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Extension of Mappings and Pseudometrics

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Abstract: The aim of this paper is to show a development of various methods of extensions of mappings and their interrelations. We shall point out some methods and relations entailing more general results than those originally stated.

Key words: Extension of function, extension of pseudometric, metric space.

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1. Introduction

1.1. General remarks. We shall use two main looks at results concerning extensions of mappings. The first one is just historical – we shall go through the original methods and divide them into three almost disjoint procedures. The second look goes into deeper analysis of the methods and tries to deduce from them as much as possible. In several cases we shall see that authors proved much more than they formulated and used (see, e.g. Theorems 3', 10', 2.6). We shall also describe some elementary relations among various extension results and some elementary procedures allowing to generalize extension results for metric spaces to more general ones (see 2.5-2.7, 3.4, Proposition 22, 4.1.3).

We shall consider those situations when *all* (uniformly) continuous mappings can be extended from closed subsets, excluding cases when extension exists for some of them only or from some more special subsets only. Also we shall not consider simultaneous or similar extensions. Including those other situations into this text would make it too long.

A predecessor of the present article is [25] devoted to F. Hausdorff. Some details were checked in the thesis [34] written under my supervision.

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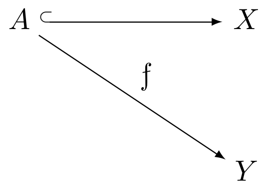
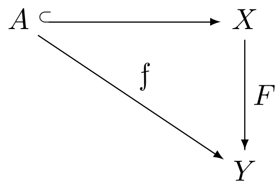
Let X and Y be topological-like structures and A be a substructure of X. To extend a morphism $f : A \to Y$ to a morphism $F : X \to Y$ means to complete the next diagram on the left to the commutative diagram on the right:



completes to



We shall call the problem described by diagrams as *extension problem*. There are two basically different situations:

1. A is *dense* in X;
2. A is *closed* in X.

We shall be interested in the second case only (exceptionally we shall use the first case, too). As for the structures, we shall restrict ourselves to topological or to uniform spaces. Our morphisms will be continuous or uniformly continuous mappings.

An extension was used and worked with already 200 hundred years ago, e.g., in solving Dirichlet problem. In that case an extension of a continuous function from a boundary to the whole set satisfied other conditions (solution of a partial differential equation). Probably it was H. Lebesgue who was the first who examined explicitly just a continuous extension in the Dirichlet problem in the plane. Proving that he noticed he proved a more general result, namely a positive answer to the above extension problem for $X = \mathbb{R}^2, Y = \mathbb{R}$, A a closed subspace of X and continuous functions.

The extension problem was shown as very important in time flow and several different methods were introduced to solve it. We shall be interested in general methods and general results only.

Basic general methods can be roughly divided into three parts:

1. Using special covers and partitions of unity.
2. Using inbetween (insertion) results.
3. Using formulas in case of metric spaces.

It is interesting to state the theorems in their original appearance (except symbols for structures and maps, where we use the notation from the above diagrams). In some cases not all assumptions for the spaces are given in the original formulations (like metrizability) since they are given in the original text somewhere before the theorems. In our citations those assumptions will be clear from the context. In the next titles of subsections, the date in brackets means the date of publication of the corresponding result.

2. Extension of continuous functions

2.1. Methods using special covers and partitions of unity.

In this section we touch results of four authors: H. Lebesgue (1907), H. Tietze (1915), L. E. J. Brouwer (1918) and J. Dugundji (1951). Only bounded continuous functions on metric spaces are mostly examined. The method used by Lebesgue, Tietze and Brouwer is a slight modification of a method used by R. Baire in 1900 [4], but only Tietze mentioned that in a footnote added in proofs. Baire was describing some functions in $\mathbb{R}^n$ being limits of sequences of continuous functions. He used cubes with vertices having coordinates equal to $k/2^p$, assigned values to vertices and extended conveniently that assignment to all points of cubes.

2.1.1. Henri Lebesgue 1875–1941 (1907).

H. Lebesgue proved the following theorem when studying Dirichlet problem in [29]:

Theorem 1. *Étant donnée une fonction continue f sur la frontière A d'un domaine fermé X, il existe une fonction F continue dans X qui est égale à f sur A.*

After proving the result he remarked the proof uses closedness of A only and the resulting function F is defined on the whole plane $\mathbb{R}^2$. We can sketch the Lebesgue's proof:

Lebesgue defines a sequence of covers of the plane by squares as Baire did. In the vertices of the largest squares disjoint with A he defines $F(x) = \liminf_{\varepsilon\to 0_+}\{f(a);\ d(x,a) < d(x,A)+\varepsilon\}$ and extends the values to the remaining points of the squares in a convenient way.

That Lebesgue's work was almost never quoted in connection with extension of functions. Reasons might be that the result is hidden in his paper and, also, the paper was addressed to a different group of mathematicians.

2.1.2. Heinrich Franz Friedrich Tietze 1880–1964 (1915).

The famous Tietze's work [38] (published in 1915, when he had a position at university of Brno) had its starting point in the Fejér's problem on plane Jordan curves (find conditions for a continuous function $x = \varphi(t)$ on a closed interval such that another such function ψ exists defining together with φ a Jordan curve) – see [39]. That problem was solved by J. Pál using an easy solution of the extension problem in $\mathbb{R}$. Tietze's feeling for abstraction led him to prove a required extension result for any Euclidean space. He needed a special case of an extension assertion of the following form:

THEOREM 2. *Zu jeder in der Menge $X \subset \mathbb{R}^n$ definierten, beschränkten und bezüglich X in allen Punkten der abgeschlossenen Teilmenge A von X stetigen Funktion $f(x)$ gibt es eine auf X stetige Funtion $F(x)$, die in A mit $f(x)$ übereinstimmt und sonst durchwegs $> f(x)$ ist.*

The proof is analogous to the above Lebesgue's proof with the exception of using lim sup instead of lim inf (because of the needed property $F(x) > f(x)$). Lebesgue is not quoted, the method is credited to Baire.

The last result is not yet the famous Tietze extension theorem. He sent his result to H. Hahn and Hahn pointed out a similarity of the result and the method with two questions:

1. *Question on extension of continuous functions from closed subsets of metric spaces to continuous functions defined on the whole space.*

2. *Question on a generalization from $\mathbb{R}^n$ to metric spaces of Baire theorem (every upper semi-continuous function is a limit of a decreasing sequence of continuous functions).*

Tietze notes that his theorem above (reproved for metric spaces – see the section on formulas) gives answer to the first question if he first extends continuous f defined on A to a function f on X, that is continuous at all points of A (in X, of course). It suffices to define (for $x \in X \setminus A$, d is the metric of X)

$$f(x) = \lim_{\varepsilon \to 0_+} \sup\{f(a);\, a \in A,\, d(a,x) < d(x,A) + \varepsilon\}\,.$$

Also the second question was answered in the affirmative using the same formula as for extension of functions (see the section on formulas).

In a footnote he discussed unbounded functions f. He says that for the proof it suffices to assume local boundedness of f. If f is unbounded, the procedure works but the extension is not defined at some points then.

2.1.3. LUITZEN EGBERTUS JAN BROUWER 1881–1966 (1919). Brouwer made a significant improvement of the previous described method. He quotes neither Baire nor Lebesgue in the paper. Only in the added remark at the end of the same volume he mentions Tietze's result and says the methods are quite near.

He used triangulations and noticed one need not use an infinite process in defining an extension (that was his motivation for the publication). Using suprema or infima poses a bound for codomains of extended mappings. His ingenious method remained unnoticed for decades.

