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Torsion Graph of Modules

Sh. Ghalandarzadeh, P. Malakooti Rad

Department of Mathematics, Faculty of Science, K. N. Toosi University of Technology, P.O. Box 16315 − 1618*, Tehran, Iran*

ghalandarzadeh@kntu.ac.ir , pmalakooti@dena.kntu.ac.ir

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Abstract: Let R be a commutative ring and M be an R-module. We associate to M a graph denoted by, $\Gamma(M)$ called the torsion graph of M, whose vertices are the non-zero torsion elements of M and two distinct elements x, y are adjacent if and only if $[x : M][y : M]M = 0$. We investigate the interplay between module-theoretic properties of M and graph-theoretic properties of $\Gamma(M)$. Among other results, we prove that $\Gamma(M)$ is connected and $\mathrm{diam}(\Gamma(M)) \leq 3$ for a faithful R-module M.

Key words: Torsion graph, multiplication modules, diameter of torsion graph.

AMS *Subject Class.* (2010): 13A99, 05C99, 13C99.

1. Introduction

The concept of a zero-divisor graph of a commutative ring was introduced by I. Beck in 1988 [7]. He let all elements of the ring be vertices of the graph and was interested mainly in colorings. This investigation of colorings of a commutative ring was then continued by Anderson and Naseer in [4]. In [3], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the non-zero zero-divisors and where $x - y$ is an edge whenever $xy = 0$. The zero-divisor graph of a commutative ring has been studied extensively by Anderson, Frazier, Lauve, Levy, Livingston and Shapiro, see [2, 3]. The zero-divisor graph concept has been extended to non-commutative rings by Redmond in [14]. The zero-divisor graph has also been introduced and studied for semigroups by DeMeyer and Schneider in [9], and for near-rings by Cannon et al. in [8].

Let R be a commutative ring with identity and M be a unitary R-module. In this paper, we will investigate the concept of a torsion-graph for modules as a natural generalization of the zero-divisor graph for rings, which has been defined by Ghalandarzadeh and Malakooti Rad in [12]. The residual of Rx by M, denoted by $[x : M]$, is the set of elements $r \in R$ such that $rM \subseteq Rx$ for $x \in M$. The annihilator of an R-module M denoted by $\mathrm{Ann}_R(M)$ is $[0 : M]$.

Let $T(M)$ be the set of elements of M such that $\operatorname{Ann}(m) \neq 0$. It is clear that if R is an integral domain, then $T(M)$ is a submodule of M, and it is called torsion submodule of M. If $T(M) = 0$, then the module M is said to be torsion-free, and it is called a torsion module if $T(M) = M$. Here the torsion graph $\Gamma(M)$ of M is a simple graph whose vertices are the non-zero torsion elements of M, and two distinct elements x, y are adjacent if and only if $[x : M][y : M]M = 0$. Thus, $\Gamma(M)$ is an empty graph if and only if M is a torsion-free R-module. In this paper, we will investigate the interplay of module properties of M in relation to the properties of $\Gamma(M)$. We believe that this study helps illuminate the structure of $T(M)$. For example, if M is a faithful multiplication R-module, we show that M is finite if and only if $\Gamma(M)$ is finite. Also, we think that torsion graphs help us study the algebraic properties of modules using graph-theoretical tools. For $x, y \in T(M)^* = T(M) - \{0\}$, define $x \sim y$ if $[x : M][y : M]M = 0$ or $x = y$. The relation $\sim$ is always reflexive and symmetric, but is usually not transitive. The torsion graph $\Gamma(M)$ measures this lack of transitivity in the sense that it is transitive if and only if $\Gamma(M)$ is complete.

An R-module M is called a multiplication module if for every submodule K of M, there exists an ideal I of R such that $K = IM$ (Barnard [6]). A proper submodule N of M is called a prime submodule of M, if $rm \in N$ (where $r \in R$ and $m \in M$) implies that $m \in N$ or $r \in [N : M]$.

