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Multiplicative Semigroups of Lipschitz Functions *

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Abstract: Given a (complete) metric space X, we denote by $\mathrm{Lip}(X)$ the space of real-valued Lipschitz functions on X and we equip it with the pointwise product. The purpose of this note is to describe those bijections $T : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$ which are "multiplicative" in the sense that whenever $f, g \in \mathrm{Lip}(Y)$ are such that $fg \in \mathrm{Lip}(Y)$ one has $T(fg) = T(f)T(g)$.

The main result of the paper states that if X has no isolated points, then every multiplicative bijection $T : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$ arises as $T(f) = f \circ \tau$, where $\tau : X \to Y$ is a Lipschitz homeomorphism and so it is automatically linear.

We also give a description of the semigroup isomorphisms $T : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$ in the case where the underlying metric spaces are compact.

Key words: Semigroups of Lipschitz functions, homomorphism, representation.

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1. Introduction

The purpose of this note is to describe the "multiplicative" bijections between spaces of Lipschitz functions. Given a metric space X, we denote by $\mathrm{Lip}(X)$ the space of real-valued Lipschitz functions on X and we equip it with the pointwise product. Our basic reference on spaces of Lipschitz functions and their relatives is Weaver booklet [14]. We hasten to remark that the product of two (in general unbounded) Lipschitz functions may fail to be Lipschitz and so $\mathrm{Lip}(X)$ is not a semigroup unless X has finite diameter.

To avoid any possible confusion, let us state the meaning in which the word "multiplicative" is used along the paper.

Definition 1. A mapping $T : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$ is said to be multiplicative if whenever $f, g \in \mathrm{Lip}(Y)$ are such that $fg \in \mathrm{Lip}(Y)$ one has $T(fg) = T(f)\,T(g)$.

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Of course, each of the spaces Y and X could be a single point, so that $\mathrm{Lip}(Y) = \mathrm{Lip}(X) = \mathbb{R}$. The simplest multiplicative automorphisms of $\mathbb{R}$ are the "powers" $t \mapsto \mathrm{sign}(t)|t|^p$, with $p \in \mathbb{R}^*$. In general, if $a : \mathbb{R} \to \mathbb{R}$ is an additive bijection, we can define a multiplicative bijection by the formula $m(t) = \mathrm{sign}(t)\exp(a(\log|t|))$. And, conversely, all multiplicative automorphisms of $\mathbb{R}$ arise in this way. As "most" additive bijections of the line are nonmeasurable, a certain degree of "pathological" behavior seems to be unavoidable. In this regard we have, as the main result of the paper, that if X has no isolated points, then every multiplicative bijection $T : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$ arises as $T(f) = f \circ \tau$, where $\tau : X \to Y$ is a Lipschitz homeomorphism and so it is automatically linear (Theorem 1).

We also give a description of the semigroup isomorphisms $T : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$ in the case where the underlying metric spaces are compact (Theorem 2) which solves a problem posed in [4].

Both Theorem 1 and Theorem 2 depend on a preliminary representation result which is the content of Section 2 (Proposition 1).

Background. The study of multiplicative bijections between rings has a relatively long history. In 1940, Eidelheit proved that any continuous multiplicative bijection between the algebras of operators of two real Banach spaces of dimension al least two is automatically linear –and so it arises as conjugation with a fixed linear homeomorphism between the underlying Banach spaces [6, Theorem 2 and Theorem 3]. A related result appears in Martindale's [10].

As for commutative rings, Milgram's classical paper [11] contains a description of the multiplicative bijections between the algebras of continuous functions on compacta. It turns out that two compact (Hausdorff) spaces are homeomorphic provided their semigroups of continuous functions are isomorphic. The papers [13, 8, 5, 9, 7, 4, 1] contain further developments, generalizations to noncompact spaces, and reiterations.

Multiplicative bijections between algebras of differentiable and smooth functions are the subject of [12] and [2]: they are all linear. Finally, [4] deals with uniformly continuous and Lipschitz functions with values in the unit interval.

2. A preliminary representation result

It this section $T : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$ will be a fixed multiplicative bijection (in the sense of Definition 1), with X and Y complete metric spaces.