He used a procedure from his proof of invariance of region from 1912: the complement of A is triangulated in a convenient way and for vertices v of the triangulation he defines $F(v) = f(a_v)$, where a_v is any point of A with $d(v, a_v) \leq 2d(v, A)$; F is then extended linearly on the simplices (in a unique way). For $x \in X \setminus A$ and a vertex v let $f_v(x)$ be the barycentric coordinate of x with respect to its support containing v (and zero if v is not a vertex of the support of x). Then $F(x) = \sum_v f(a_v)f_v(x)$. The family $\{f_v\}$ is a partition of unity subordinated to the cover of stars of vertices in the triangulation.

His result was as follows.

THEOREM 3. *Ist A eine abgeschlossene Punktmenge des n-dimensionalen Raumes und f eine beschränkte Funktion, dia auf A definiert und in jedem Häufungspunkte von A stetig is, so kann man eine im ganzen Raum stetig, beschränkte Funktion F finden, die in jedem Punkte von A gleich f ist.*

His method gives the following result without any change of the proof (notice that boundedness of f is not needed in the procedure):

THEOREM 3'. *Every continuous mapping f from a closed subset A of $\mathbb{R}^n$ to a locally convex space can be extended continuously on $\mathbb{R}^n$. The codomain of the extension is contained in the convex hull of codomain of f.*

Very probably, nobody became aware of the preceding reformulation for several decades (of course, in the years around 1920 and 1930 not for locally convex spaces as codomains but for special normed spaces only). In fact, there was not a big interest in a topological study of mappings into such spaces.

2.1.4. JAMES DUGUNDJI 1919–1985 (1951). A generalization of the Brouwer's method (if known) needs a partition of unity that was not known before 40'th and it was explicitly proved by A. H. Stone in [37] (although P. S. Alexandrov knew that covers of separable metric spaces have locally finite refinements already in 1922 – see [1]). Knowing Brouwer's approach and the fact that metric spaces are paracompact, it would be easy to prove the following result. But Dugundji in [15] was probably not aware of the Brouwer's paper.

THEOREM 4. *Let X be an arbitrary metric space, A a closed subset of X, Y a locally convex linear space, and $f : A \to Y$ a continuous map. Then there exists an extension $F : X \to Y$ of f; furthermore, $F(X) \subset$ convex hull of $f(A)$.*

The proof cannot be automatically carried over to paracompact spaces instead of metric spaces. One needs that the complement of the closed set A is paracompact. It was R. Arens who found a way how to go around that restriction in [2]. He used extensions of pseudometrics and had to assume that the codomain is complete. We shall come to his result in the section devoted to extensions of pseudometrics.

2.2. INBETWEEN (INSERTION) THEOREMS. The insertion results were proved directly for unbounded functions but their application to extension uses upper and lower bounds (if we do not want to use the Hahn's approach: he used functions with values $\pm\infty$).

2.2.1. FELIX HAUSDORFF 1868–1942 (1919). F. Hausdorff (see [22]) gave a simpler proof of Hahn's result ([18]):

THEOREM 5. *Ist $g(x)$ unterhalb, $h(x)$ oberhalb stetig und überall $g(x) \geq h(x)$, so gibt es eine stetige Funktion $F(x)$ mit $g(x) \geq F(x) \geq h(x)$.*

His using of the above result for extension of maps is ingenious and has no predecessor:

THEOREM 6. *Eine in der abgeschossenen Menge A definierte, stetige Funktion lässt sich zu einer in ganzen Raume stetigen Funktion ergänzen.*

For bounded f on A one takes

$$g(x) = \begin{cases} f(x), & \text{for } x \in A; \\ \text{an upper bound of } f, & \text{for } x \in X \setminus A. \end{cases}$$

$$h(x) = \begin{cases} f(x), & \text{for } x \in A; \\ \text{a lower bound of } f, & \text{for } x \in X \setminus A. \end{cases}$$

The functions g, h satisfy the conditions of Theorem 5 and the resulting function F extends f from Theorem 6.

That gives extension for bounded functions. Hausdorff was the second author considering extension for unbounded functions, too (after the consideration in the book by Carathéodory [10]).

If f is unbounded, then the function $f(x)/(1+|f(x)|)$ can be extended to a continuous function $G_1(X)$ on X with values in $[-1,1]$. Taking $G(x) = G_1(x)/(1+d(x,A))$ one gets an extension with values in $(-1,1)$. Now, the function $F(x) = G(x)/(1-|G(x)|)$ is the required continuous extension of f.

The method just described gives the following assertion: If Theorem 5 holds for a space X, also the extension Theorem 6 holds for X and bounded functions (or mappings into convenient codomains). In 1944, such an insertion result was proved by J. Dieudonné in [13] for paracompact spaces. For normal spaces (as their characterization) it was shown by M. Katětov and H. Tong.

2.2.2. Miroslav Katětov 1918–1995 (1951), Hing Tong 1922-2007 (1952). Katětov and Tong proved independently the above Hahn's insertion theorem for normal spaces; the Katětov's formulation:

Theorem 7. *If X is a normal space, g and h are functions in X, g is upper semi-continuous, h is lower semi-continuous, and $g(x) \leq h(x)$ for any $x \in X$, then there exists a continuous function f in X such that, for any $x \in X$, $g(x) \leq f(x) \leq h(x)$.*

Tong announced his result in 1948 ([40]) and submitted it more than 3 years later ([41]). Katětov had a mistake in his basic Lemma in [27] and corrected it in 1953. Both authors use an abstract approach via ordered sets or lattices. In some sense, the procedures generalize the Urysohn's idea how to construct a continuous function in normal spaces (Urysohn lemma). It is true that a simple direct proof of the insertion theorem for normal spaces can be done using the Urysohn's procedure.

It is clear that the insertion theorem, as stated, characterizes normal spaces. Both Katětov and Tong (also Dieudonné, Dowker, Michael and others) showed characterizations of certain special normal spaces using various combinations of $\leq$ and $<$ in the theorem. But that has no consequences to extension of functions we are interested in.

The insertion theorems are strong and can be generalized to other structures (we shall touch that later in uniformly continuous extensions). Of course, they have disadvantage that the codomain must be ordered. That excludes general normed linear spaces as codomains. But some generalizations with lattices as codomains can give a kind of extensions into lattices (see, e.g., [31]).

2.3. Formulas. Formulas for the extended function can be given in metric spaces. In addition to his proof using covers in $\mathbb{R}^n$, Tietze used a formula for generalizing the result to metric spaces. Later on, more formulas appeared looking optically simpler than the previous ones.

2.3.1. Heinrich Franz Friedrich Tietze 1880–1964 (1915). To prove Theorem 2 for metric spaces, Tietze first set

$$g(x) = \begin{cases} f(x), & \text{if } x \in A; \\ \sup\limits_{y \in X} \left\{ \dfrac{f(y)}{(1 + d(x,y))^{\frac{1}{d(x,A)}}} \right\}, & \text{if } x \notin A, \end{cases}$$

and then $F(x) = g(x) + rd(x, A)$, where r is a positive real number. He remarks that one can take supremum for accumulation points of $X \setminus A$ only. Another remark asserts that if $f(x) < s$ and $\limsup_{y \to x} f(y) < s$, than one may require $F(x) < s$. Although the formula looks not nicely, the proof of continuity is short and simple.

After generalizing Baire theorem ([5] – every bounded upper semi-continuous function in $\mathbb{R}^n$ is a limit of non-increasing sequence of continuous functions) to metric spaces, Tietze then proves:

Theorem 8. *Eine auf einer abgeschlossenen Menge A einer Fréchetschen Klasse X definierte, beschränkte und stetige Funktion f lässt sich zu einer in allen Elementen x von X definierten und stetigen Funktion ergänzen.*

It is stated that the same proof as above works for the formula

$$g(x) = \begin{cases} f(x), & \text{if } x \in A; \\ \sup\limits_{y \in A} \left\{ \dfrac{f(y)}{(1 + d^2(x,y))^{\frac{1}{d(x,A)}}} \right\}, & \text{if } x \notin A. \end{cases}$$

Then Tietze remarks that one can easily deduce Theorem 8 from Theorem 2 – see the part about Tietze in the section *Methods using special covers and decompositions of unit.*

2.3.2. Charles Jean de la Vallée-Poussin 1866–1962 (1916). In his book on integration and Baire functions [43] Vallé-Poussin gave a very interesting formula for continuous extensions of functions in Euclidean spaces. It is almost clear he could not be aware of the Tietze's result. He does not mention Lebesgue's paper [29].

