EXTRACTA MATHEMATICAE Vol. **25**, Núm. 3, 249 – 261 (2010)

Lip-density and Algebras of Lipschitz Functions on Metric Spaces *

I. Garrido, J.A. Jaramillo, Y.C. Rangel

Departamento de Geometría y Topología, Universidad Complutense de Madrid, 28040 *Madrid, Spain, maigarri@mat.ucm.es*

Departamento de Análisis Matemático, Universidad Complutense de Madrid, 28040 *Madrid, Spain, jaramil@mat.ucm.es*

Departamento de Matemática, Universidad Centroccidental Lisandro Alvarado, Barquisimeto, Venezuela, yrangel25@hotmail.com

Presented by Francisco Montalvo Received December 21, 2010

Abstract: Our aim in this note is to give an extension of the classical Myers-Nakai theorem in the context of Finsler manifolds. To achieve this, we provide a general result in this line for subalgebras of bounded Lipschitz functions on length metric spaces. We also establish some connection with the uniform approximation of bounded Lipschitz functions by functions in the subalgebra, keeping control on the Lipschitz constants.

Key words: Algebras of Lipschitz functions, approximation, Finsler manifolds.

AMS *Subject Class.* (2010): 54C35, 46E25, 58B20.

1. Introduction

There is a wide range of results in the literature asserting that the topological, metric or differentiable structure of a given space X can be characterized in terms of a suitable algebraic or topological-algebraic structure on the space $C(X)$ of continuous real functions on X, or on a certain subfamily of $C(X)$. We should mention, as fundamental prototype, the classical Banach-Stone theorem, from which the topology of a compact space X is determined by the linear metric structure of $C(X)$ (endowed with the sup-norm). We refer to [3] and references therein for further information about different extensions, generalizations and variants of this result.

We are especially interested here in the theorem of Myers-Nakai, giving that the Riemannian structure of a Riemannian manifold M is determined by

* Isabel Garrido and Jesús A. Jaramillo have been supported in part by D.G.I. (Spain) Grant MTM2009-07848. Yenny Rangel has been associated to the Proyect 004-RCT-2010 (UCLA) (Venezuela).

the natural normed algebra structure on the space $C_b^1(M)$ of all bounded C^1 functions on M which have bounded derivative (or, equivalently, which are Lipschitz on M with respect to the geodesic distance). This was proved by Myers [10] in the case that M is compact, and later on by Nakai [12] for general (finite-dimensional) manifolds. The case of infinite-dimensional Riemannian manifolds is considered in [6]. Furthermore, by the Myers-Steenrod theorem (see [11]), the Riemannian structure of a connected Riemannian manifold is in turn determined by the purely metric structure of the manifold endowed with its geodesic distance.

In [4] we have obtained a result of this type connecting the isometric type of a complete length metric space X with the normed algebra structure of the corresponding space $\mathrm{Lip}^*(X)$ of all bounded Lipschitz functions on X (see also [5] for related results in the class of the so-called small-determined metric spaces). Our first aim in this note is to give an extension of this result for certain subalgebras of bounded Lipschitz functions on length metric spaces. In order to achieve this, we consider the notion of *Lip-density* for a subalgebra A of $\mathrm{Lip}^*(X)$, which provides uniform approximation of an arbitrary function $f \in \mathrm{Lip}^*(X)$ by functions in A, in such a way that the Lipschitz constants of the approximating functions also approach the Lipschitz constant of f. Finally, as a consequence, we obtain an extension of the Myers-Nakai theorem to the context of Finsler manifolds.

This note is based on the talk delivered by the second author at the “Seventh Italian-Spanish Conference in General Topology and its Applications” which took place in Badajoz (Spain), September 2010. It is a great pleasure to thank the organizers for their invitation to participate in the Conference.

2. Lip-density in metric spaces

Let (X, d) be a metric space. We denote by $\mathrm{Lip}(X)$ the space of all real Lipschitz functions defined on X, and by $\mathrm{Lip}^*(X)$ the space of all bounded real Lipschitz functions defined on X. For $f \in \mathrm{Lip}(X)$, its *Lipschitz constant* is defined, as usual, by

$$\mathrm{Lip}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} \; : \; x, y \in X \, , \; x \neq y \right\}.$$

The space $\mathrm{Lip}^*(X)$ is a complete normed algebra endowed with the natural norm:

$$\|f\|_{\mathrm{Lip}^*(X)} = \|f\|_\infty \vee \mathrm{Lip}(f)$$

where $\| \cdot \|_\infty$ denotes the sup-norm and $\vee$ denotes the supremum.

