists a Banach space of density $\mathfrak{c}$ which is of universal disposition for separable Banach spaces.*

The arguments from the last part of the proof of Theorem 4.2 show that the construction could be somewhat optimized. Namely, since we know that $\mu^{<\kappa}$ has cofinality $\geqslant \kappa$ and clearly $\mu^{<\kappa} \geqslant 2^{<\kappa} \geqslant \kappa$, we conclude that either $\mu^{<\kappa} = \kappa$ and κ is a regular cardinal, or else $\mu^{<\kappa} \geqslant \kappa^+$ and κ^+ is always a regular cardinal. Thus, the space Y can be constructed as the union of a continuous chain of length either κ (if κ is regular) or κ^+ (if κ is singular). On the other hand, it is not clear whether taking the shorter chain we really obtain a different Banach space.

The theorem above does not say anything about uniqueness. The only known fact, coming from the general Fraïssé-Jónsson theory, is as follows.

THEOREM 4.4. *Let κ be an uncountable cardinal satisfying $\kappa^{<\kappa} = \kappa$. Then there exists a unique, up to isometry, Banach space $\mathbb{V}_\kappa$ of density κ and of universal disposition for Banach spaces of density $< \kappa$. Furthermore, every isometry between subspaces of $\mathbb{V}_\kappa$ of density $< \kappa$ extends to a bijective isometry of $\mathbb{V}_\kappa$.*

Proof. The existence of $\mathbb{V}_\kappa$ is an application of Theorem 4.2 with $\mu = \kappa$. The second statement and the uniqueness of $\mathbb{V}_\kappa$ follow from Proposition 4.1(2). ∎

Note that Theorem 4.2 shows the existence of strong Gurariĭ spaces. In fact, all spaces that are $\mathrm{UD}(\mathfrak{D}_{<\kappa})$ are strong Gurariĭ, but on the other hand one can construct a strong Gurariĭ space using pushouts with finite-dimensional spaces only. As proved in [3], such a space is not $\mathrm{UD}(\mathrm{sep})$. We explain the details in Section 6, showing that it is even not universal for spaces of density $\aleph_1$.

Note that the "pushout construction" can be continued "forever". In other words, there is no upper bound for the density of a strong Gurariĭ space. In fact, a well-known property of infinite cardinals is that if $\mu^{<\kappa} = \mu$ then $(\mu^+)^{<\kappa} = \mu^+$, therefore one can use Theorem 4.2 to construct spaces of $\mathrm{UD}(\mathfrak{D}_{<\kappa})$ that are of densities μ^+, μ^{++}, and so on. The problem of existence arises when one reaches a limit cardinal, however it can always be "skipped", replaced by its successor.

In view of the recent results of Avilés and Brech [2], a strong Gurariĭ space of density $\mathfrak{c}$ constructed by pushouts is in some sense unique, as long as $\mathfrak{c}$ is a regular cardinal.

5. ON THE STRUCTURE OF STRONG GURARIĬ SPACES

The following had already been observed by Gurariĭ. The proof comes from his work [7].

PROPOSITION 5.1. *No separable Banach space can be a strong Gurariĭ space.*

Proof. Suppose U is a separable strong Gurariĭ space. For every two points a, b on the unit sphere of U there exists a unique linear isometry $f \colon X_a \to X_b$

satisfying $f(a) = b$, where X_a, X_b are linear spans of $\{a\}$ and $\{b\}$ respectively. Applying Proposition 4.1(2), we conclude that for every two points a, b on the unit sphere of U there exists a bijective isometry h of U such that $h(a) = b$.

Now, using a theorem of Mazur on the existence of smooth points on the unit sphere in every separable Banach space, we deduce that every point on the unit sphere of U is smooth. Recall that $p \in \mathrm{S}_U$ is *smooth* if there exists only one functional $\varphi \in U^*$ such that $\|\varphi\| = 1 = \varphi(p)$.