Recall that a graph is finite if both its vertices set and edge set are finite, and we use symbol $|\Gamma(M)|$ to denote the number of vertices in the graph $\Gamma(M)$. Also, a graph G is connected if there is a path between any two distinct vertices. The distance $d(x, y)$ between connected vertices x, y is the length of a shortest path from x to y ($d(x, y) = \infty$ if there is no such path). The diameter of G is the diameter of a connected graph, which is the supremum of the distances between vertices. The diameter is 0 if the graph consists of a single vertex. The girth of G, denoted by $\operatorname{gr}(G)$, is defined as the length of the shortest cycle in G ($\operatorname{gr}(G) = \infty$ if G contains no cycles).

In section 2, we give many examples and we show that $\Gamma(M)$ is always connected with $\operatorname{diam}(\Gamma(M)) \leq 3$ if M is a faithful R-module.

Throughout the paper, for $N \subseteq M$, we let $N^* = N - \{0\}$. As usual, the rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively. We use the symbol (x, y) or $x + y$ to denote the elements of $M = M_1 \oplus M_2$. Also, we use the symbol $(M)_R$ to denote M as an R-module.

$$\operatorname{Nil}(M) := \bigcap_{N \in \operatorname{Spec}(M)} N\,,$$

where $\mathrm{Spec}(M)$ is a set of the prime submodules of M, and finally, let

$$D(M) := \left\{ m \in M \,:\, [m : M][m' : M]M = 0 \text{ for some non-zero } m' \in M \right\}.$$

To avoid trivialities when $\Gamma(M)$ is empty, we will assume implicitly when necessary that M is not torsion-free.

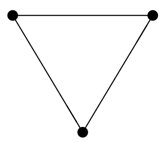
2. Properties of $\Gamma(M)$

In this section, we show that if M is faithful then $\Gamma(M)$ is connected and has small diameter and girth.

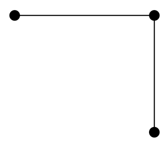
Example 2.1. (a) Let $M = M_1 \oplus M_2$ be an R-module, where M_1 is a torsion-free module. So $T(M)^* = \{(0, m_2) : m_2 \in T(M_2)^*\}$ and $[(0, m_2) : M] = 0$. Hence $\Gamma(M)$ is a complete graph. Below are the torsion graphs for several modules. Note that these examples show that non-isomorphic modules may have the same torsion graph.

$(\mathbb{Z}_4)_{\mathbb{Z}_4}$

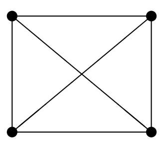
$(\mathbb{Z} \oplus \mathbb{Z}_3)_{\mathbb{Z}}$ $(\mathbb{Z}_9)_{\mathbb{Z}_9}$



$(\mathbb{Z} \oplus \mathbb{Z}_4)_{\mathbb{Z}}$ $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)_{\mathbb{Z}}$

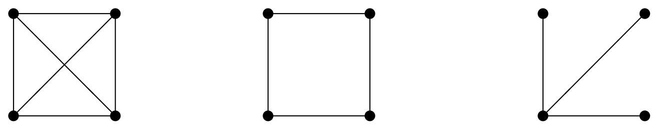


$(\mathbb{Z}_2 \oplus \mathbb{Z}_3)_{\mathbb{Z}_6}$ $(\mathbb{Z}_8)_{\mathbb{Z}_8}$



$(\mathbb{Z}_{25})_{\mathbb{Z}_{25}}$ $(\mathbb{Z} \oplus \mathbb{Z}_5)_{\mathbb{Z}}$

(b) By part (a) above, all connected graphs with less than four vertices may be regarded as $\Gamma(M)$. Of the eleven graphs with four vertices, only six graphs are connected. Of these six graphs, only the following three graphs may be realized as $\Gamma(M)$, when M is a faithful multiplication R-module, by [3, Example 2.1], there are three $\Gamma(R)$.