Definition 1. Let (X, d) be a metric space. We say that a subset $\mathcal{G} \subset \mathrm{Lip}(X)$ is *Lip-dense* in $\mathrm{Lip}(X)$ if for every $f \in \mathrm{Lip}(X)$ and every $\varepsilon > 0$ there exists some $g \in \mathcal{G}$ such that:

(i) $|f(x) - g(x)| \leq \varepsilon$ for each $x \in X$,

(ii) $\mathrm{Lip}(g) \leq \mathrm{Lip}(f) + \varepsilon$.

In the same way, we say that $\mathcal{G} \subset \mathrm{Lip}^*(X)$ is *Lip-dense* in $\mathrm{Lip}^*(X)$ if for every $f \in \mathrm{Lip}^*(X)$ and every $\varepsilon > 0$ there exists some $g \in \mathcal{G}$ such that conditions (i) and (ii) above hold.

The most remarkable examples of Lip-density are provided by subsets of smooth functions defined on a Riemannian manifold or, more generally, on a Finsler manifold. The notion of Finsler manifold can be traced back to Riemann in his 1854 Habilitationschrift, where he suggested the possibility of considering more general, non-quadratic, norms for measuring the length of an element of arc. The systematic study was initiated by P. Finsler in his thesis (1919). We refer to the book by Bao, Chern and Shen [1] for a detailed introduction to this subject. The basic idea of a Finsler structure on a smooth manifold is a smooth assignment, on each tangent space, of a so-called Minkowski norm, which is a kind of norm with nice properties of convexity and smoothness. More precisely:

Definition 2. Let V be a finite-dimensional real vector space. We say that a functional $F : V \to [0, \infty)$ is a *Minkowski norm* on V if it satisfies:

(1) Positivity: $F(v) = 0$ if, and only if, $v = 0$.

(2) Triangle inequality: $F(u + v) \leq F(u) + F(v)$, for every $u, v \in V$.

(3) Positive homogeneity: $F(\lambda v) = \lambda F(v)$, for every $v \in V$ and every $\lambda > 0$.

(4) Regularity: F is continuous on V and C^∞ on $V \setminus \{0\}$.

(5) Strong convexity: For every $v \in V \setminus \{0\}$, the quadratic form g_v associated to the second derivative of the function F^2 at v, that is

$$g_v = \frac{1}{2} \, \mathrm{d}^2 \big[F^2 \big](v) \,,$$

is positive-definite on V.

We note that conditions (1) and (2) in the above definition are, in fact, consequence of conditions (3) − (5) (see [1, Theorem 1.2.2]). It is clear that every norm associated to an inner product is a Minkowski norm. We also recall that, in general, a Minkowski norm needs not be symmetric, and there are indeed very interesting examples of nonsymmetric Minkowski norms, such as for example Randers spaces (see [1]). We say F is *symmetric* or *absolutely homogeneous* if:

$$F(\lambda v) = |\lambda|\, F(v) \qquad \text{for every } v \in V \text{ and every } \lambda \in \mathbb{R}\,.$$

In this case, F is a norm in the usual sense.

Now the definition of Finsler manifold is as follows:

Definition 3. A Finsler manifold is a pair (M, F), where M is a finite-dimensional C^∞-smooth manifold and $F : TM \to [0, \infty)$ is a continuous function defined on the tangent bundle TM, satisfying:

(i) F is C^∞-smooth on $TM \setminus \{0\}$.

(ii) For every $x \in M$, $F(x, \cdot) : T_xM \to [0, \infty)$ is a Minkowski norm on the tangent space T_xM.

In particular, a Riemannian manifold is a special case of Finsler manifold, where the Minkowski norm on each tangent space T_xM is given by an inner product. The Finsler structure F is said to be *reversible* if, for every $x \in M$, $F(x, \cdot)$ is symmetric. This is of course the case of Riemannian manifolds.