Finally, we get a contradiction by applying Proposition 4.1(1) which says that every separable Banach space is isometric to a subspace of U; in particular the unit sphere of U must contain non-smooth points. Note that a point that is non-smooth in a subspace of U cannot be smooth in U, by the Hahn-Banach extension theorem. ∎

A Banach space X is called *transitive* if for every a, b in the unit sphere of X there exists a bijective isometry $h \colon X \to X$ such that $h(a) = b$. The argument above shows that a transitive separable space must be smooth. This is closely related to Mazur's rotation problem: Does there exist a separable transitive Banach space, different from the Hilbert space? According to our knowledge, this problem is still open.

Recall that a Banach space X is 1-*injective for finite-dimensional spaces* if for every pair $E \subseteq F$ of finite-dimensional spaces, every bounded linear operator $f \colon E \to X$ extends to an operator $g \colon F \to X$ with $\|g\| = \|f\|$.

PROPOSITION 5.2. *The Gurariĭ space is not 1-injective for finite-dimensional Banach spaces.*

Proof. According to [21, Example 6.2], there exists a Banach space $E = C(K)$, where K is a metric compact space, that is not 1-injective for finite-dimensional Banach spaces. Every $C(K)$ space is a π_1^∞ space (see [18]), therefore by Theorem 2.18 the space E is 1-complemented in the Gurariĭ space $\mathbb{G}$. Finally, if $\mathbb{G}$ were 1-injective for finite-dimensional spaces, then so would be E, a contradiction. ∎

The following negative result is in contrast to Theorem 3.7.

THEOREM 5.3. *Let E be a non-separable strong Gurariĭ space and let $\mathcal{G}$ be a skeleton in E. Then there exists $G \in \mathcal{G}$ that is not 1-complemented in E.*

Proof. Suppose $\mathcal{G}$ is a skeleton in E such that each $G \in \mathcal{G}$ is 1-complemented in E. By Theorem 3.4 and Proposition 3.1, we may assume that each member

of $\mathcal{G}$ is linearly isometric to the Gurariĭ space $\mathbb{G}$. We now claim that $\mathbb{G}$ is 1-injective for finite-dimensional spaces, which in view of Proposition 5.2 is a contradiction.

Fix finite-dimensional spaces $X \subseteq Y$ and fix an operator $f \colon X \to E$ with $\|f\| \leqslant 1$. By the Pushout Lemma, there are a finite-dimensional space W, an isometric embedding $j \colon f[X] \to W$ and a linear operator $g \colon Y \to W$ such that $\|g\| \leqslant 1$ and $g \restriction X = j \circ f$. There exists $G \in \mathcal{G}$ such that $f[X] \subseteq G$. Let $P \colon E \to E$ be a projection such that $\|P\| = 1$ and $P[E] = G$. Using the fact that E is a strong Gurariĭ space, we find an isometric embedding $k \colon W \to E$ such that $k \circ j$ is the inclusion $f[X] \subseteq E$. The operator $P \circ k \circ g$ is an extension of f and has norm $\leqslant 1$. ∎

Note that exactly the same proof shows that $\mathbb{G}$ is not a strong Gurariĭ space. This argument does not use Mazur's theorem on the existence of smooth points.

Recall that a Banach space is *weakly Lindelöf determined* if its dual has a weak star continuous one-to-one linear operator into some Σ-product of the real lines, i.e., a linear topological space of the form

$$\Sigma(\Gamma) = \big\{ x \in \mathbb{R}^\Gamma : |\{\gamma \colon x(\gamma) \neq 0\}| \leqslant \aleph_0 \big\},$$

endowed with the product topology. This class of Banach spaces contains all weakly compactly generated (in particular, all reflexive) spaces. It is well known (see, e.g., [10, Ch. 19]) that a weakly Lindelöf determined Banach space always contains a skeleton of 1-complemented subspaces and this does not depend on the norm of the space (i.e. it holds after any renorming). Thus, Theorem 5.3 gives the following

COROLLARY 5.4. *No strong Gurariĭ space can be weakly Lindelöf determined.*

One can go further and conclude that no strong Gurariĭ space has a monotone (transfinite) Schauder basis (see, e.g., [16] for the definition and results on transfinite Schauder bases). The reason is again that such a space has a skeleton of 1-complemented spaces (with standard monotone Schauder bases). This property, however, is not preserved after renormings and indeed it is not clear whether there exists a strong Gurariĭ space with any transfinite Schauder basis, or more generally, isomorphic to a space with a projectional resolution of the identity.