LEMMA 1. *The inverse of T is multiplicative.*

Proof. We must check that if $f, g \in \mathrm{Lip}(X)$ are such that $fg \in \mathrm{Lip}(X)$, then $T^{-1}(f)\,T^{-1}(g) = T^{-1}(fg)$. It clearly suffices to check that

$$u\,T^{-1}(f)\,T^{-1}(g) = u\,T^{-1}(fg)$$

for every $u \in \mathrm{Lip}(Y)$ whose support has finite diameter. Please notice that the product of any Lipschitz function by such an u is automatically Lipschitz – actually the support of the product has finite diameter again. But

$$\begin{aligned} T\big(u\,T^{-1}(f)\,T^{-1}(g)\big) &= T(u)\,T\big(T^{-1}(f)\big)\,T\big(T^{-1}(g)\big) \\ &= T(u)\,fg \,=\, T\big(u\,T^{-1}(fg)\big) \end{aligned}$$

and we are done. ∎

The main result of this section is the following.

PROPOSITION 1. *There is a uniform homeomorphism $\tau : X \to Y$ and a mapping $: X \to \mathrm{Aut}(\mathbb{R})$ such that*

$$T(f)(x) = \mathfrak{t}_x(f(\tau(x))) \qquad (f \in \mathrm{Lip}(Y)\,,\ \ x \in X)\,. \tag{1}$$

Proof. An open set is said to be regular if it is the interior of its closure. The class of all regular open subsets of X is denoted by $R(X)$. The support of a continuous $f : X \to \mathbb{R}$ is the closure of the (cozero) set $\{x \in X \,:\, f(x) \neq 0\}$ and we define U_f as the interior of $\mathrm{supp}\, f$. Quite clearly, U_f is a regular open set and each regular open set arises in this way. Indeed, if $U \in R(X)$, then $U = U_f$, where $f(x) = \mathrm{dist}(x, U^c)$.

We will consider the order given by inclusion in $R(X)$.

CLAIM 1. The map $\mathfrak{T} : R(Y) \to R(X)$ given by $\mathfrak{T}(U_f) = U_{T(f)}$ is correctly defined and it is an order isomorphism. Moreover, given $f, g \in \mathrm{Lip}(Y)$ and $U \in R(Y)$ one has $f = g$ in U if and only if $T(f) = T(g)$ in $\mathfrak{T}(U)$.

Proof of Claim 1. First, the condition $U_f \subset U_g$ can be expressed within the multiplicative structure of $\mathrm{Lip}(X)$. To see this, following Shirota [13], let us declare $f \subset g$ if, whenever $h \in \mathrm{Lip}(X), hg = 0$ implies $hf = 0$. It is easily seen that, given $f, g \in \mathrm{Lip}(X)$, one has $f \subset g$ if and only if $U_f \subset U_g$. It follows that, given $f, g \in \mathrm{Lip}(X)$ one has $U_f = U_g$ if and only if $f \subset g$ and $g \subset f$.

As for the “moreover” part, let $M(Y)$ be the set of those h in $\mathrm{Lip}(Y)$ such that $hf \in \mathrm{Lip}(Y)$ for every $f \in \mathrm{Lip}(Y)$. Quite clearly, $M(Y)$ is an algebra containing every Lipschitz function whose support has finite diameter. First notice that if $U = U_h$ for some $h \in M(Y)$, one has $f = g$ on U if and only if $fh = gh$. For arbitrary $U \in R(Y)$ one has $f = g$ on U if and only if $fh = gh$ for every $h \in M(Y)$ such that $U_h \subset U$. *End of proof of Claim* 1.

Given $x \in X$ and $y \in Y$ we write $x \sim y$ provided

$$x = \bigcap_{y\in U} \mathfrak{T}(U) \qquad \text{and} \qquad y = \bigcap_{x\in V} \mathfrak{T}^{-1}(V)\,,$$

where $U \in R(Y)$ and $V \in R(X)$. Please note that if $x \sim y$ and $x \sim y'$, then $y = y'$. Similarly, if $x \sim y$ and $x' \sim y$, then $x = x'$. Write $X_0 = \{x \in X : x \sim y \text{ for some } y \in Y\}$ and $Y_0 = \{y \in Y : x \sim y \text{ for some } x \in X\}$.

Claim 2. (Cf. [3, Lemma 6 and the proof of Theorem 3].) X_0 and Y_0 are dense in X and Y respectively. The map $\tau : X_0 \to Y_0$ sending each x into the only y such that $x \sim y$ is a uniform homeomorphism.

Thus τ extends to a uniform homeomorphism between X and Y we denote again τ.