Now let (M, F) be a Finsler manifold and $1 \le k \le \infty$. If $f : M \to \mathbb{R}$ is a C^k function, we define as usual the *norm* of its differential $\mathrm{d}f(x)$ at a point $x \in M$ by:

$$\|\mathrm{d}f(x)\|_F = \sup\left\{|\mathrm{d}f(x)(v)| \,:\, v \in T_xM\,,\ F(x, v) \le 1\right\}.$$

We denote by $C_b^k(M)$ the space of all real bounded C^k-functions defined on M whose derivative has uniformly bounded norm. We endow $C_b^k(M)$ with the natural norm:

$$\|f\|_{C_b^k(M)} = \sup_{x\in M} |f(x)| \vee \sup_{x\in M} \|\mathrm{d}f(x)\|_F.$$

Endowed with this norm, $C_b^k(M)$ is a normed algebra, and in particular $C_b^1(M)$ is a complete normed algebra.

Now suppose that (M, F) is a connected Finsler manifold. The *Finsler distance* d_F on M is defined by:

$$d_F(x, y) = \inf \{\ell_F(\sigma) : \sigma \text{ piecewise } C^1 \text{ path from } x \text{ to } y\},$$

where the *Finsler length* of a piecewise C^1 path $\sigma : [a, b] \to M$ is defined as:

$$\ell_F(\sigma) = \int_a^b F(\sigma(t), \sigma'(t))\, \mathrm{d}t.$$

In this way we have (see Section 6.2 of [1]) that the Finsler distance d_F is an *asymmetric distance* on M, in the sense that it verifies:

- $d_F(x, y) \geq 0$;
- $d_F(x, y) = 0$ if, and only if, $x = y$;
- $d_F(x, y) \leq d_F(x, z) + d_F(z, y)$ for every $x, y, z \in M$.

In general, d_F needs not be symmetric. Nevertheless, if F is reversible the Finsler distance d_F is symmetric, and therefore (M, d_F) is a metric space in the usual sense. In this case, as a consequence of the corresponding mean value theorem, we obtain in [7] the following:

LEMMA 4. *Let (M, F) be a connected, reversible Finsler manifold, and consider the associated Finsler distance d_F on M. For each C^1 function $f : M \to \mathbb{R}$ we have that $f : (M, d_F) \to \mathbb{R}$ is Lipschitz if, and only if, f has bounded derivative. In fact,*

$$\mathrm{Lip}(f) = \sup_{x \in M} \|\mathrm{d}f(x)\|_F.$$

In particular, from the above lemma we deduce that if (M, F) is a connected, reversible Finsler manifold, then $C^k(M) \cap \mathrm{Lip}^*(M) = C^k_b(M)$.

We recall now the following approximation result due to Greene and Wu (see [8, Proposition 2.1]):

THEOREM 5. ([8]) *Let M be a connected and second countable Riemannian manifold, endowed with its Riemannian distance. Then $C^\infty(M) \cap \mathrm{Lip}(M)$ is Lip-dense in $\mathrm{Lip}(M)$, and therefore $C^\infty_b(M)$ is Lip-dense in $\mathrm{Lip}^*(M)$.*

In the case of Finsler manifolds, we do not know whether it is possible to obtain in general Lip-approximation of Lipschitz functions by C^∞-smooth functions. Nevertheless, we have the following result from [7] giving Lip-approximation by C^1-smooth functions. In the proof, we use the exponential map associated to the Finsler structure, which in general is only C^1-smooth. There is a relevant class of Finsler manifolds which are characterized by the fact that the exponential map is C^∞-smooth. These are called *Berwald spaces*, and include, of course, the Riemannian manifolds (see [1, Section 5.3]). In the class of Berwald spaces, the same proof gives Lip-approximation by C^∞-smooth functions.

Theorem 6. ([7]) *Let M be a connected, reversible and second countable Finsler manifold, endowed with its Finsler distance. Then $C^1(M) \cap \mathrm{Lip}(M)$ is Lip-dense in $\mathrm{Lip}(M)$, and therefore $C^1_b(M)$ is Lip-dense in $\mathrm{Lip}^*(M)$.*

We also note that the above approximation theorem is proved in [7] in the general case of non-reversible Finsler manifolds, using a suitable definition of Lipschitz functions in this context.