He first proves a result on extension of α-Baire functions for $\alpha > 0$ on page 126:

Theorem 9. *Si f est de classe $\alpha > 0$ sur A parfait, une fonction F, égale à f sur A et à la constante a partout ailleurs, est de classe α sur le continu.*

The nontrivial case is $\alpha = 1$. For $\alpha = 0$ the result is not true and Vallé-Poussin proves the following result (there is no assumption on boundedness of the function in the formulation, but the proof needs boundedness of f) (see [43], p.127):

Theorem 10. *Si f est continu sur A, on peut définir une fonction F partout continu et égale à f sur A.*

To prove the result, Vallé-Poussin first takes a countable dense subset $\{a_k\}$ in A and defines $(B(x,r) = \{y; d(x,y) < r\})$

$$r_k(x) = d\big(a_k, \mathbb{R}^n - B(x, 2d(x,A))\big)\,,$$

$$F(x) = \begin{cases} \dfrac{\sum\limits_{k=1}^{\infty} r_k(x) f(a_k)/k^2}{\sum\limits_{k=1}^{\infty} r_k(x)/k^2}, & \text{if } x \in \mathbb{R}^n \setminus A, \\ f(x), & \text{if } x \in A. \end{cases}$$

Both the numerator and denominator are continuous functions on $\mathbb{R}^n \setminus A$ since $r_k(x)$ are continuous functions bounded on a neighborhood of x. Thus F is continuous on $\mathbb{R}^n \setminus A$. If x_i from $\mathbb{R}^n \setminus A$ converge to a point $a \in A$, then the values of $f(a_k)$ for k with $r_k(x) \neq 0$ converge to $f(a)$ and, thus, $F(x_i)$ converges to $f(a)$.

For the formula to work one needs the codomain to be complete and linear. For the domain one needs that A is separable. To prove continuity, one needs local convexity of the codomain. Therefore, Vallé-Poussin proved the following result:

THEOREM 10'. If f is a continuous mapping on a separable subset A of a metric space X into a complete locally convex space Y, it is possible to define a continuous mapping F on X into Y coinciding with f on A.

2.3.3. HARALD AUGUST BOHR 1887–1951 (1918). Carathéodory in his book [10] on real functions gives an interesting proof of the extension theorem in $\mathbb{R}^n$. In Preface he expresses his thanks to H. Bohr for written information about that result:

THEOREM 11. *Ist auf einer abgeschlossenen Punktmenge A des n-dimensionalen Raumes eine endliche Funktion f definiert, die in allen Häufungspunkten von A stetig ist, so kann man eine in ganzen Raume endliche und stetige Funktion F finden, die in jedem Punkte von A gleich f ist.*

For $X = \mathbb{R}^n$ and $0 \le f \le 1$ one puts

$$\psi(x,r) = \begin{cases} 0, & \text{if } A \cap B(x,r) = \emptyset, \\ \sup\{f(y) : y \in A \cap B(x,r)\}, & \text{if } A \cap B(x,r) \ne \emptyset. \end{cases}$$

The function ψ is nondecreasing and bounded in the variable r and, thus, the next integral converges.

$$F(x) = \begin{cases} \dfrac{1}{d(x,A)} \displaystyle\int_{d(x,A)}^{2d(x,A)} \psi(x,r)\, dr, & \text{if } x \notin A, \\ f(x), & \text{if } x \in A. \end{cases}$$

Adding to F the function $d(x,A)$ one can assume that F is everywhere positive whenever f is.

Any continuous $f \ge 0$ can be written as f_1/f_2, where f_1, f_2 are continuous and have values in $[0,1]$. Using the previous extensions F_1, F_2 for those two functions, one gets a continuous extension F_1/F_2 for f.

Any continuous f can be written as $f_1 - f_2$, where f_1, f_2 are continuous and have values in $[0,\infty)$. Using the extensions from the preceding paragraph for f_1, f_2, one gets a continuous extension for f.

This method gives directly continuous extensions of maps into $\mathbb{R}$ from closed subsets of metric spaces:

2.3.4. Felix Hausdorff 1868–1942 (1919). In his paper [22], Theorem 6, Hausdorff gives, in addition to application of insertion result, a formula for direct definition of the extended function F for a bounded function f:

$$F(x) = \begin{cases} \inf_{y \in A} \left\{ f(y) + \dfrac{d(x,y)}{d(x,A)} - 1 \right\}, & \text{if } x \notin A, \\ f(x), & \text{if } x \in A. \end{cases}$$

Later on, Hausdorff used modifications of that formula for extending metrics and extending continuous mappings into metric spaces.

He quotes Tietze, Bohr and Brouwer for their extension results. He probably noticed the Bohr's procedure is valid not only in Euclidean spaces but in metric spaces because he adds a remark to his quotation of Brouwer saying his proof is for Euclidean spaces only. He also says the Bohr's procedure is a little artificial.

2.3.5. Hans Hahn 1879–1934 (1921). Hahn in his book [19] presents a generalization of Tietze's extension theorem in the following form (X is a metric space):

Theorem 12. *Ist A ein in X abgeschlossener Teil von X, und f eine Funktion auf A, so gibt es auf X eine Funktion F, die auf A mit f übereinstimmt, und stetig auf X ist in allem Punkten von $X - A$, sowie in allen denjenigen Punkten von A, in denen f stetig ist auf A.*

The proof goes as follows. Assume $0 \leq f \leq 1$ and for $x \in X \setminus A$ define a continuous function $h_x(r)$ on $[0, \infty)$ having value 1 on $[0, 2d(x,A))$, 0 on $[3d(x,A), \infty)$ and being linear on $[2d(x,A), 3d(x,A)]$. Then

$$F(x) = \begin{cases} \sup_{a \in A} \{ f(a)\, h_x(d(x,a)) \}, & \text{if } x \notin A, \\ f(x), & \text{if } x \in A. \end{cases}$$

Hahn quotes Tietze, Bohr (in Carathéodory's book), Brouwer and Hausdorff ("with especially simple proof"). So, only Vallé-Poussin is missing.

As a consequence he gets the Tietze's extension theorem. He also formulates an extension theorem for functions f defined on general subsets A of metric spaces – first he extends them to the closure, which is equivalent to $\liminf_{a \to x} f(a) = \limsup_{a \to x} f(a)$.

After Hahn reproves his insertion theorem from [18] (he uses the Hausdorff's proof now) he repeats the Hausdorff's procedure for simple proof of extension theorem. He allows functions to have infinite values, and takes (see the paragraph following Theorem 6) $+\infty$ for the value of g on $X \setminus A$ and $-\infty$ for the value of h on $X \setminus A$. It follows from the proof of the insertion theorem that the resulting inbetween function has values in $\mathbb{R}$ in this case.

2.3.6. Frigyes Frédéric Riesz 1880–1956 (1923). A similar situation as with the Bohr's procedure happened with the F. Riesz' one – it was published in the Kerékjártó's book [28], p.75. His formula seems to be the simplest one from optical point of view. As for its proof, it seems to be practically at the same level as the other formulas.