Claim 3. Given $f \in \mathrm{Lip}(Y)$ and $x \in X$, the value of $T(f)$ at x depends only on $f(\tau(x))$.

Proof of Claim 3. Suppose $f, g \in \mathrm{Lip}(Y)$ agree at $y = \tau(x)$ and let us see that $T(f)(x) = T(g)(x)$. It is possible to find a new Lipschitz function h having the following property: every neighborhood of y contains an open set where h agrees with f and another open set where h agrees with g (see [3, Lemma 3]). It follows that every neighborhood of x contains an open set where $T(h)$ agrees with $T(f)$ and another open set where $T(h)$ agrees with $T(g)$ and so $T(f)(x) = T(g)(x) = T(h)(x)$. *End of proof of Claim* 3.

To complete the proof of the proposition, just take $\mathfrak{t}_x : \mathbb{R} \to \mathbb{R}$ by letting $\mathfrak{t}_x(c) = T(c)(x)$, where c is treated first as a real number and then as a constant function on Y. ∎

We can continue our analysis assuming $T : \mathrm{Lip}(X, d) \to \mathrm{Lip}(X, d')$ has the form

$$T(f)(x) = \mathfrak{t}_x(f(x)) \qquad (f \in \mathrm{Lip}(X, d)\,,\ \ x \in X)\,, \tag{2}$$

where d and d' are uniformly equivalent metrics on X, both making it complete. The general case reduces to this one just taking $d'(x,x') = d(\tau(x),\tau(x'))$.

LEMMA 2. *The set of those $x \in X$ for which $\mathfrak{t}_x$ is not a positive power is at most finite and contains only isolated points.*

Proof. First, observe that a semigroup automorphism of $\mathbb{R}$ is either a positive power or it maps any neighborhood of the origin into an unbounded subset of the line [11, Lemma 4.3].

Suppose there is a sequence (x_n) such that $\mathfrak{t}_{x_n}$ is not a positive power. Passing to a subsequence if necessary we may assume every point in $S = \{x_n : n \in \mathbb{N}\}$ is (relatively) isolated. It is easily seen that there is a sequence (a_n) of strictly positive numbers in so that, if $|t_n| \le a_n$, then the map sending each x_n to t_n is Lipschitz on (S,d) and so ([14, Theorem 1.5.6(a)]) it extends to a bounded Lipschitz function on (X,d).

Given $n \in \mathbb{N}$, pick $t_n \in [-a_n, a_n]$ so that $|\mathfrak{t}_{x_n}(t_n)| > nd'(x_n,x_1)$ an let $f \in \mathrm{Lip}(X,d)$ such that $f(x_n) = t_n$. Obviously, $T(f)$ cannot be Lipschitz on (X,d').

As for the second part, let X^+ be the set of points in X where $\mathfrak{t}_x$ is a positive power, so that there is a function $p : X^+ \to (0,\infty)$ such that

$$T(f)(x) = \mathrm{sign}(f(x))|f(x)|^{p(x)} \qquad (f \in \mathrm{Lip}(X,d)\,,\ \ x \in X^+)\,. \tag{3}$$

Notice that p is uniformly continuous on X^+. Indeed $p(x) = \log T(e)$ is even d'-Lipschitz on X^+. Suppose (x_n) is a sequence in X^+ converging to $x \in X$. Then $p(x_n)$ converges, say to $q \ge 0$. Thus for $c > 0$ one has

$$\mathfrak{t}_x(c) = T(c)(x) = \lim_n T(c)(x_n) = \lim_n c^{p(x_n)} = c^q.$$

It follows that $q > 0$ and so $\mathfrak{t}_x$ is a positive power. ∎

LEMMA 3. *The metrics d and d' are locally Lipschitz equivalent on X. (And so, in the general case, $\tau : X \to Y$ is locally a Lipschitz homeomorphism.)*

Proof. Let $x \in X$. If the identity fails to be Lipschitz in every neighborhood of x, then there exist two sequences (x_n) and (y_n) converging to x such that $\frac{d(x_n,y_n)}{d'(x_n,y_n)} \to \infty$. In particular, x is non-isolated, so there is a neighborhood of x where $T(f)$ is given by (3).