6. THE ROLE OF c_0

A well known theorem of Sobczyk [19] says that c_0 is complemented in every separable Banach space. More precisely, for every isometric embedding $i\colon c_0 \to X$ with X separable, there exists a linear operator $T\colon X \to c_0$ satisfying $T \circ i = \mathrm{id}_{c_0}$ and $\|T\| \leqslant 2$ (see, e.g., the proof of Sobczyk's theorem in [10, Thm. 17.2]). We are going to prove the same for the class of "pushout generated" Banach spaces that includes some strong Gurariĭ spaces (see [3] or remarks after the proof of Theorem 4.4 above). As a consequence, we answer Problem 1 from [3].

The next fact explains why complementability of c_0 forces the space not to be of universal disposition for Banach spaces of density $\leqslant \aleph_1$. For this aim we need to know the fact that Sobczyk's theorem fails for Banach spaces of density $\aleph_1$ (regardless of the validity of the continuum hypothesis).

Recall that a family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ is *almost disjoint* if $A \cap B$ is finite for every $A \neq B$ in $\mathcal{A}$. There is a natural locally compact topology on $\mathbb{N} \cup \mathcal{A}$ whose base consists of all the singletons of $\mathbb{N}$ and all sets of the form $\{A\} \cup (A \setminus F)$ with $F \subseteq \mathbb{N}$ finite. Let $K_{\mathcal{A}}$ be the one-point compactification of this space. In the literature, spaces of the form $K_{\mathcal{A}}$ are often called *Mrówka compacta*, although they were considered first by Alexandroff and Urysohn [1]. Notice that $C(K_{\mathcal{A}})$ has a natural isometric copy of c_0; the standard basis consists of all characteristic functions of the singletons of $\mathbb{N}$. This copy of c_0 is not complemented in $C(K_{\mathcal{A}})$, unless $\mathcal{A}$ is countable. For the proof, see [10, Cor. 17.4]. Clearly, $\mathcal{A}$ can be taken so that $|\mathcal{A}| = \aleph_1$ and therefore c_0 is not complemented in some Banach space of density $\aleph_1$.

PROPOSITION 6.1. *Let X be a Banach space of* UD(*sep*). *Then no copy of c_0 can be complemented in X.*

Proof. Let $Z = C(K_{\mathcal{A}})$ for some almost disjoint family $\mathcal{A}$ of cardinality $\aleph_1$ and consider c_0 as the canonical non-complemented copy of Z. Let $E \subseteq X$ be isomorphic to c_0 and let $f\colon c_0 \to E$ be an isomorphism. Using Lemma 1.1, find an equivalent norm on Z such that f becomes an isometry. By Proposition 4.1(0), there is an isometry $g\colon Z \to X$ such that $g \restriction c_0 = f$. It is now clear that E cannot be complemented in $g[Z] \subseteq X$, therefore it cannot be complemented in X. ∎

We are now going to show that Sobczyk's theorem holds in a class of Banach spaces containing strong Gurariĭ spaces of arbitrarily large density.

DEFINITION 6.2. Let $\mathfrak{PO}_{fd}$ denote the class of all Banach spaces that can be obtained as the limit (i.e. the completion of the union) of a transfinite chain $\{X_\alpha\}_{\alpha<\varrho}$ such that X_0 is separable, $X_\delta = \mathrm{cl}(\bigcup_{\xi<\delta} X_\xi)$ for every limit ordinal $\delta < \varrho$ and for each $\alpha < \varrho$, the space $X_{\alpha+1}$ comes from the pushout square

$$
\begin{array}{ccc}
X_\alpha & \overset{\subseteq}{\longrightarrow} & X_{\alpha+1} \\
{\scriptstyle j_\alpha}\big\uparrow & & \big\uparrow \\
E_\alpha & \underset{\subseteq}{\longrightarrow} & F_\alpha
\end{array}
$$

where $E_\alpha \subseteq F_\alpha$ are finite-dimensional spaces and j_α is an isometric embedding. More specifically, we shall write $X \in \mathfrak{PO}_{fd}(Y)$ if X is the limit of a chain as above, in which $Y = X_0$.