The original formulation of the result is as follows:

Theorem 13. *Sei f eine eindeutige stetige Funktion auf einer beschränkten abgeschlossenen Punktmegne A. Es läßt sich eine in der ganzen Ebene stetige eindeutige Funktion F angeben, die in der Punkten von A mit der gegebenen Funktion f übereinstimmt.*

Assume that $1 \le f \le 2$ on A. Then its continuous extension F to X is described as follows:

$$F(x) = \begin{cases} \inf_{y \in A} \left\{ f(y) \dfrac{d(x,y)}{d(x,A)} \right\}, & \text{if } x \notin A, \\ f(x), & \text{if } x \in A. \end{cases}$$

The extended function is bounded again by 1 and by 2. The proof is valid in any metric space.

2.4. Urysohn's procedure. The Urysohn's extension theorem for normal spaces was a qualitative jump in extension results and his method does not fit into the previous boxes. It is true that insertion theorem for normal spaces (Katětov and Tong) gives the Urysohn's extension theorem, but its proofs usually use the Urysohn's procedure.

For completeness we briefly describe the Urysohn's method from [42]. First he proves the result (known as Urysohn lemma nowadays) that every two disjoint closed subspace A, B of a normal space X can be functionally separated (there is a continuous function $X \to [0, 1]$ with values 0 on A and 1 on B). Repeating separation of disjoint closed sets by their neighborhoods one gets a dyadic system $\{G_r;\, r = k/2^n,\, n \in \mathbb{N},\, k = 0, 1, \ldots, 2^n\}$ of open sets such that $\overline{G_r} \subset G_s$ whenever $r < s$, $G_0 \supset A, G_1 = X \backslash B$. Defining $f(x) = \inf\{r,\, x \in G_r\}$ one gets the required function ($\inf \emptyset = 1$ in our case). That procedure became a standard in many constructions of continuous functions.

The extension theorem, e.g., for $f : A \to [-1, 1]$, is then proved by using Urysohn lemma to construct a sequence $\{f_n\}$ of continuous functions having the properties $f_n(x) \le 2^{n-1}/3^n$, $|f(x) - (f_1(x) + \cdots + f_n(x))| \le 2^n/3^n$ (by induction: the sets $A = \big(f(x) - (f_1(x) + \cdots + f_n(x))\big)^{-1}[-1, -1/3]$ and $B = \big(f(x) - (f_1(x) + \cdots + f_n(x))\big)^{-1}[1/3, 1]$ are separated by a function g and then $f_{n+1}(x) = 2/3(g(x) - 1/2)$. The requesting extension is the sum of the functions f_n.

What about a method using covers or formulas? If f is bounded continuous on a closed subspace of a normal space X, then $d(x, y) = |f(x) - f(y)|$ is a continuous pseudometric on A. If we know that d can be extended to a continuous pseudometric e on X (such that (A, d) is closed in (X, e)), and we know an extension theorem for metric spaces, then f can be extended continuously on (X, e), thus on X. For the needed extension of continuous pseudometrics see the corresponding section.

2.5. Extension theorem for normal spaces using covers. The extension theorem for normal spaces follows easily from metrization theorem for uniformities. One must know that in normal spaces every finite open cover is uniformizable (normal in another terminology), i.e., belongs to the fine uniformity of the space – that is a characterization of normal spaces in the realm of completely regular spaces.

The weak uniformity on A generated by bounded f (in fact, the uniformity of the pseudometric $d(x, y) = |f(x) - f(y)|$) has for its base finite open covers and, thus, is a uniform subspace of a uniformity on X (since every finite open cover of A extends to a finite open cover of X) with a countable base, therefore is pseudometrizable by a pseudometric e. The restriction of e to A is uniformly equivalent to d and f is uniformly continuous on (A, d). Consequently, f can be extended to a uniformly continuous function on the closure of A in (X, e) and, using an extension result for metric spaces, to a continuous function on (X, e), thus continuous on X.

So, knowing the metrization for uniform spaces, extension of uniformly continuous maps to completions and extension theorem for metric spaces,

Observation. One can prove Urysohn extension theorem for normal spaces using the extension result for metric spaces.

If one realizes that instead of finite open covers one can use countable locally finite open covers (a completely regular space is normal iff every locally finite open cover is uniformizable) then the preceding procedure (using Dugundji theorem now) gives directly the following result for normal spaces (see [14], [20] and [3] for different proofs):

Observation. Every continuous mapping f from a closed subspace of a normal space into a separable metric subspace of a complete locally convex space Y can be continuously extended to $X \to Y$ (with its image contained in the closed convex hull of the image of f).

It follows from a result by C. H. Dowker (see the last but one section) that one cannot remove separability from the previous result.

2.6. Equicontinuous families. As mentioned above, the previous result does not hold for all normal spaces X and all locally convex spaces. Nevertheless, it follows from Brouwer's procedure for Euclidean spaces and from Dugundji's result for metric spaces that the result holds for Euclidean or for metric spaces X without assuming separability of the image. Both authors used partitions of unity. Can that result be proved without using partitions of unity?

Some introductory observations: Every locally convex (Hausdorff) space can be embedded in a product of Banach spaces as a topological linear subspace. Every Banach space can be embedded into some $C^*(Z)$ (endowed with the supremum norm) as a closed linear subspace (Z can be chosen compact).

A mapping $f : A \to C(Z)$ is a continuous iff the corresponding family $\{f_z; z \in Z\}$ is equicontinuous (here f_z is the composition of f with the z-projection of $C(Z) \subset \mathbb{R}^Z$ into $\mathbb{R}$). So, to prove Dugundji's extension theorem it suffices to show that every equicontinuous family of functions on a closed subspace A of a metric space X can be extended to an equicontinuous family of functions on X, and also that the image of extensions is contained in linear hulls of the images of the original maps.

A look at formulas used for proving extension result for metric spaces (like those of Tietze, Hausdorff, Riesz) shows that the "equicontinuous" extension is possible. To get equicontinuity from the insertion results is not possible.

So, we are able to deduce a special case of Theorem 4 from extension procedure for metric spaces into function spaces $C(Z)$:

Observation. Every continuous mapping from a closed subspace of a metric space X into a function space $C(Z)$ can be continuously extended to X.

To show that the extension preserves linearity does not follow from the formulas. So, to get Dugundji theorem, one cannot use just insertion results or the formulas.

2.7. Summary. Using Tietze theorem for extension of continuous functions in metric spaces, one can show in an elementary way the extension remains true in normal spaces.

Using formulas for extension of continuous functions in metric spaces one can show in an elementary way the extension remains valid for mappings into function spaces.

Using Dugundji theorem for extension of continuous functions in metric spaces into locally convex spaces one can show elementarily the theorem is valid for extension of continuous mappings from normal spaces into separable metrizable complete locally convex spaces.

3. Extension of uniformly continuous functions

At first we should recall the fact that extending uniformly real-valued continuous functions from subspaces or from closed subspaces is the same task (because a uniformly continuous map from a subspace A of a uniform space X into a complete space Y can always be extended to a uniformly continuous map on the closure of A in X). Secondly, not all unbounded uniformly continuous functions can be extended to uniformly continuous functions; we shall omit results showing special cases when such extensions exist.

For a long time, there was no interest to extend uniformly continuous functions to uniformly continuous functions. Even Hausdorff in his famous book [21] or Hahn in [19] were not interested in such extensions even to completions of metric spaces. They state that every uniformly continuous mapping into a complete metric space defined on a subspace of a metric space A can be extended to a continuous function on the closure $\overline{A}$ (although their proofs give uniform continuity of the extension). That result generalizes the results of several authors for extension from dense subspaces of sets in $\mathbb{R}$ (e.g., L. Scheefer in 1884, S. Pincherle in 1893, T. Brodén in 1897, E. Steinitz in 1899). In case of dense subsets of a compact interval (like the Brodén's case) one gets uniformly continuous extension from continuity - that was noticed. In 1923

Boulingand [8] again extended any uniformly continuous mapping defined on the subset of polynomials in $C[a,b]$ to a continuous function only. It was M. Fréchet who noticed that those extensions to completions are uniformly continuous, see [16].