By the triangle inequality we can find a sequence (z_n) such that $\frac{d(x,z_n)}{d'(x,z_n)} \to \infty$ (actually each z_n can be chosen to be x_n or y_n). Now, $h_0(y) = d(x,y)$ and $h_1(y) = 1 + d(x,y)$ are d-Lipschitz but fail to be d'-Lipschitz. Each $T(h_i) = h_i^p$ is d'-Lipschitz, and it is straightforward that p cannot be greater than 1 if h_1^p is d'-Lipschitz, nor lower or equal if h_0^p is, so we have a contradiction and so d and d' are locally Lipschitz equivalent. ∎

Lemma 4. *If x is a cluster point of X, then $p(x) = 1$.*

Proof. Let x be a cluster point of X. We know that, in a certain neighborhood of x, d and d' are Lipschitz equivalent and any function is d-Lipschitz if and only if it is d'-Lipschitz there. Taking $f(y) = d(x,y)$, we have that $T(f)(y) = d(x,y)^{p(y)}$ must be d'-Lipschitz, hence d-Lipschitz which forces $p(x) \geq 1$. By symmetry, it must be $p(x) = 1$. ∎

3. Applications

We state now the main results in this note which should be compared to [4, Theorem 5 and Corollary 2].

Theorem 1. *Let $T : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$ be a multiplicative bijection, where X and Y are complete metric spaces. If X (and so Y) has no isolated points, then T has the form $T(f)(x) = f(\tau(x))$, where $\tau : X \to Y$ is a Lipschitz homeomorphism and so it is automatically linear.*

Proof. In fact we only need to prove that τ is bi-Lipschitz, but if $f \circ \tau$ is Lipschitz on X whenever f is Lipschitz on Y, then τ must be Lipschitz. By symmetry (see Lemma 1) the inverse is also Lipschitz. ∎

The following result solves a problem (number 2) posed in [4].

Theorem 2. *Let X and Y be compact metric spaces, $\tau : X \to Y$ a Lipschitz homeomorphism, F a finite set of isolated points of X and $\mathfrak{t} : F \to \mathrm{Aut}(\mathbb{R})$ an arbitrary map. Let, further, $p : X \backslash F \to (0,\infty)$ be a Lipschitz function such that $p(x) = 1$ for every cluster point $x \in X$. Then the map $T : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$ given by*

$$T(f)(x) = \begin{cases} \mathrm{sign}(f(\tau(x))|f(\tau(x))|^{p(x)} & (x \in X\backslash F), \\ \mathfrak{t}_x(f(\tau(x))) & (x \in F), \end{cases}$$

is a semigroup isomorphism. And, conversely, all semigroup isomorphisms arise in this way.

Proof. The "conversely" part has been already proved.

As for the other part, we may and do assume F is empty and τ is the identity on X and we must prove that if $p : X \to \mathbb{R}_+$ is a Lipschitz function whose value at every cluster point is 1, then $f \mapsto \operatorname{sign}(f)|f|^p$ defines a semigroup isomorphism of $\operatorname{Lip} X$. By symmetry, it suffices to check that $\operatorname{sign}(f)|f|^p$ is Lipschitz if f is.

Observe that a continuous function g is locally Lipschitz provided $|g|$ is and that, by compactness, locally Lipschitz functions on X are Lipschitz. Hence we may assume f to be nonnegative.

Suppose f^p fails to be Lipschitz. Then there is a point x and a sequence (x_n) converging to x such that

$$\frac{f(x_n)^{p(x_n)} - f(x)^{p(x)}}{d(x_n,x)} \longrightarrow \infty \qquad (n \to \infty)\,.$$

But $p(x) = 1$ and quite clearly, $f(x) = 0$, so one actually has

$$\frac{f(x_n)^{p(x_n)}}{d(x_n,x)} \longrightarrow \infty \qquad (n \to \infty)\,.$$

However, the logarithm of the above expression is bounded since

$$\begin{aligned} p(x_n)\log f(x_n) - \log d(x_n,x) & \\ &\le p(x_n)\log(\Lambda(f)d(x_n,x)) - \log d(x_n,x) \\ &\le \|p\|_\infty |\log\Lambda(f)| + p(x_n)\log d(x_n,x) - \log d(x_n,x) \\ &\le \|p\|_\infty |\log\Lambda(f)| + |p(x_n)-1||\log d(x_n,x)| \\ &\le \|p\|_\infty |\log\Lambda(f)| + \Lambda(p)\, d(x_n,x)\, |\log d(x_n,x)| \end{aligned}$$

and $t\log t \to 0$ as $t \to 0^+$. ∎

Let us present another application of the results proved in Section 2.