To finish this section we are going to show that Lip-density corresponds with the usual notion of density for a suitable metric on $\mathrm{Lip}(X)$. Namely, the metric defined by

$$\rho(f,g) = \sup_{x\in X}\{|f(x)-g(x)| \wedge 1\} \vee |\,\mathrm{Lip}(f) - \mathrm{Lip}(g)|,$$

where $\vee$ and $\wedge$ denotes the *sup* and *inf*, respectively. Indeed, it is clear that density for the metric ρ implies Lip-density. Conversely, let $\mathcal{G}$ be Lip-dense in $\mathrm{Lip}(X)$ and $f \in \mathrm{Lip}(X)$. Given $0 < \varepsilon < 1$, choose $x_0 \neq y_0$ in X such that

$$\mathrm{Lip}(f) - \frac{\varepsilon}{2} < \frac{|f(x_0) - f(y_0)|}{d(x_0,y_0)}\,.$$

Now, take $0 < \varepsilon' < \inf\{\varepsilon,\, (\frac{1}{4}\,\varepsilon)\, d(x_0,y_0)\}$. By Lip-density we can find $g \in \mathcal{G}$ such that $|f(x) - g(x)| \leq \varepsilon'$ for each $x \in X$, and $\mathrm{Lip}(g) \leq \mathrm{Lip}(f) + \varepsilon'$. Then,

$$\begin{aligned}\mathrm{Lip}(g) &\geq \frac{|g(x_0)-g(y_0)|}{d(x_0,y_0)} \geq \frac{|f(x_0)-f(y_0)| - 2\varepsilon'}{d(x_0,y_0)}\\ &\geq \mathrm{Lip}(f) - \frac{\varepsilon}{2} - \frac{2\varepsilon'}{d(x_0,y_0)} > \mathrm{Lip}(f) - \varepsilon\,.\end{aligned}$$

And, therefore $\rho(f,g) < \varepsilon$.

Moreover, note that on $\mathrm{Lip}^*(X)$ the norm $\|\cdot\|_{\mathrm{Lip}^*}$ induces a topology finer than the metric topology given by ρ. In general these topologies are not equivalent. For instance, if M is a connected and second countable Riemannian (or, more generally, Finsler) manifold, then $C_b^1(M)$ is $\|\cdot\|_{\mathrm{Lip}^*}$-closed but Lip-dense in $\mathrm{Lip}^*(M)$.

3. The structure space

In this section we consider a metric space (X, d) and a unital and Lip-dense subalgebra A of $\mathrm{Lip}^*(X)$. We are going to recall the construction of the *structure space* associated to A, following the lines of Isbell in [9]. First of all note that, by Lip-density, A separates points and closed sets of X, and this implies in particular that X is endowed with the weak topology given by A.

As usual, we say that $\varphi : A \to \mathbb{R}$ is an *algebra homomorphism* whenever it satisfies:

$$\varphi(\lambda f + \mu g) = \lambda\varphi(f) + \mu\varphi(g)\,,$$
$$\varphi(f \cdot g) = \varphi(f) \cdot \varphi(g)\,,$$

for all $f, g \in A$ and for all $\lambda, \mu \in \mathbb{R}$. Note that an algebra homomorphism φ is nonzero if, and only if, $\varphi(1) = 1$. On the other hand, we say that φ is *positive* if $\varphi(f) \geq 0$ whenever $f \geq 0$.

We define the *structure space* $\mathfrak{M}(A)$ as the set of all nonzero and positive algebra homomorphisms $\varphi : A \to \mathbb{R}$, considered as a topological subspace of the product $\mathbb{R}^A$. It is not difficult to see that $\mathfrak{M}(A)$ is closed in $\mathbb{R}^A$. Moreover, since every function in A is bounded, $\mathfrak{M}(A)$ is in particular a compact space.

Now, consider the natural map $\delta : X \to \mathfrak{M}(A)$ given by $\delta(x) = \delta_x$, where δ_x is the *point evaluation* homomorphism, i.e., $\delta_x(f) = f(x)$, for every $f \in A$. Clearly, δ is a continuous map. On the other hand, the subspace $\delta(X)$ is dense in $\mathfrak{M}(A)$. Indeed, given $\varphi \in \mathfrak{M}(A)$, $f_1, \ldots, f_n \in A$, and $\varepsilon > 0$, there exists some $x \in X$ such that $|\delta_x(f_i) - \varphi(f_i)| < \varepsilon$, for all $i = 1, \ldots, n$. Otherwise the function $g = \sum_{i=1}^n (f_i - \varphi(f_i))^2 \in A$ would satisfy $g \geq \varepsilon^2$ and $\varphi(g) = 0$, and this is impossible since φ is positive.