$\mathbb{Z}_2$ $\qquad$ $\mathbb{Z}_3 \times \mathbb{Z}_3$ $\qquad$ $\mathbb{Z}_2 \times \mathbb{F}_4$

It is clear that every ring R is a multiplication R-module. We next explain a proof that the graph G with vertices $\{a, b, c, d\}$ and edges $a - b$, $b - c$, $c - d$ cannot be realized as $\Gamma(M)$. Suppose that there is a ring R and faithful multiplication R-module M with $T(M)^* = \{a, b, c, d\}$ together with only the above torsion relations. Observe that

$$[a : M][b : M]M \;=\; 0 \;=\; [b : M][c : M]M\,,$$

so

$$[b : M]a \;=\; 0 \;=\; [b : M]c$$

and $[b : M][a + c : M]M = 0$. Hence $a + c \in T(M)^*$ and so $a + c$ must be either a, b, c or d. If $a + c = a$ or $a + c = c$, then $a = 0$ or $c = 0$ and we have a contradiction. Also, if $a + c = d$, then $[d : M][b : M]M = 0$, which is a contradiction. Therefore, $a + c = b$ is the only possibility. Similarly, $b + d = c$. Hence $b = a + c = a + b + d$; so $[a : M][d : M]M = 0$, which is a contradiction. The proofs for the other two non-realizable connected graphs on four vertices are similar.

Theorem 2.2. *If M is a multiplication R-module, then $\Gamma(M)$ can not be an n-gon for $n \geq 5$.*

Proof. Let G be the graph with vertices $\{a_1, a_2, \ldots, a_n\}$ and $\Gamma(M)$ be a graph with edges

$$\{a_1 - a_2, a_2 - a_3, \ldots, a_{n-2} - a_{n-1}, a_{n-1} - a_n, a_n - a_1\}\,.$$

Since

$$[a_1 : M][a_2 : M]M \;=\; 0 \;=\; [a_1 : M][a_n : M]M\,,$$

it follows that $[a_1 : M][a_2 + a_n : M]M = 0$. Thus, $a_2 + a_n$ must be either a_1, a_2, $\ldots$, a_{n-1} or a_n. A simple check yields that $a_2 + a_n = a_1$ is the only possibility. Similarly, $a_1 + a_{n-1} = a_n$. Hence

$$a_n = a_1 + a_{n-1} = a_2 + a_n + a_{n-1}\,;$$

so $[a_3 : M][a_{n-1} : M]M = 0$, is a contradiction. Consequently $\Gamma(M)$ can not be an n-gon for $n \geq 5$. ∎

An r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is in another subset. The complete bipartite graph (i.e., 2-partite graph) with vertex sets having m and n elements, respectively, will be denoted by $K_{m,n}$. A complete bipartite graph of the form $K_{1,n}$ is called a star graph.

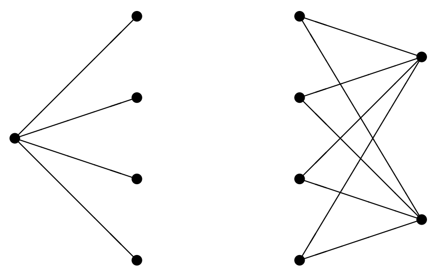
Example 2.3. Let M_1 be a multiplication torsion-free R_1-module and M_2 be a multiplication torsion-free R_2-module, then $M = M_1 \times M_2$ is $R = R_1 \times R_2$ module with multiplication

$$R \times M \longrightarrow M\,, \qquad (r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)\,.$$

$\Gamma(M)$ is a complete bipartite graph (i.e., $\Gamma(M)$ may be partitioned into two disjoint vertex sets

$$V_1 = \{(m_1, 0)\,:\, m_1 \in M_1^*\}\,, \qquad V_2 = \{(0, m_2)\,:\, m_2 \in M_2^*\}$$

and two vertices x and y are adjacent if and only if they are in distinct vertex sets), with $|\Gamma(M)| = |M_1| + |M_2| - 2$. Here are two specific examples:



$(\mathbb{Z}_2 \oplus \mathbb{Z}_5)_{\mathbb{Z}_{10}}$ $\qquad$ $(\mathbb{Z}_3 \oplus \mathbb{Z}_5)_{\mathbb{Z}_{15}}$

We know that $\Gamma(M)$ may be infinite(i.e., R-module M has infinite torsion elements). But an interesting case occurs when $\Gamma(M)$ is finite, because in a finite case a drawing of the graph is possible. First, we focus one's attention on the infinite graphs and later in this article we will consider the interesting case of finite graphs, and the next theorem shows that $\Gamma(M)$ is finite (except when $\Gamma(M)$ is empty) if and only if M is finite.