3.1. Miroslav Katětov 1918–1995 (1951). Before 1950, probably nobody was interested in extension of uniformly continuous functions from subspaces of metric spaces or, after introducing uniformities, of uniform spaces. Katětov was the first who proved the next result in his paper [27] in the most general setting.

Theorem 14. *Let X be a uniform space and let f be a bounded uniformly continuous function in a subspace $A \subset X$. Then there exists a bounded uniformly continuous function F on X which coincide with f on A.*

The construction of F follows from his procedure for insertion theorem. A direct proof may be given using the Urysohn procedure. Instead of Urysohn lemma one must use the functional separation of far sets due to J. M. Smirnov: *If A, B are subsets of a uniform space $(X,\mathcal{U})$ and $U[A] \cap B = \emptyset$ for some $U \in \mathcal{U}$ then there is a uniformly continuous function $f : X \to [0,1]$ with the value 0 on A and 1 on B.*

3.2. Some generalities. We shall now look at other approaches related to those from the preceding section. Do some of the previous methods for extension of continuos functions in metric spaces give a uniformly continuous extension when one starts with a uniformly continuous function? That was not noticed probably till 1990 when Mandelkern showed in [30] that the Riesz' formula gives uniformly continuous F in case f is uniformly continuous.

We shall use the following characterization of uniform continuity:

Proposition 15. *Let $(X,\mathcal{U})$ be a uniform space (defined by means of neighborhoods of diagonal), A its subspace and $F : X \to \mathbb{R}$. The mapping F is uniformly continuous iff it has the following property:*

F is uniformly continuous on any $X \setminus U[A]$, $U \in \mathcal{U}$, and for every $\varepsilon > 0$ there is $U \in \mathcal{U}$ such that if $x \in A$, $y \in U[A]$, $(x,y) \in U$, then $|F(x)-F(y)| < \varepsilon$.

Proof. At first we observe that one can require also $x \in U[A]$ in the condition. Indeed, take U from the condition and take $V \in \mathcal{U}$ such that

$V \circ V \circ V \subset U$. For $x, y \in V[A]$, $(x, y) \in V$, there are $a, b \in A$ such that $(x, a) \in V$, $(y, b) \in V$ and, thus $(a, b) \in U$. Consequently, $|F(x) - F(y)| < 3\varepsilon$.

The necessity is clear. Assume that F satisfies our condition and take $\varepsilon > 0$ and $W, V \in \mathcal{U}$, $W \subset V$, $V \circ V \subset U$ (where U is from our generalized condition for $x \in U[A]$) such that $|F(x) - F(y)| < \varepsilon$ whenever $x, y \in X \setminus V[A]$, $(x, y) \in W$. Taking now $x, y \in X$, $(x, y) \in W$ we have either $(x, y) \in U[A]$ or $(x, y) \in X \setminus V[A]$, which both imply $|F(x) - F(y)| < \varepsilon$. ∎

It is not difficult to reformulate Theorem 14 for mappings into uniform spaces instead of into $\mathbb{R}$. In metric spaces the result has the following form:

COROLLARY 16. *Let (X, d) be a metric space, A its subspace and $F : X \to \mathbb{R}$. The mapping F is uniformly continuous iff it has the following property:*

For any $\varepsilon > 0$, F is uniformly continuous on $\{x \in X; d(x, A) > \varepsilon\}$ and there is $\delta > 0$ such that if $x \in A$, $y \in X$, $d(x, y) < \delta$, then $|F(x) - F(y)| < \varepsilon$.

3.3. METHODS USING SPECIAL COVERS AND PARTITIONS OF UNITY. Does the “Baire method” give a uniform function on $\mathbb{R}^n$ when one starts with a bounded uniform function on closed $A \subset \mathbb{R}^n$? From the procedures described earlier (Lebesgue, Tietze, Brouwer) it is convenient to consider the Brouwer's one. In the corresponding partition of unity described just before Theorem 3 all the functions f_v are uniformly continuous because they have compact supports. The question is, whether the function $F(x) = \sum_v f(a_v) f_v(x)$ is uniformly continuous provided f is uniformly continuous.

The triangulation used in the Brouwer's proof has the following properties:

1. The simplices with vertices not belonging to a uniform neighborhood of A have diameters bigger than a fixed positive number.

2. There is a nondecreasing function φ on $(0, \infty)$ with $\lim_{t \to 0_+} \varphi(t) = 0$ such that if s is a simplex of the triangulation having distance to A less than $t > 0$ then its diameter is less than $\varphi(t)$.

We shall check the condition from Corollary 16. Take $\varepsilon > 0$. Because of the previous property 1, the partition of unity f_v forms an equi-uniformly continuous family outside any uniform neighborhood of A. Consequently, the function F is uniformly continuous on the complement of the ε-neighborhood of A.

For $x \in A$ and $y \in X$ we have $|F(x) - F(y)| = |f(x) - \sum_v f(a_v) f_v(y)| \leq \sum_v |f(x) - f(a_v)| f_v(y)$. Take δ such that $|f(x) - f(a)| < \varepsilon$ whenever $d(x, a) < \delta$ and $\eta > 0$ such that $\eta + \varphi(\eta) < \delta/3$. If $d(x, y) < \eta$ and $f_v(y) \neq 0$ then $d(x, a_v) \leq d(x, y) + d(y, v) + d(v, a_v) < \eta + \varphi(\eta) + 2d(v, A)$. Since $d(v, A) \leq$

$\eta+\varphi(\eta)$ we get that $d(x,a_v) < 3\eta+3\varphi(\eta) < \delta$. Consequently, $|f(x)-f(a_v)| \le \varepsilon$ and $\sum_v |f(x) - f(a_v)| f_v(y) \le \varepsilon \sum_v f_v(y) = \varepsilon$.

Observation. The method for extensions of mappings using partitions of unity (or special covers) works for uniformly continuous mappings in metric spaces: it gives a uniformly continuous function F provided that f is uniformly continuous bounded.

The method will work in uniform spaces if we know that certain covers have equi-uniformly continuous partition of unity (which is true for every uniform cover but the partition of unity need not be locally finite).

3.4. Inbetween (insertion) theorems. A corresponding insertion result for uniformly continuous functions was proved by D. Preis and J. Vilímovský in [35]:

Theorem 17. *If $g \ge h$ are bounded functions on a uniform space X, the following conditions are equivalent:*

1. *There exists f uniformly continuous on X such that $g \ge f \ge h$.*
2. *For each $r < s \in \mathbb{R}$, the sets $\{x \in X;\, g(x) \le r\}$, $\{x \in X;\, h(x) \ge s\}$ are proximally far.*

If $A \subset X$ and $f : A \to [a,b]$ is uniformly continuous, then the functions

$$g(x) = \begin{cases} f(x), & \text{for } x \in A; \\ b, & \text{for } x \in X \setminus A, \end{cases}$$

$$h(x) = \begin{cases} f(x), & \text{for } x \in A; \\ a, & \text{for } x \in X \setminus A, \end{cases}$$

satisfy the condition (2) of the previous theorem, so that the resulting f is a uniformly continuous extension of f.

Consequently, *the method of insertion functions works in uniform spaces*, too.

The authors derived from their results also conditions for extension of unbounded uniformly continuous functions and some known insertion theorems in topological spaces (including Katětov-Tong theorem). It would be interesting to find out whether the original Katětov-Tong's procedures work in uniform spaces, too.

3.5. Formulas. Going through the proofs of continuity of all the formulas from the previous section, one can easily show that the condition of Corollary 16 is satisfied. Consequently, *all the formulas work in extending uniformly continuous functions in metric spaces*, too.