Finally, from the fact that A separates points and closed sets of X, we can derive that δ is a topological embedding, and therefore $\mathfrak{M}(A)$ can be considered as a compactification of X. In addition this compactification has the property that each $f \in A$ admits a continuous extension $\hat{f}$ to $\mathfrak{M}(A)$, namely by defining $\hat{f}(\varphi) = \varphi(f)$, for all $\varphi \in \mathfrak{M}(A)$. Note that this extension $\hat{f}$ coincides on $\mathfrak{M}(A)$ with the corresponding projection map $\pi_f : \mathbb{R}^A \to \mathbb{R}$.

Next we will see that, thanks to Lip-density, the points in X can be topologically distinguished into $\mathfrak{M}(A)$.

Lemma 7. *Let (X,d) be a complete metric space, and A a unital and Lip-dense subalgebra of* $\mathrm{Lip}^*(X)$. *Then $\varphi \in \mathfrak{M}(A)$ has a countable neighborhood basis in $\mathfrak{M}(A)$ if, and only if, $\varphi \in X$.*

Proof. Assume first that $\varphi \in \mathfrak{M}(A) \setminus X$ has a countable neighborhood basis. Since X is dense in $\mathfrak{M}(A)$ there exists a sequence (x_n) in X converging to φ. From the completeness of X it follows that (x_n) has no Cauchy subsequence, and therefore there exist some $\varepsilon > 0$ and a subsequence (x_{n_k}) of (x_n) such that $d(x_{n_k}, x_{n_j}) \geq \varepsilon$ for $k \neq j$. Consider the sets $P = \{x_{n_{2k}} : k \in \mathbb{N}\}$ and $Q = \{x_{n_{2k+1}} : k \in \mathbb{N}\}$, and define the function $f(x) = \min\{\varepsilon, d(x,P)\}$. It is clear that $f \in \mathrm{Lip}^*(X)$ and we have that $f(x) = 0$ for every $x \in P$ and $f(x) \geq \varepsilon$ for every $x \in Q$. By Lip-density, there exists $g \in A$ such that $|f(x) - g(x)| \leq \frac{\varepsilon}{4}$, for each $x \in X$. Therefore the extended function $\hat{g}$ defined on the whole space $\mathfrak{M}(A)$ takes values $\leq \frac{\varepsilon}{4}$ on $\mathrm{cl}_{\mathfrak{M}(A)}P$ and values $\geq \frac{3\varepsilon}{4}$ on $\mathrm{cl}_{\mathfrak{M}(A)}Q$ (where $\mathrm{cl}_{\mathfrak{M}(A)}$ denotes the closure operator on the space $\mathfrak{M}(A)$). But this is a contradiction since $\varphi \in \mathrm{cl}_{\mathfrak{M}(A)}P \cap \mathrm{cl}_{\mathfrak{M}(A)}Q$.

Conversely, if $\varphi \in X$, consider B_n the open ball in X with centre φ and radius $1/n$. Then the family $\{\mathrm{cl}_{\mathfrak{M}(A)}B_n\}$ is easily seen to be a countable neighborhood basis as required. ∎

Note that along this section it would not be necessary to suppose that A is Lip-dense, but only uniformly dense.

4. Isometries between length spaces

Recall that a metric space (X,d) is said to be a *length space* if for every $x, y \in X$

$$d(x,y) = \inf\ \{\ell(\gamma) : \gamma \text{ is a continuous path from } x \text{ to } y\}.$$

Here the *length* of a continuous path $\gamma : [a,b] \to X$ is defined, as usual, by

$$\ell(\gamma) = \sup\left\{\sum_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i-1}))\right\} \in [0, +\infty]$$

where $a = t_0 < t_1 < \cdots < t_k = b$ runs over all partitions of the interval $[a,b]$.