Theorem 2.4. *Let R be a commutative ring and M be a faithful multiplication R-module. Then $\Gamma(M)$ is finite if and only if either M is finite or M is a torsion free R-module. If $1 \leq |\Gamma(M)| < \infty$, then M is finite and not torsion-free.*

Proof. Suppose that $\Gamma(M)(= T(M)^*)$ is finite and nonempty. Then there exists $x \in T(M)^*$ such that $rx = 0$ for some $r \in R$. Let $N := [x : M]M$ and let $0 \neq s \in [x : M]$; we know $rn = 0$ for all $n \in N$. Hence $N \subseteq T(M)^*$ is finite and $sm \in N$ for all $m \in M$. If M is infinite, then there is a $n \in N$ with $H = \{m \in M : sm = n\}$ infinite. For all distinct elements $m_1, m_2 \in H$, $s(m_1 - m_2) = 0$. So $m_1 - m_2 \in T(M)^*$ is infinite, is a contradiction. Thus, M must be finite. ∎

The following example shows that the multiplication condition is not superfluous.

EXAMPLE 2.5. Let $M = \mathbb{Z} \oplus \mathbb{Z}_3$ as $\mathbb{Z}$-module. Clearly M is not a finite multiplication module, but $T(M)^* = \{(0, \bar{1}), (0, \bar{2})\}$ and so $\Gamma(M)$ is finite.

Example 2.1 (a), gives several R-modules with $\operatorname{diam}(\Gamma(M)) = 0, 1$ or 2. In $R = \mathbb{Z}_2 \oplus \mathbb{Z}_4$, the path $(\bar{0}, \bar{1}) - (\bar{1}, \bar{0}) - (\bar{0}, \bar{2}) - (\bar{1}, \bar{2})$ shows that $\operatorname{diam}(\Gamma(R_R)) = 3$. Now we next show that the torsion graphs of faithful R-modules are all connected with diameter ≤ 3.

THEOREM 2.6. *Let R be a commutative ring and M be a faithful R-module. Then $\Gamma(M)$ is connected and $\operatorname{diam}(\Gamma(M)) \leq 3$. Moreover, if $\Gamma(M)$ contains a cycle, then $\operatorname{gr}(\Gamma(M)) \leq 7$.*

Proof. Let $x, y \in T(M)^*$ be two distinct elements. If $[x : M]$ or $[y : M]$ or $[x : M][y : M]$ is zero, then $d(x, y) = 1$. Therefore, we suppose that $[x : M][y : M]$ is nonzero, so there is a non-zero element $\alpha \in [x : M][y : M]$.

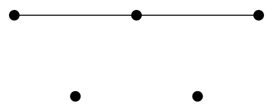
If $[x : M]^2 = [y : M]^2 = 0$, then there exists $m \in M$ such that $\alpha m \in T(M)^*$, and hence $x - \alpha m - y$ is a path of length 2; thus, $d(x, y) = 2$. Now suppose that $[x : M]^2 = 0$ and $[y : M]^2 \neq 0$, since $y \in T(M)^*$, $sy = 0$ for some $0 \neq s \in R$. Now we consider the case $[x : M] \operatorname{Ann}(y) = 0$. In this case $sm_0 \in T(M)^*$ for some $m_0 \in M$ and so $x - sm_0 - y$ is a path of length 2. In the other case, if $[x : M] \operatorname{Ann}(y) \neq 0$, then $m_1 := \alpha_1 tm \in T(M)^*$ for some non-zero elements $\alpha_1 \in [x : M]$, $t \in \operatorname{Ann}(y)$, $m \in M$ and $x - m_1 - y$ is a path of length 2. A similar argument holds if $[x : M]^2 \neq 0$, $[y : M]^2 = 0$. Thus, we may assume that $[x : M]^2, [y : M]^2$ and $[x : M][y : M]$ are all nonzero.