There are two different parts of the proof. The first one uses just the definition of $F(x)$ for $x \in X \setminus A$ and proves that F is uniformly continuous on complement of every uniform neighborhood of A (in fact, even Lipschitz continuous). The proofs of continuity of F at $x \in A$ gives for every $\varepsilon > 0$ some $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ when $d(x, y) < \delta$, $y \in A$; if there exists some k (not depending on x) such that $|f(x) - F(y)| < \varepsilon$ whenever $d(x, y) < \delta/k$, $y \in X$, then also the second part of the condition of Proposition 16 is satisfied, so that F is uniformly continuous on X.

Details were checked in a thesis [34] written under my supervision.

3.6. From extensions in metric spaces to those in uniform spaces. For uniformly continuous functions it is possible to repeat extension procedure for continuous functions described just before this section.

Let X be a uniform space and A its subspace, $f : A \to [a, b]$ a uniformly continuous function. Take the precompact pseudometric $d(x, y) = |f(x) - f(y)|$ on A. Every its ε-neighborhood of diagonal can be extended to a member of uniformity on X, thus determining a uniformly continuous pseudometric ρ that is uniformly equivalent to d on A. The function f is uniformly continuous on (A, ρ_A). If we know an extension theorem for metric spaces (e.g. using formulas), we know that f can be extended to a uniformly continuous function F on (X, ρ), therefore uniformly continuous on X.

If we do not want to use extensions for metric spaces, it is possible to use extensions from compact subspaces. The following result (again we state it in its original formulation) was proved by E. Čech in his famous paper [11].

Theorem 18. *Let X be a completely regular space. Let A be a closed subset of X; let $\overline{A}$ denote the closure of A in the space βX. Then we have $\overline{A} = \beta A$ if and only if every bounded continuous function in the domain A admits of a continuous extension to the domain X.*

The Čech's proof uses the Urysohn's extension result for normal spaces that we cannot use in our situation. A completely different approach gives Theorem 18 as an easy consequence of Stone-Weierstrass theorem. Then the procedure for extension of uniformly continuous map from a subspace A of a uniform space X is as follows: Take again the precompact pseudometric

$d(x,y) = |f(x) - f(y)|$ on A and extend f onto the completion of (A, ρ_A), which is a compact subspace of the completion of (X, ρ). Consequently, f extends to a uniformly continuous function on that last completion, which implies its uniform continuity on X.

One cannot prove an analogy of Dugundji's theorem for uniformly continuous mappings because not every closed convex subset of a Banach space is its uniform retract, as was shown by Lindenstrauss.

4. Extensions of pseudometrics.

In the preceding parts we dealt with (uniformly) continuous extensions of (uniformly) continuous mappings. Metrics are special functions but one cannot expect that the methods used for extension of functions preserve the axioms for metrics.

It is clear that extending mappings into Hausdorff spaces from metrizable spaces is the same task as extending mappings from pseudometrizable spaces into Hausdorff spaces. The situation for extending metrics is similar: extending metrics from metric spaces is the same task as extending pseudometrics from pseudometric spaces.

There are two problems related to such extension. We recall that (uniform) continuity of a pseudometric d on a topological (or uniform) space X means (uniform) continuity of d on $X \times X$ or, equivalently, the topology (or the uniformity) of d is coarser than that of X.

Questions.

1. We have a metric space (X, d) and a metric e on its subspace A topologically (or uniformly) equivalent to d_A. Does there exist a metric E on X extending e and topologically (or uniformly, resp.) equivalent to d?
2. We have a pseudometric d that is (uniformly) continuous on a subspace A of a space X. Does there exist a pseudometric D on X extending d that is (uniformly) continuous on X?

As we mentioned above, it is equivalent to speak about pseudometrics and pseudometric spaces in the above questions.

Notice that the first question has a sense for pseudometrizable spaces X and the second question has a sense for any topological (or uniform) space X.

As in the previous sections we shall first deal with continuous pseudometrics and then with uniformly continuous pseudometrics.

4.1. Continuous pseudometrics. For continuous case, the above Questions should be formulated for closed subsets of A. If X is assumed to be metric in (2), then both questions are equivalent in the sense that a positive answer to one of them gives a positive answer to the other. I do not know who was the first to consider that equivalence. The idea of the next proof was used in the book [12] and is a modification of that in [33].

Proposition 19. *Every metric on a closed subset A of a metric space (X,d) that is equivalent to d_A can be extended to a metric on X equivalent to d iff every continuous pseudometric on A can be extended to a continuous metric on (X,d).*

Proof. Necessity: If e is a continuous pseudometric on A, then $\sup\{e,d_A\}$ is a metric equivalent to d_A. Our condition implies existence of a metric M on X extending $\sup\{e,d_A\}$ and equivalent to d. Denote by $h(x,y)$ the function on $X\times X$ having values $e(x,y)$ on $A\times A$ and $M(x,y)$ elsewhere. The pseudometric

$$E(x,y)=\sup\{\rho;\ \rho \text{ is a pseudometric on } X,\ \rho(x,y)\le h(x,y)\}$$

is the required one.

Sufficiency: Let e be a metric on A equivalent to d_A. Our condition implies existence of a pseudometric M extending e and continuous on (X,d). We may assume that A is closed in (X,M) (otherwise we add $|d(x,A)-d(y,A)|$ to $M(x,y)$). Let u be a canonical isometric embedding of (X,d) into $C^*(X,d)$ (endowed with sup norm), i.e., $u(x)(y)=d(x,y)-d(y,a_0)$ where a_0 is a fixed point of X. Using Dugundji extension theorem, take a continuous extension $v:(X,M)\to C^*(X,d)$ of the restriction of u to A having the same supremum on X as u has on A. The metric

$$E(x,y)=\max\big\{M(x,y),\ \|(u-v)(x)-(u-v)(y)\|\big\}$$

is the required metric. ∎

For sufficiency we used Dugundji theorem for mappings into $C^*(X)$ endowed with the supremum norm. Dugundji theorem in this case means that an equicontinuous collection of bounded continuous maps $A\to\mathbb{R}$ extends to an equicontinuous collection of bounded continuous maps $X\to\mathbb{R}$. As we mentioned in the preceding sections, such an implication follows from the methods of the proofs of extending single functions. So, it is not necessary to use Dugundji theorem for that special case of maps into $C^*(X)$.

In [32] it is written that the remark in the previous paragraph for uniformly continuous functions follows directly from the Katětov extension theorem – that is not true since one may easily extend a uniformly equicontinuous collection to a collection that is not uniformly equicontinuous.

We should mention that, as one would expect, existence of extensions of continuous pseudometrics implies existence of extensions of continuous functions. We shall see later that the converse is not true.

PROPOSITION 20. *If every continuous pseudometric on a subspace A of a topological space X can be extended to a continuous pseudometric on X, then every continuous function on A can be extended to a continuous function on X.*

Proof. Let f be a continuous function on A. We may assume that $f \geq 0$ (otherwise we take f_+, f_-) and $\inf f(x) = 0$. The mapping $d(x, y) = |f(x) - f(y)|$ is a continuous pseudometric on A. Suppose it can be extended to a continuous pseudometric D on X. Take $a_n \in A$ such that $f(a_n)$ converges monotonically to 0 ($a_n = a$ provided a exists with $f(a) = 0$) and define $F(x) = \inf_{\mathbb{N}}\{D(x, a_n) + f(a_n)\}$. It is easy to show that F is a continuous extension of f to X. ∎

The above result comes from [17].

4.1.1. FELIX HAUSDORFF 1868–1942 (1930). As far as I know, the first one examining extensions of metrics was F. Hausdorff in [23]. He gave no motivation for such a result (it may well be true that his motivation was to find an easier proof for Niemytzki and Tikhonov result – se below).