Definition 8. Let (X, d_X) and (Y, d_Y) be metric spaces and consider two subsets $A_X \subset \mathrm{Lip}^*(X)$ and $A_Y \subset \mathrm{Lip}^*(Y)$, respectively. We say that a bijection $h : Y \to X$ is stable for A_X and A_Y if, for every function $f : X \to \mathbb{R}$, we have that $f \in A_X$ if, and only if, $f \circ h \in A_Y$.

The next result shows in particular that if a bijection $h : Y \to X$ between length metric spaces is stable for some Lip-dense subalgebras, then h is in fact a bi-Lipschitz mapping.

LEMMA 9. *Let (X, d_X) and (Y, d_Y) be length metric spaces, $h : Y \to X$ a mapping, and consider two Lip-dense subalgebras $A_X \subset \mathrm{Lip}^*(X)$ and $A_Y \subset \mathrm{Lip}^*(Y)$, respectively. Suppose that $T : A_X \to A_Y$ is a continuous homomorphism for the corresponding Lip*-norms given by $T(f) = f \circ h$, for every $f \in A_X$. Then h is $\|T\|$-Lipschitz.*

Proof. Firstly, note that h is continuous. Indeed, as we said before, the Lip-density of A_X in $\mathrm{Lip}^*(X)$ implies that X is endowed with the weak topology given by A_X, and therefore h is continuous because $f \circ h$ is continuous on Y, for every $f \in A_X$.

Now, in order to see that h is $\|T\|$-Lipschitz, let $p, q \in Y$, and suppose first that $d_X(h(p), h(q)) \leq 1$. Fix $\varepsilon > 0$, and consider the function $f : X \to \mathbb{R}$, defined by $f(x) = \min\{1, d_X(h(p), x)\}$. It is clear that f is 1-Lipschitz and $0 \leq f \leq 1$, so by Lip-density there exists $g \in A_X$ such that

(1) $|f(x) - g(x)| \leq \varepsilon$ for every $x \in X$,

(2) $\mathrm{Lip}(g) \leq \mathrm{Lip}(f) + \varepsilon = 1 + \varepsilon$.

Since, in particular, $|g(x)| \leq f(x) + \varepsilon \leq 1 + \varepsilon$, for every $x \in X$, then we have that $\|g\|_{\mathrm{Lip}^*(X)} \leq 1 + \varepsilon$. Now, the continuity of T gives that:

$$\mathrm{Lip}(g \circ h) \leq \|g \circ h\|_{\mathrm{Lip}^*(Y)} \leq \|T\| \cdot \|g\|_{\mathrm{Lip}^*(X)} \leq (1 + \varepsilon) \cdot \|T\| .$$

Therefore,

$$\begin{aligned} d_X(h(p), h(q)) &= f(h(q)) = |f(h(q)) - f(h(p))| \\ &\leq |g(h(q)) - g(h(p))| + 2\varepsilon \\ &\leq \mathrm{Lip}(g \circ h) \cdot d_Y(p, q) + 2\varepsilon \\ &\leq (1 + \varepsilon) \cdot \|T\| \cdot d_Y(p, q) + 2\varepsilon . \end{aligned}$$

Taking $\varepsilon \to 0$ we obtain that $d_X(h(p), h(q)) \leq \|T\| \cdot d_Y(p, q)$.

Next, given arbitrary $p, q \in Y$ consider $h(p), h(q) \in X$. For every $\varepsilon > 0$, let $\sigma : [a, b] \to Y$ a continuous path from p to q with $\ell(\sigma) \leq d_Y(p, q) + \varepsilon$ and let $\hat{\sigma} = h \circ \sigma$. Consider a partition $P = \{a = t_0 < t_1 < \cdots < t_k = b\}$ of the interval $[a,b]$ such that $d_X(h(\sigma(t_{i-1})), h(\sigma(t_i))) < 1$, for each $i = 1, 2, \dots, k$. By the previous case, we have that

$$\begin{aligned} d_X(h(p), h(q)) &\leq \sum_{i=1}^{k} d_X\big(h(\sigma(t_{i-1})), h(\sigma(t_i))\big) \\ &\leq \sum_{i=1}^{k} \|T\| \cdot d_Y\big(\sigma(t_{i-1}), \sigma(t_i)\big) \\ &\leq \sum_{i=1}^{k} \|T\| \cdot \ell\big(\sigma|_{[t_{i-1}, t_i]}\big) \\ &= \|T\| \cdot \sum_{i=1}^{k} \ell\big(\sigma|_{[t_{i-1}, t_i]}\big) \\ &= \|T\| \cdot \ell(\sigma) \leq \|T\| \cdot \big(d_Y(p,q) + \varepsilon\big) \,. \end{aligned}$$