If $\operatorname{Ann}(x) \nsubseteq \operatorname{Ann}(y)$ and $\operatorname{Ann}(y) \nsubseteq \operatorname{Ann}(x)$, then there are non-zero elements $r, s \in R$ such that $rx = 0$, $ry \neq 0$ and $sx \neq 0$, $sy = 0$; hence $ry, sx \in T(M)^*$. Now if $ry \neq sx$, then $x - ry - sx - y$ is a path of length 3. In the other case, if $ry = sx$, then $x - ry - y$ is a path of length 2. Therefore, $d(x, y) \leq 3$. Thus, we may assume that $\operatorname{Ann}(x) \subseteq \operatorname{Ann}(y)$

or $\mathrm{Ann}(y) \subseteq \mathrm{Ann}(x)$, if $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(y)$, then $rm \in T(M)^*$ for some $r \in \mathrm{Ann}(x)$, $m \in M$ and $x - rm - y$ is a path of length 2. A similar argument holds if $\mathrm{Ann}(y) \subseteq \mathrm{Ann}(x)$. Hence $d(x, y) \leq 3$; thus, $\mathrm{diam}(\Gamma(M)) \leq 3$. The "moreover" statement follows from [10, Proposition 1.3]. ∎

The following example shows that the faithful condition is not superfluous.

Example 2.7. Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$ as $\mathbb{Z}$-module; then $\Gamma(M)$ is not connected.



Anderson and Livingston [3], proved that if $\Gamma(R)$ contains a cycle, then $\mathrm{gr}(R) \leq 7$. They also proved that $\mathrm{gr}(R) \leq 4$ when R is Artinian. The following theorem shows that $\mathrm{gr}(R) \leq 4$ for any commutative rings.

Theorem 2.8. *Let M be a multiplication R-module. If $\Gamma(M)$ contains a cycle, then* $\mathrm{gr}(\Gamma(M) \leq 4$.

Proof. Let
$$m_0 - m_1 - m_2 - \cdots - m_n - m_0$$
be the shortest cycle of $T(M)$ for $n > 4$. If
$$[m_1 : M][m_{n-1} : M]M = 0\,,$$
then $\Gamma(M)$ contains a cycle
$$m_1 - m_2 - \cdots - m_{n-1}\,,$$
which is a contradiction. So there exist non-zero elements $\alpha \in [m_1 : M]$, $\beta \in [m_{n-1} : M]$ and $m \in M$ such that $\alpha\beta m \in V(\Gamma(M))$. If $\alpha\beta m \neq m_0$ and $\alpha\beta m \neq m_n$, then $\Gamma(M)$ contains a cycle $m_0 - \alpha\beta m - m_n - m_0$ is a contradiction. Therefore,
$$\alpha\beta m = m_0 \qquad \text{or} \qquad \alpha\beta m = m_n\,.$$
So, without loss of generality, assume $\alpha\beta m = m_0$; thus $[m_0 : M]m_0 = 0$. Now we show that $Rm_0 = \{0, m_0\} \subset Rm_1$. If there exists a non-zero element $x \in Rm_0$ such that $x \neq m_0$, then $m_0 - m_1 - x - m_0$ is a cycle of length 3,

which is a contradiction. Hence, there exists $y \in Rm_1$ such that $y \neq 0$ and $y \neq m_1$. By a routine argument we obtain $y \neq m_0$ and $y \neq m_2$; therefore, $m_0 - m_1 - m_2 - y - m_0$ is a cycle of length 4, which is a contradiction. Consequently, $\mathrm{gr}(\Gamma(M) \leq 4$. ∎

Theorem 2.9. *Let M be a multiplication R-module. Then there is a vertex of $\Gamma(M)$ which is adjacent to every other vertex if and only if either $M = M_1 \oplus M_2$ is a faithful R-module, where M_1 and M_2 are two submodules of M such that M_1 has only two elements, M_2 is finitely generated with $T(M) = \{(x, 0), (0, m_2) : x \in M_1, m_2 \in M_2\}$, or $T(M) = IM$, where I is an annihilator ideal of R (and hence if $T(M) \neq M$, then $T(M)$ is a prime submodule).*

Proof. $(\Longrightarrow)$ Suppose that $T(M) \neq IM$ for all annihilator ideal I of R and let $x \in T(M)^*$ be adjacent to every other vertex. Since $T(M) \neq \mathrm{Ann}(x)M$, we have $x \notin \mathrm{Ann}(x)M$. We divide the proof of the theorem into 6 claims, which are of some interest in their own right.