As mentioned before, it has been proved in [3] that the class $\mathfrak{PO}_{fd}$ contains strong Gurariĭ spaces (see the proof of Theorem 4.2 and comments in the end of Section 4).

Before proving our result, we need the following lemma, which can be easily deduced from a variant of [2, Lemma 20] involving finite-dimensional spaces.

LEMMA 6.3. *Let Z be a separable subspace of a space $X \in \mathfrak{PO}_{fd}$. Then there exists a separable space $Y \subseteq X$ such that $Z \subseteq Y$ and $X \in \mathfrak{PO}_{fd}(Y)$.*

THEOREM 6.4. *Let $X \in \mathfrak{PO}_{fd}$. Then every copy of c_0 is complemented in X.*

Proof. Let $C \subseteq X$ be isometric to c_0. By Lemma 6.3, we may assume that $C \subseteq X_0$ for some separable space X_0 such that $X = \mathrm{cl}(\bigcup_{\xi<\varrho} X_\xi)$, where the chain $\{X_\xi\}_{\xi<\varrho}$ satisfies the conditions in Definition 6.2. By Sobczyk's theorem, there exists a projection $P\colon X_0 \to C$ with $\|P\| \leqslant 2$.

Set $P_0 = P$. We now construct inductively projections $P_\alpha\colon X_\alpha \to C$ so that P_β extends P_α whenever $\beta > \alpha$ and $\|P_\alpha\| = \|P\|$ for every α. Suppose P_ξ have been constructed for $\xi < \alpha$. If α is a limit ordinal, we define P_α to be the pointwise limit of $\{P_{\xi_n}\}_{n\in\omega}$, where $\xi_0 < \xi_1 < \cdots < \alpha$ converges to α. Here we have used the fact that X_α is the closure of $\bigcup_{n\in\omega} X_{\xi_n}$.

Now suppose $\alpha = \eta + 1$ and fix a pushout square

$$
\begin{array}{ccc}
X_\eta & \overset{\subseteq}{\longrightarrow} & X_\alpha \\
{\scriptstyle j}\big\uparrow & & \big\uparrow{\scriptstyle k} \\
E & \underset{\subseteq}{\longrightarrow} & F
\end{array}
$$

defining X_α, with finite-dimensional spaces E, F. Using the fact that c_0 is 1-injective for finite-dimensional spaces, we find a linear operator $T\colon F \to C$ satisfying $T \upharpoonright E = P_\eta \circ j$ and $\|T\| = \|P_\eta \circ j\| = \|P_\eta\|$. By the pushout property, there exists a unique operator $P_\alpha\colon X_\alpha \to C$ satisfying $P_\alpha \upharpoonright X_\eta = P_\eta$, $P_\alpha \circ k = T$ and $\|P_\alpha\| = \|P_\eta\|$.

Finally, $P = \lim_{\xi < \varrho} P_\xi$ is the required projection. ∎

It has been shown in [3] (with almost the same arguments) that if $X \in \mathfrak{PO}_{fd}(Y)$, where Y is linearly isometric to c_0, then Y is 1-complemented in X.

COROLLARY 6.5. *Let* $X \in \mathfrak{PO}_{fd}$. *Then* X *cannot contain any isomorphic copy of* $C(K_{\mathcal{A}})$, *where* $\mathcal{A}$ *is an almost disjoint family of infinite subsets* $\mathbb{N}$ *and* $|\mathcal{A}| = \aleph_1$.

This answers Problem 1 from [3]: There exist strong Gurariĭ spaces (of arbitrarily large density) that are not universal for Banach spaces of density $\aleph_1$.