Taking $\varepsilon \to 0$, we deduce that $d_X(h(p), h(q)) \leq \|T\| \cdot d_Y(p,q)$. ∎

We are going to establish a connection between isometries of length metric spaces which are stable for some Lip-dense subalgebras, and linear isometries between the corresponding subalgebras. We first recall the following definition.

Definition 10. Let $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$ be normed algebras. We say that $T : A \to B$ is an *isometry of normed algebras* if:

(i) T is bijective, linear and multiplicative.

(ii) $\|T(f)\|_B = \|f\|_A$, for every $f \in A$.

Now we give our main result:

Theorem 11. *Let (X, d_X) and (Y, d_Y) be complete length metric spaces, and consider two Lip-dense subalgebras $A_X \subset \mathrm{Lip}^*(X)$ and $A_Y \subset \mathrm{Lip}^*(Y)$, respectively. Then the following are equivalent:*

(a) *The mapping $T : \big(A_X, \|\cdot\|_{\mathrm{Lip}^*(X)}\big) \to \big(A_Y, \|\cdot\|_{\mathrm{Lip}^*(Y)}\big)$ is an isometry of normed algebras.*

(b) *There exists an isometry $h : Y \to X$ stable for A_X and A_Y, such that $T(f) = f \circ h$, for every $f \in A_X$.*

In particular, if A_X and A_Y are isometric as normed algebras, then X and Y are isometric.

Proof. Firstly, suppose that $T : A_X \to A_Y$ is an isometry of normed algebras. Since $T(1) = T(1{\cdot}1) = T(1)^2$, then the function $T(1) \in A_Y$ can only take the values 0 and 1. Taking into account that Y is path-connected, we deduce that $T(1)$ is a constant function, and since $\|T(1)\|_{\mathrm{Lip}^*(Y)} = \|1\|_{\mathrm{Lip}^*(X)} = 1$, we obtain that $T(1) = 1$.

On the other hand, T must be positive. Otherwise, there would exist $0 \leq f \in A_X$ such that $T(f)$ is not positive, i.e., there is some $y_0 \in Y$ with $T(f)(y_0) < 0$. Take a real number $R > \|f\|_{\mathrm{Lip}^*(X)}$, and consider $g = -f + R$. Then $0 \leq g \in A_X$ and $\|g\|_{\mathrm{Lip}^*(X)} \leq R$. But,

$$\begin{aligned}\|T(g)\|_{\mathrm{Lip}^*(Y)} &= \|-T(f)+R\|_{\mathrm{Lip}^*(Y)} \\ &\geq \|-T(f)+R\|_\infty \\ &\geq |-T(f)(y_0)+R| \;>\; R\end{aligned}$$

which would be a contradiction.

In this way, we can consider the mapping $h : \mathfrak{M}(A_Y) \to \mathfrak{M}(A_X)$ between the corresponding structure spaces by defining $h(\varphi) = \varphi \circ T$, for every $\varphi \in A_Y$. It is clear that h is a bijection, and in fact h is a homeomorphism since $\pi_f \circ h = \pi_{T(f)}$, for every $f \in A_X$, where π_f and $\pi_{T(f)}$ denote the corresponding projection maps on the product spaces. By Lemma 7, we have that a point $\varphi \in \mathfrak{M}(A_X)$ has a countable neighborhood basis in $\mathfrak{M}(A_X)$ if, and only if, $\varphi \in X$, and the same is true for $\mathfrak{M}(A_Y)$. Therefore, h takes Y onto X.