Claim 1: $N = \mathrm{Ann}(x)M$ is a prime submodule of M. It is clear that $N \neq M$, let $rm \in N$ and $m \notin N$ for non-zero elements $m \in M$, $r \in R$, therefore, $r[m : M][x : M]M = 0$, so $rkx = 0$ for all $k \in [m : M]$, hence $r \in \mathrm{Ann}(kx)$. But there is $k \in [m : M]$ such that $kx \in T(M)^*$, so $\mathrm{Ann}(kx)M \subseteq \mathrm{Ann}(x)M$. Thus, $rM \subseteq N$ and $r \in [N : M]$. Therefore, N is a prime submodule and as a consequence $[N : M]$ is a prime ideal.

Claim 2: $[x : M]M = [x : M]^2M$. If $[x : M]M \neq [x : M]^2M$, then $x \notin [x : M]^2M$, so $x \neq \alpha x$ for all $\alpha \in [x : M]$. Since $\alpha x = 0$ or $\alpha x \in T(M)^*$ and $[\alpha x : M][x : M]M = 0$, hence $\alpha^3M = 0$, therefore, $\alpha^3 \in \mathrm{Ann}(x)$. Since N is a prime submodule, $\alpha M \subseteq N$, thus, $x \in N$, which is a contradiction. Therefore, $[x : M]M = [x : M]^2M$.

Claim 3: $M = Rx \oplus \mathrm{Ann}(x)M$. Since $[x : M]M = [x : M]^2M$, we have $Rx = [x : M]x$. We know that Rx is a weakly cancellation R-module and so $R = [x : M] + \mathrm{Ann}(x)$. A simple check yields $M = Rx \oplus \mathrm{Ann}(x)M$. Hence, we may assume that $M = Rx \oplus M_2$ with $(x, 0)$ adjacent to every other vertex.

Claim 4: $Rx = \{0, x\}$. Let $c \in Rx$, then $(c, 0) \in T(M)^*$ and $[(c, 0) : M][(x, 0) : M]M = 0$ and hence $[(c, 0) : M]x = 0$, so $c = 0$.

Claim 5: M_2 is finitely generated. We claim that $D(M_2) = 0$. If $D(M_2) \neq 0$, then there is a $0 \neq m_2 \in M_2$, such that

$$[m_2 : M_2][m_2' : M_2]M_2 = 0$$

for some $0 \neq m_2' \in M_2$. Thus, $(x, m_2') \in T(M)^*$ which is adjacent to $(x, 0)$; therefore, $x = 0$, which is a contradiction, consequently $D(M_2) = 0$. Let $st \in \mathrm{Ann}(M_2)$ for $s, t \in R$. So $stM_2 = 0$ hence

$$[sM_2 : M_2][tM_2 : M_2]M_2 = 0\,.$$

Since $D(M_2) = 0$ we have $sM_2 = tM_2 = 0$. Thus, $\mathrm{Ann}(M_2)$ is prime ideal of R. Hence M_2 is a faithful $\frac{R}{\mathrm{Ann}(M_2)}$-module and $\frac{R}{\mathrm{Ann}(M_2)}$ is an integral domain and by [1, p. 572], M_2 is a finitely generated $\frac{R}{\mathrm{Ann}(M_2)}$-module, and so M_2 is a finitely generated R-module.

Claim 6: M is a faithful module. Now suppose that $0 \neq r \in \mathrm{Ann}(x) \cap \mathrm{Ann}(m_2)$ for some $m_2 \in M_2$, hence $[(x, 0) : M][(x, m_2) : M]M = 0$ and $[x : M]^2M = 0$, is a contradiction. Therefore, $\mathrm{Ann}(x) \cap \mathrm{Ann}(M_2) = 0$. So M is a faithful module and $T(M) = \{(x, 0),\, (0, m_2) : x \in M_1,\, m_2 \in M_2\}$.

($\Longleftarrow$) If $M = M_1 \oplus M_2$, where M_1 has only two elements $\{0, x\}$ and $T(M) = \{(x, 0),\, (0, m_2) : m_2 \in M_2\}$, then $(x, 0)$ is adjacent to every other vertex. And if $T(M) = \mathrm{Ann}(x)M$ for some non-zero $x \in M$, then x is adjacent to every other vertex. ∎

If R is reduced, M is a faithful multiplication R-module and $\Gamma(M)$ has a vertex adjacent to every other vertex, then M must have the form $M = M_1 \oplus M_2$ where M_1 has only two elements and M_2 is finitely generated.