Hausdorff solved positively the first question in the next result, where X is a metric space:

THEOREM 21. *Ist A in X abgeschlossen, so lässt sich eine Homöomorphie zwischen A and B zu einer Homöomorphie zwischen X und einem geigneten Raum Y erweitern.*

Another formulation given by Hausdorff is: *Every metrizable topological space X can be metrized by a metric, restriction of which is a given metric on A determining its topology.*

Hausdorff uses a modification of his formula for proving Tietze extension theorem. Let d be a metric on X and e a metric on A equivalent to d_A. He defines for $x \in X \setminus A$, $a \in A$, c is a fixed selected point in A:

$$\varphi(x,a) = \inf_{p\in A}\left\{e(p,c) + 2\frac{d(p,x)}{d(x,A)} - 2\right\} + \sup_{p\in A}\left\{e(a,p) - e(c,p) - \frac{d(x,p)}{d(x,A)} + 1\right\}$$

and then the required metric

$$E(x,y) = \max\left\{\min\{d(x,y), d(x,A) + d(y,A)\}, \sup_{a\in A}\{|\varphi(x,a) - \varphi(y,a|\}\right\}.$$

The main role in the proof is played by the function φ. Unlike the proof for extending functions, the proof of the above theorem is more complicated.

At the end of the paper, Hausdorff applies his result to a very simple proof of the result by Niemytzki and Tikhonov (a metrizable space is compact iff every its metric is complete) and shows another result of similar kind: *A metrizable space is compact iff every its metric is bounded.* If X is not compact, it contains a countable subset without accumulation point that is, therefore, homeomorphic to both $\mathbb{N}$ and to $\{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$ bearing the usual metrics. Extending that homeomorphism to X one gets the assertions.

F. Hausdorff generalized his previous result in his last published paper [24]. We may notice that Theorem 21 together with the equivalent formulation given in Proposition 19 implies the following result. Conversely, the following result easily implies extension of continuous pseudometrics.

Proposition 22. *If $f : A \to B$ is a continuous mapping on a closed subspace A of a metric space X onto a metric space B, then there exists a metric space Y containing B as a closed subspace and a continuous mapping $F : X \to Y$ that extends f and maps $X \setminus A$ onto $Y \setminus B$. If f is a homeomorphism then F is a homeomorphism.*

Proof. Let the metrics on B, X be e, d resp. and denote by e_f the continuous pseudometric $e \circ (f \times f)$ on (A, d_A). Then $f : (A, e_f) \to B$ is uniformly continuous and the metric modification $\widetilde{A}$ of (A, e_f) is isometric to (B, e). Now we extend e_f to a continuous pseudometric E on (X, d) (adding $|d(x,A) - d(y,A)|$ to it we may assume that A is closed in (X, E)) and define Y to be the metric modification of (X, E). Up to the mentioned isometry, B is a closed metric subspace of Y and the required F is the composition of the identity mapping $(X, d) \to (X, E)$ and the metric modification map $(X, E) \to Y$.

The second part of the assertion now follows from the above original Hausdorff's formulation. ∎

Proposition 22 was proved (in a stronger formulation that cannot be easily obtained by the above method) in Hausdorff's last published paper [24]. The proof uses again a modification of his formula for extension of function and is complicated. Hausdorff in his paper reacted to a previous K. Kuratowski's paper from 1935 showing the above result for separable metric spaces (again in a stronger formulation). After reading Hausdorff's generalization, Kuratowski gave a more elegant proof for separable spaces in the same volume of Fund. Math., where the Hausdorff's result was published (via maps into function spaces). Later on, the Kuratowski's proof was shown (after some modifications) to work without assuming separability (see [2]).

I do not know who first used the above proof of Theorem 22 (maybe it was a folklore) but it is very probable that J. Isbell had it in mind when he stated without proof a modification of Theorem 22 for uniform spaces as a corollary of his extension of uniformly continuous pseudometrics in uniform spaces.

F. Hausdorff proved the following result in [24]. The substantial improvement in comparison with Proposition 22 is the second part of II.

THEOREM 23. I. *Eine stetige Abbildung der im metrischen Raum X abgeschlossenen Menge A auf den metrischen Raum B lässt sich zu einer stetigen Abbildung von X auf einen geigneten metrischen Raum $Y \supset B$ erweitern.*

II. *Diese Erweiterung ist insbesondere so möglich, dass B in Y abgeschlossen ist und $X \setminus A$ topologisch auf $Y \setminus B$ abgebildt wird.*

III. *Eine topologische Abbildung von A lässt sich zu einer topologischen Abbildung von X erweitern.*

4.1.2. R. H. BING 1914–1986 (1947). Another proof of Theorem 21 was given by R. H. Bing in [6] without quoting Hausdorff.

THEOREM 24. *Suppose that A is a closed subset of a metrizable space X and that $d(x, y; A)$ is a metric on A. Then there is a metric $d(x, y)$ on X that preserves the metric $d(x, y; A)$ on A ; that is, if x and y are points of A, then $d(x, y) = d(x, y; A)$.*

Bing assumes at first that $d(x, y; A)$ is bounded and constructs a metric d' on X such that $d(x, y) \le d'(x, y)$ whenever $x, y \in A$. The required metric is then defined as follows:

$$d(x,y) = \min\left\{d'(x,y),\ \inf_{a,b\in A}\left\{d'(x,a)+d(a,b)+d'(b,y)\right\}\right\}.$$

The metric d' is constructed by means of sequence of covers from his modification of Alexandrov-Urysohn metrization theorem (it follows easily from the metrization procedure for uniform spaces, too). If $d(x, y; A)$ is unbounded, X can be expressed as a union of an increasing sequence of closed sets intersecting A in balls with the same center and radiuses equal to $n, n \in \mathbb{N}$. Using the first part of the proof, $d(x, y; A)$ can be successively extended to X.

In [7] Bing used his previous method to show that some compact (e.g. all finite-dimensional) compact locally connected continua have a convex metric.

The Bing's method is simpler than those of Hausdorff and can be used in extension of uniformly continuous pseudometrics, too.

4.1.3. Interrelations, Richard Friederich Arens (1919–2000), Clifford Hugh Dowker (1912–1982). We have now results about extensions of continuous functions, extensions of continuous maps into locally convex spaces and extensions of continuous pseudometrics. A deeper look at their interrelations started around 1951 in the works by R. Arens, C. H. Dowker, J. Dugundji and O. Hanner. Some special cases were know earlier (e.g., Kuratowski in 1935, Kakutani in 1940), additions and various combinations were done later, for instance by Alo and Shapiro. We shall describe the situation without giving all the details and references.

Pseudometrics are in a close relation with uniformities. Dealing now with topological spaces, we shall use fine uniformities on completely regular spaces formed by the collection of all uniformizable covers (or normal covers, in another terminology), and generated by all continuous pseudometrics. We should know that a uniformity is pseudometrizable iff it has a countable base and that pseudometrizable spaces are paracompact. Then it is easy to show that *all locally finite cozero covers of a space X form a base of all the uniformizable covers of X.*

Clearly, if every continuous pseudometric on a subspace A of a completely regular space X extends to a continuous pseudometric on X, then every uniformizable cover of A extends to a uniformizable cover of X. Conversely, it follows from the metrization theorem for uniform spaces that if every uniformizable cover of A extends to a uniformizable cover of X, then every continuous pseudometric on A is equivalent to a restriction to A of a continuous pseudometric on X. The above Bing's construction gives even that instead of "equivalent" we may require "equal". So we have the three equivalences $1 \leftrightarrow 2 \leftrightarrow 3$ in the following theorem.