Also, for every $f \in A_X$ and for every $y \in Y$ we have that

$$\begin{aligned}T(f)(y) = \delta_y(T(f)) &= (\delta_y \circ T)(f) \\ &= h(\delta_y)(f) = f(h(\delta_y)) = (f \circ h)(y)\end{aligned}$$

so it follows that $T(f) = f \circ h$. In particular, we have that $f \circ h \in A_Y$ whenever $f \in A_X$. Now by Lemma 9 we deduce that $h : Y \to X$ es 1-Lipschitz, since T is an isometry and $\|T\| = 1$. On the other hand, working with T^{-1} and h^{-1}, we also obtain that $h^{-1} \circ g \in A_X$ whenever $f \in A_Y$ and h^{-1} is also 1-Lipschitz. In this way we deduce that $h : Y \to X$ is an isometry stable for A_X and A_Y.

Finally, it is easy to check that condition (b) implies (a). ∎

5. A Myers-Nakai theorem

In this section we obtain from our previous results a version of the classical Myers-Nakai theorem in the context of Finsler manifolds. Recall that a mapping $h : (M, F) \to (N, G)$ between Finsler manifolds is said to be a *Finsler isometry* if h is a diffeomorphism which preserves the Finsler structure, that is, for every $x \in M$ and every $v \in T_x M$:

$$F(x, v) = G\big(h(x), \mathrm{d}h(x)(v)\big) .$$

A classical result due to Myers and Steenrod [11] asserts that, if M and N are connected Riemannian manifolds, a mapping $h : M \to N$ is a Riemannian isometry if, and only if, it is a metric isometry for the corresponding Riemannian distances. This result has been extended by Deng and Hou [2] to the context of Finsler manifolds:

Theorem 12. ([2]) *Let (M, F) and (N, G) be connected Finsler manifolds. Then $h : M \to N$ is a Finsler isometry if, and only if, h is bijective and preserves the corresponding Finsler distances, that is, for every $x, y \in M$:*

$$d_F(x, y) = d_G(h(x), h(y)) .$$

Now combining Theorem 12 above with Theorem 11 and the approximation result in Theorem 6, we obtain the following Myers-Nakai theorem for Finsler manifolds.

Theorem 13. *Let M and N be connected, reversible, complete and second countable Finsler manifolds. Then M and N are equivalent as Finsler manifolds if, and only if, $C_b^1(M)$ and $C_b^1(N)$ are equivalent as normed algebras. Moreover, every normed algebra isometry $T : C_b^1(N) \to C_b^1(M)$ is of the form $T(f) = f \circ h$, where $h : M \to N$ is a Finsler isometry.*

References

[1] D. Bao, S.S. Chern, Z. Shen, "An Introduction to Riemann-Finsler Geometry", Graduate Texts in Mathematics, 200, Springer-Verlag, New York, 2000.

[2] S. Deng, Z. Hou, The group of isometries of a Finsler space, *Pacific J. Math.* **207** (2002), 149–155.

[3] M.I. Garrido, J.A. Jaramillo, Variations on the Banach-Stone theorem, *Extracta Math.* **17** (2002), 351–383.

[4] M.I. Garrido, J.A. Jaramillo, Homomorphisms on function lattices, *Monatsh. Math.* **141** (2004), 127 – 146.

[5] M.I. Garrido, J.A. Jaramillo, Lipschitz-type functions on metric spaces, *J. Math. Anal. Appl.* **340** (2008), 282 – 290.

[6] M.I. Garrido, J.A. Jaramillo, Y. Rangel, Algebras of Differentiable Functions on Riemannian Manifolds, *Bull. Lond. Math. Soc.* **41** (2009), 993 – 1001.

[7] M.I. Garrido, J.A. Jaramillo, Y. Rangel, Smooth approximation of Lipschitz functions on Finsler Manifolds, *preprint*.

[8] R.E. Greene, H. Wu, C^∞ approximations of convex, subharmonic and plurisubharmonic functions, *Ann. Sci. École Norm. Sup. (4)* **12** (1979), 47 – 84.

[9] J.R. Isbell, Algebras of uniformly continuous functions, *Ann. of Math.(2)* **68** (1958), 96 – 125.

[10] S.B. Myers, Algebras of differentiable functions, *Proc. Amer. Math. Soc.* **5** (1954), 917 – 922.

[11] S.B. Myers, N.E. Steenrod, The group of isometries of a Riemannian manifold, *Ann. of Math. (2)* **40** (1939), 400 – 416.

[12] M. Nakai, Algebras of some differentiable functions on Riemannian manifolds, *Japan. J. Math.* **29** (1959), 60 – 67.