Corollary 1. *Let $T : \operatorname{Lip}(Y) \to \operatorname{Lip}(X)$ be a multiplicative bijection. If Y has finite diameter, then so X does and τ is a Lipschitz homeomorphism.*

Proof. Clearly, $\operatorname{Lip} Y$ is a semigroup if and only if Y has finite diameter, and the same applies to X. On the other hand, $\operatorname{Lip} Y$ is a semigroup if and only if $\operatorname{Lip} X$ is, by Lemma 1 and so X has finite diameter. According to Lemma 2 we may and do assume $\mathfrak{t}_x$ is a positive power for every $x \in X$ and therefore T preserves order in both directions and the main result of [3] applies. ∎

We close with a counterexample showing that most of the hypotheses appearing in our statements are really necessary.

EXAMPLE 1. Two complete metric spaces X and Y which are not Lipschitz homeomorphic, yet there is a multiplicative bijection between $\operatorname{Lip}(Y)$ and $\operatorname{Lip}(X)$.

Proof. Set $X = \{4^n : n \in \mathbb{N}\}$ and $Y = \{2^n : n \in \mathbb{N}\}$ endowed with their standard metrics. These spaces are not Lipschitz homeomorphic. Indeed, let $\tau : Y \to X$ be any injective mapping. The set $S = \{n \in \mathbb{N} : \tau(2^n) \geq 4^n\}$ is infinite. If $n \in S$ is large enough one has $\tau(2^n) - \tau(2) \geq 4^n - \tau(2)$, which makes an estimate of the form $|\tau(2^n) - \tau(2)| \leq L|2^n - 2|$ impossible.

However, the formula $T(f)(4^n) = (f(2^n))^2 \operatorname{sign}(f(2^n))$ defines a multiplicative bijection between $\operatorname{Lip}(Y)$ and $\operatorname{Lip}(X)$. Quite clearly, T defines a semigroup isomorphism between $\mathbb{R}^Y$ and $\mathbb{R}^X$ whose inverse is given by $T^{-1}(g)(2^n) = \operatorname{sign} g(4^n)\sqrt{|g(4^n)|}$ and so the point is to check that T restricts to a bijection between $\operatorname{Lip} Y$ and $\operatorname{Lip} X$.

For $Z = X, Y$ we consider the space Z_0 obtained by adding the point 0 to Z. If f is any function on Z we extend it to a function f_0 on Z_0 just taking $f_0(0) = 0$. Obviously f_0 is Lipschitz on Z_0 provided $f \in \operatorname{Lip} Z$.

Let $f \in \operatorname{Lip}(Y)$ and let L a Lipschitz constant for f_0. Let us check that $T(f)$ is Lipschitz on X. Working separately with the positive and negative parts of f we may and do assume $f \geq 0$. We have

$$\begin{aligned}
\frac{\big|T(f)(4^n) - T(f)(4^m)\big|}{|4^n - 4^m|} &= \frac{\big|(f(2^n))^2 - (f(2^m))^2\big|}{|(2^n)^2 - (2^m)^2|} \\
&= \frac{|(f(2^n) - f(2^m))(f(2^n) + f(2^m))|}{|(2^n - 2^m)(2^n + 2^m)|} \\
&\leq L_f \frac{|f(2^n) + f(2^m)|}{2^n + 2^m} \leq L \frac{L2^n + L2^m}{2^n + 2^m} = L^2
\end{aligned}$$

and L^2 is a Lipschitz constant for $T(f)$.

For $g \in \mathrm{Lip}(X)$, take a Lipschitz constant K for g_0. If $m < n$, then

$$\frac{\left|T^{-1}(g)(2^n) - T^{-1}(g)(2^m)\right|}{|2^n - 2^m|} \leq \frac{\sqrt{|g(4^n)|} + \sqrt{|g(4^m)|}}{2^n - 2^m} \leq \frac{\sqrt{K4^n} + \sqrt{K4^m}}{2^n - 2^m}$$

$$\leq \frac{\sqrt{K4^n} + \sqrt{K4^{(n-1)}}}{2^n - 2^{n-1}} \leq \sqrt{K}\frac{2^n + 2^{n-1}}{2^{n-1}} = 3\sqrt{K}$$

and $3\sqrt{K}$ is a Lipschitz constant for $T^{-1}(g)$. ∎

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