7. FINAL REMARKS AND OPEN PROBLEMS

Below we collect some open questions; some of them are motivated by the results described in previous sections.

MINIMAL DENSITY. It is not clear what the minimal density of a strong Gurariĭ space is. The only known bound is the continuum. A more concrete question is:

QUESTION 7.1. Does there exist, without extra set-theoretic assumptions, a strong Gurariĭ space of density $\aleph_1$?

QUESTION 7.2. Assuming $\mathfrak{c} < \aleph_\omega$, does there exist a strong Gurariĭ space of density $\aleph_\omega$?

Note that $\aleph_\omega$ is the smallest singular cardinal and it has cofinality ω; therefore always $\mathfrak{c} \neq \aleph_\omega$.

SCHAUDER BASES. A Banach space with a PRI and of density $\aleph_1$ has a countably 1-norming Markushevich basis (see, e.g., [10, Section 17.8]). A Markushevich basis can be viewed as a natural "non-separable" generalization of Schauder bases, although, contrary to Schauder bases, it exists in every separable Banach space. Theorem 3.7 motivates the following

QUESTION 7.3. Does there exist a Gurariĭ space of density $\aleph_1$ with a monotone transfinite Schauder basis?

Note that by Theorem 5.3 such a space cannot be strong Gurariĭ. Let us mention that some of the "generic" Banach spaces constructed in [16] are Gurariĭ, although none of them has a transfinite Schauder basis.

QUESTION 7.4. Does there exist a strong Gurariĭ space, isomorphic to a Banach space with a PRI?

QUESTION 7.5. Does there exist a non-separable weakly Lindelöf determined (or better: weakly compactly generated) Gurariĭ space?

Again, this cannot be a strong Gurariĭ space. Note that every weakly Lindelöf determined Banach space has a countably 1-norming Markushevich basis.

RENORMINGS. Recall that a norm $\|\cdot\|$ is *rotund* if $\|x+y\| = 2\|x\| = 2\|y\|$ implies $x = y$. A *rotund renorming* is an equivalent norm that is rotund. Many non-separable Banach spaces have rotund renormings, for a general treatment we refer to the book [4]. A result of Zizler [22] says that the existence of a renorming stronger than rotund (namely: locally uniformly rotund) is preserved by a PRI. In particular, every Banach space of density $\aleph_1$ and with a PRI has a rotund renorming. In view of Theorem 3.7, there exist non-separable Gurariĭ spaces admitting a rotund renorming. This suggests:

QUESTION 7.6. Does there exist a strong Gurariĭ space with a rotund renorming?

A typical example of a Banach space with no rotund renorming is ℓ_∞/c_0 (see [4]). Unfortunately, this space has density $\mathfrak{c}$ and the following interesting question, due to Antonio Avilés, seems to be open.

QUESTION 7.7. Does there exist, without extra set-theoretic assumptions, a Banach space X of density exactly $\aleph_1$ and with no rotund renorming?

A positive answer to this question would yield a simple and direct proof of the following result.

THEOREM 7.8. *No Banach space of universal disposition for separable spaces can have a rotund renorming.*

Indeed, a space of UD(sep) contains copies of all Banach spaces of density $\aleph_1$, so all of them would have to admit rotund renormings. Assuming CH, this gives a contradiction. Still, the statement above is a theorem. For readers familiar with the technique of forcing, we sketch a "metamathematical" proof, involving absoluteness.

Proof. Suppose the statement above is not a theorem, i.e. it is not a consequence of the usual axioms of set theory. By Gödel's completeness, there exists a model of set theory $\mathbb{V}$ that contains a Banach space X of UD(sep) with rotund renorming. There exists an extension $\mathbb{W}$ of $\mathbb{V}$ (obtained by forcing) such that $\mathbb{W}$ is a model of set theory in which the continuum hypothesis holds and, moreover, for every function $\varphi \colon \omega \to S$ in $\mathbb{W}$ if $S \in \mathbb{V}$ then $\varphi \in \mathbb{V}$. The last property of $\mathbb{W}$ implies that X is a Banach space in $\mathbb{W}$ and it is of UD(sep). The latter fact is because $\mathbb{W}$ does not contain "new" separable Banach spaces. Finally, X still has a rotund renorming, since this property is preserved. This leads to a contradiction, since in $\mathbb{W}$ the space X contains a copy of ℓ_∞/c_0. ∎