Let $\mathrm{Spec}(M) = \{N < M : N \text{ is a prime submodule}\}$ and $\mathrm{Max}(M) = \{H < M : H \text{ is a maximal submodule }\}$, by [11, Theorem 2.5], for a multiplication R-module M, $H \in \mathrm{Max}(M)$ if and only if $M \neq H = QM$ for some maximal ideal Q of R.

Corollary 2.10. *Let M be a finite multiplication R-module. Then there is a vertex of $\Gamma(M)$ that is adjacent to every other vertex if and only if either $M = M_1 \oplus M_2$ is a faithful R-module, where M_1, M_2 are two submodules of M such that M_1 has only two elements and M_2 is simple, or R is a local ring (and hence $|\mathrm{Max}(M)| = 1$).*

Proof. ($\Longrightarrow$) Let M be a finite multiplication R-module. By Theorem 2.9, either $M = M_1 \oplus M_2$ or $T(M) = IM$ where I is an annihilator ideal of R. Let $M = M_1 \oplus M_2$, so M_2 is finite, hence M_2 is an Artinian R-module, also by [11, Theorem 2.2 and Corollary 2.9], M_2 is cyclic, so $M_2 \cong \frac{R}{\mathrm{Ann}(M_2)}$. Similar to the proof of Theorem 2.9, $\mathrm{Ann}(M_2)$ is a prime ideal. Thus, $\frac{R}{\mathrm{Ann}(M_2)}$ is a finite integral domain and so is a field and so $\mathrm{Ann}(M_2)$ is a maximal ideal of R;

therefore, M_2 is a simple R-module. Now assume $M \neq M_1 \oplus M_2$. By Theorem 2.9, $T(M) = \mathrm{Ann}(x)M$ for some $x \in M$. Let $M = \{m_1, m_2, \ldots, m_n\}$, so M is an Artinian multiplication R-module and by [11, Corollary 2.9], M is cyclic; thus, $M = Ry$ for some $y \in M$. Let $s \in R$ be a non-unit element, and $sM = \{sm_1, sm_2, \ldots, sm_n\}$. If $sm_i = sm_j$ for some $i \neq j$, then $s \in \mathrm{Ann}(m_i - m_j)$. But $m_i - m_j \in M = Ry$ and there is a $0 \neq t \in R$ such that $sty = 0$; therefore, $sy \in T(M)$. If $sm_i \neq sm_j$ for all $i \neq j$, then $sM = M$, so $Rsy = Ry$. Hence $sy \in T(M) = \mathrm{Ann}(x)M$. We know $s \in \mathrm{Ann}(x)$ since Ry is weakly cancellation R-module. By [5, Proposition 1.6], R is a local ring with only maximal ideal $\mathrm{Ann}(x)$ so $|\mathrm{Max}(M)| = 1$.

($\Longleftarrow$) Let R be a local ring, $M = Ry$ for some $y \in M$, and $s \in R$ be a non-unit, with a similar argument as above $sy \in T(M)$; hence $s \in Z(M)$, so $Z(M)$ is a maximal ideal and by [13, Theorem 82], $Z(M) = \mathrm{Ann}(x)$ for some $x \in M$. Therefore, $\mathrm{Ann}(x)M = T(M)$. Consequently, by Theorem 2.9, there is a vertex of $\Gamma(M)$ which is adjacent to every other vertex. ∎

Let M be a multiplication R-module; we next determine when $\Gamma(M)$ is a complete graph (i.e., any two vertices are adjacent). By definition, $\Gamma(M)$ is complete if and only if $[x : M][y : M]M = 0$ for all distinct $x, y \in T(M)^*$. Except for the case when $M = M_1 \oplus M_2$, with M_1, M_2 having only two elements, the proof of our next theorem shows that we must also have $[x : M]^2M = 0$ for all $x \in T(M)$, when $\Gamma(M)$ is complete. So except for the mentioned case, $\mathrm{Nil}(M)$ is detected by complete graphs, because $T(M) = \mathrm{Nil}(M)$.