ULTRA-HOMOGENEITY.   Let $\mathfrak{K}$ be a class of Banach spaces. We say that a Banach space X is *homogeneous* with respect to $\mathfrak{K}$ if every bijective isometry between two subspaces of X that are in class $\mathfrak{K}$ extends to an isometry of X onto itself. If $\mathfrak{K}$ contains all 1-dimensional subspaces of X, homogeneity implies transitivity. In fact, the difficulty of Mazur's problem on rotations exhibits the fact that so far the Hilbert space is the only known example of a separable Banach space homogeneous for finite-dimensional spaces. Now let $\mathfrak{K} = \mathfrak{D}_{<\kappa}$, the class of all Banach spaces of density $< \kappa$. Proposition 4.1(2) says that every space of UD($\mathfrak{K}$) and of density κ is homogeneous with respect to $\mathfrak{K}$. On the other hand, in view of the results of [2], there exist (arbitrarily large) Banach spaces of UD(sep) that are homogeneous with respect to separable subspaces. It is not clear what happens with strong Gurariĭ spaces.

QUESTION 7.9. Does there exist a strong Gurariĭ space, homogeneous with respect to finite-dimensional subspaces and not of universal disposition for separable spaces?

QUESTION 7.10. Does there exist a strong Gurariĭ space that is not homogeneous for finite-dimensional spaces?

In fact, we do not know the answer to a more general question:

QUESTION 7.11. Does there exist a Banach space of $\mathrm{UD}(\mathfrak{D}_{<\kappa})$ that is not homogeneous with respect to $\mathfrak{D}_{<\kappa}$?

We finish with the following problem whose solution may lead to a better understanding of Mazur's rotation problem.

PROBLEM 7.12. Find a class $\mathfrak{K}$ of finite-dimensional Banach spaces with the following properties:

 (i) $\mathfrak{K}$ is hereditary (i.e. $X \subseteq Y \in \mathfrak{K}$ implies $X \in \mathfrak{K}$).
 (ii) All spaces in $\mathfrak{K}$ are smooth.
 (iii) For each $n \in \mathbb{N}$, $\mathfrak{K}$ contains a space of dimension n.
 (iv) $\mathfrak{K}$ has the amalgamation property. That is, given isometric embeddings $i\colon Z \to X$, $j\colon Z \to Y$ with $X, Y \in \mathfrak{K}$, there exist $W \in \mathfrak{K}$ and isometric embeddings $i'\colon X \to W$, $j'\colon Y \to W$ satisfying $j' \circ j = i' \circ i$.
 (v) $\mathfrak{K}$ is not dense (with respect to the Banach-Mazur distance) in the class of all finite-dimensional Banach spaces.
 (vi) $\mathfrak{K}$ is not the class of Euclidean spaces.

Actually, it is desirable to replace condition (vi) by a formally stronger one: $\mathfrak{K}$ contains a chain $\{X_n\}_{n\in\omega}$ such that the completion of $\bigcup_{n\in\omega} X_n$ is not isomorphic to the Hilbert space.

Having such a class $\mathfrak{K}$, one would be able to construct a Banach space $\mathbb{G}_{\mathfrak{K}}$ satisfying the definition of the Gurariĭ space for finite-dimensional spaces from class $\mathfrak{K}$ only. If the class $\mathfrak{K}$ had an additional property that $\mathbb{G}_{\mathfrak{K}}$ remains smooth (which does not follow from condition (ii)), Gurariĭ's argument would not be applicable for showing that $\mathbb{G}_{\mathfrak{K}}$ is not transitive. In any case, $\mathbb{G}_{\mathfrak{K}}$ would be a new Banach space "almost" homogeneous with respect to its finite-dimensional subspaces and not isomorphic to the Hilbert space.

Acknowledgements

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