THEOREM 2.11. *Let M be a multiplication R-module. Then $\Gamma(M)$ is complete if and only if, either $M = M_1 \oplus M_2$ is faithful, with submodules M_1, M_2 having only two elements, or $[x : M][y : M]M = 0$ for all $x, y \in T(M)$.*

Proof. ($\Longleftarrow$) By definition.

($\Longrightarrow$) Suppose that $\Gamma(M)$ is connected, but assume there is $x \in T(M)$ with $[x : M]^2M \neq 0$, so $x \notin \mathrm{Ann}(x)M$ and by Theorem 2.9, $M = M_1 \oplus M_2$ where M_1 has only two elements. Similar to the proof of Theorem 2.9, $[(0, y) : M]^2M = [(0, y) : M]M$ for all $y \in T(M)$. Therefore,

$$Ry \subseteq [(0, y) : M]y \subseteq (\mathrm{Ann}(M_1) \cap [y : M_2])y \subseteq [y : M_2]y\,.$$

Hence $Ry = [y : M_2]y$ and $y = sy$ for some $s \in [y : M_2]$. Let $m_2 \in M_2$, so

$$[y : M_2][(1 - s)m_2 : M_2]M_2 = 0$$

and similar to the proof of Theorem 2.9, $D(M_2) = 0$, thus, $y = 0$ or $m_2 = sm_2 \in Ry$. Hence $M_2 = Ry$. On the other hand, $(0, m_2) \in T(M)$ for all $y \neq m_2 \in M_2$ and

$$[(0, m_2) : M][(0, y) : M]M = 0\,,$$

therefore, $m_2 = 0$ and Ry has only two elements. ∎

Corollary 2.12. *Let M be a finite multiplication R-module. If $\Gamma(M)$ is complete, then either $M = M_1 \oplus M_2$, where M_1, M_2 are two submodules of M such that M_1, M_2 has only two elements or R is a local ring (and hence $|\operatorname{Max}(M)| = 1$).*

References

[1] M.M. Ali, D.J. Smith, Finite and infinite collections of multiplication modules, *Beiträge Algebra Geom.* **42** (2) (2001), 557 – 573.

[2] D.F. Anderson, A. Frazier, A. Lauve, P.S. Livingston, The zero-divisor graph of a commutative ring, II, in "Ideal Theoretic Methods in Commutative Algebra" (Columbia, MO, 1999), Lecture Notes in Pure and Appl. Math., Marcel Dekker, New York, 220, 2001, 61 – 72

[3] D.F. Anderson, P.S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** (2) (1999), 434 – 447.

[4] D.D. Anderson, M. Naseer, Beck's coloring of a commutative rings, *J. Algebra* **159** (2) (1993), 500 – 514.

[5] M.F. Atiyah, I.G. Macdonald, "Introduction to Commutative Algebra", AddisonûWesley, Reading, MA, 1969.

[6] A. Barnard, Multiplication modules, *J. Algebra* **71** (1) (1981), 174 – 178.

[7] I. Beck, Coloring of commutative rings, *J. Algebra* **116** (1) (1988), 208 – 226.

[8] A. Cannon, K. Neuerburg, S.P. Redmond, Zero-divisor graphs of nearrings and semigroups, in "Nearrings and Nearfields" (A. Kreuzer, M.J. Thomsen, Eds.), Springer, Dordrecht, 2005 189 – 200.

[9] F.R. DeMeyer, T. McKenzie, K. Schneider, The zero-divisor graph of a commutative semigroup, *Semigroup Forum* **65** (2) (2002), 206 – 214.

[10] R. Diestel, "Graph Theory", Springer-Verlag, New York, 1997.

[11] Z.A. El-Bast, P.F. Smith, Multiplication modules, *Comm. Algebra* **16** (4)(1988), 755 – 779.

[12] Sh. Ghalandarzadeh, P. Malakooti Rad, Torsion graph over multiplication modules, *Extracta Math.* **24** (3) (2009), 281 – 299.

[13] I. Kaplansky, "Commutative Rings", The University of Chicago Press, Chicago-London, 1974.

[14] S.P. Redmond, The zero-divisor graph of a non-commutative ring, *Int. J. Commut. Rings* **1** (2002), 203 – 211. (in "Commutative Rings", Nova Sci. Publ., Hauppauge, NY, 2002, 39 – 47.)