THEOREM 25. *If A is a subspace of a completely regular space X, then the following conditions are equivalent:*

1. *Every continuous pseudometric on A extends to a a continuous pseudometric on X.*
2. *Every uniformizable cover of A extends to a uniformizable cover of X.*
3. *Every locally finite cozero cover of A is refined by the trace on A of a locally finite cozero cover of X.*
4. *Every continuous mapping on A into a Banach space Y (or into a complete locally convex space) extends to a continuous mapping on X into Y.*

The implication $1 \to 4$ can be proved as follows: if $f : A \to Y$, Y a Banach space, is continuous, then $d(x,y) = \|f(x) - f(y)\|$ is a continuous pseudometric, extend it to a continuous pseudometric D on X, completeness of Y guaranties continuous extension of f to the closure of A in (X, D), and Dugundji theorem finished the implication for Banach spaces Y (the case of complete locally convex spaces follows from their embedding onto a closed convex subspace of a product of Banach spaces).

Since fine uniformities are determined by continuous maps into metric spaces (thus into Banach spaces since every metric space embeds isometrically into some $C^*(X)$), the implication $4 \to 2$ is an easy consequence of that consideration.

As we know, completeness of the range in (4) is not necessary to assume if X is metric and A is closed. The completeness is needed for extension of functions to the closure of A in the constructed continuous pseudometric.

Since in paracompact spaces all open covers are uniformizable, it follows (first stated by R. Arens in [2] for condition 4) that the conditions hold for X paracompact and A closed. This result was strengthened by C. H. Dowker in [14]; he showed that *the condition* (4) *holds for every closed subset A of X iff X is collectionwise normal.* That means the condition (4) does not hold for normal spaces that are not collectionwise normal. Consequently, Proposition 20 cannot be converted. Nevertheless, it follows from a result of O. Hanner ([20]) and J. Dugundji that *the condition* (4) *with separable Banach spaces holds for any closed subset A of X iff X is normal* (see our second Observation in 2.5). In [3] (without being aware of [14]) the above equivalent conditions were formulated when corresponding cardinalities of covers or of dense sets are restricted. For countability case the result is as follows (Arens does not mention the condition (2)).

THEOREM 26. *If A is a subspace of a completely regular space X, then the following conditions are equivalent:*

1. *Every (bounded) continuous separable pseudometric on A extends to a (bounded) continuous pseudometric on X.*
2. *Every countable uniformizable cover of A extends to a countable uniformizable cover of X.*
3. *Every countable locally finite cozero cover of A is refined by the trace on A of a countable locally finite cozero cover of X.*
4. *Every continuous mapping on A into a separable Banach space Y (or into a separable complete locally convex space) extends to a continuous mapping on X into Y.*

Arens then proves that the conditions (for countable case) hold provided A is a closed subspace of a normal space X, and without cardinality restriction, they hold provided A is a closed subspace of a paracompact space X. In case the space X is normal and A is closed, one can write "open" instead of "cozero" in the condition (3) since a completely regular space is normal iff every its finite (or locally finite) open cover is uniformizable.

4.2. UNIFORMLY CONTINUOUS PSEUDOMETRICS.

4.2.1. JOHN R. ISBELL 1931-2005. A corresponding theorem for extension of uniformly continuous pseudometrics waited until 1959 when J. Isbell proved the next result in [26]. He proved the result directly for general uniform spaces without any predecessor with extension from metric spaces.

THEOREM 27. *Every bounded uniformly continuous pseudometric on a subspace of a uniform space may be extended to a bounded uniformly continuous pseudometric on the whole space.*

A basic idea of the Isbell's procedure is that of Bing but using a simpler metrization procedure for uniform spaces. For a bounded uniformly continuous pseudometric e on a subspace A of a uniform space X he first finds a uniformly continuous pseudometric d on X with $d \geq e$ on A (the sequence of 2^{-n}-covers of (A, e) extends to a convenient sequence of uniform covers of X – the corresponding uniformly continuous pseudometric d on X generated by that sequence may be chosen having bigger values on A than e). The requested pseudometric is then defined by the equality

$$E(x,y) = \min \Big\{ d(x,y),\ \inf_{a,b\in A} \big\{ d(x,a) + e(a,b) + d(b,y) \big\} \Big\} .$$

The following corollary is then stated without proof in [26] (equivalently, one may assume both B and Y to be metric spaces). See also a discussion about continuous case after Theorem 22.

Corollary 28. *For every uniformly continuous mapping f of a subspace A of a uniform space X into a uniform space B, there exist a uniform space Y containing B and a mapping $g : X \to Y$ extending f.*

The same proof as for the topological case shows that extension of uniformly continuous pseudometrics implies extension of uniformly continuous functions ([17]. Unlike the topological case, here one can deduce the Isbell's result from the Katětov's result on extension of maps (see [12] and [33]):

Proposition 29. *Every bounded uniformly continuous pseudometric on a subspace A of a uniform space X can be extended to a bounded uniformly continuous pseudometric on X iff every bounded uniformly continuous function on A can be extended to a bounded uniformly continuous function on X.*

Proof. The proof of necessity is the same as in the proof of Proposition 20. We shall prove the sufficiency. Assume that e is a bounded uniformly continuous pseudometric on A. We can extend the function $e : A\times A \to [0, r]$ to a uniformly continuous function $f : X \times X \to [0, r]$ and take a bounded uniformly continuous pseudometric $D(x, y)$ on X defined as $\sup_{z\in X} |f(x, z) - f(y, z)|$. Then $E(x, y) = \sup\{\rho;\ \rho$ is a pseudometric on $X,\ \rho \le D,\ \rho \le d$ on $A\}$ is a bounded uniformly continuous pseudometric on X extending e. ∎

The original Hausdorff's version of extension of uniformly equivalent metrics was proved by Nhu in [33]:

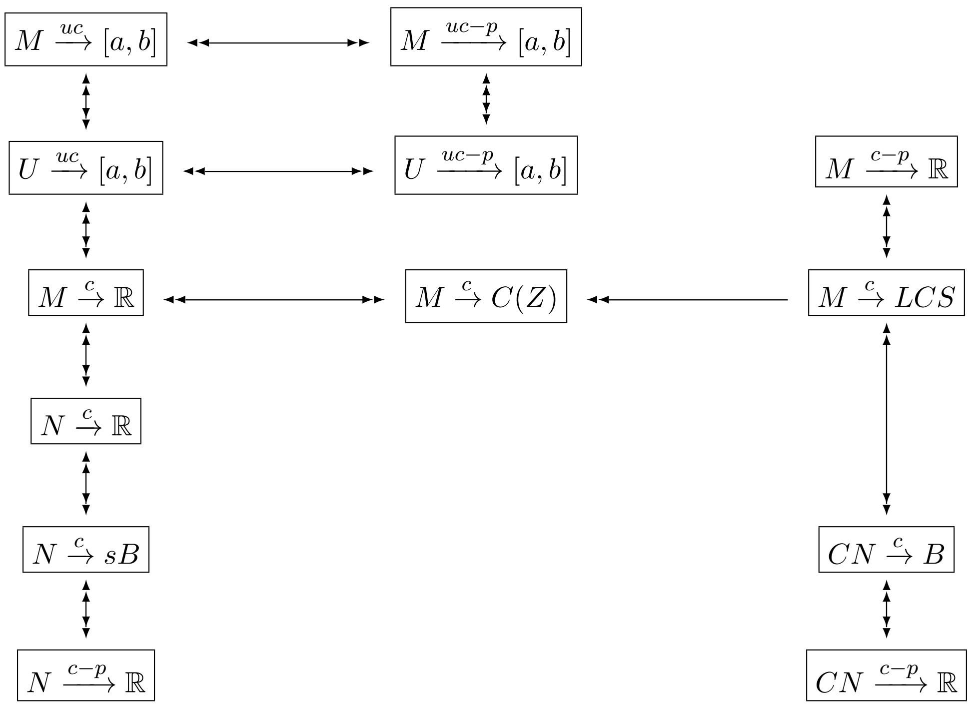
Theorem 30. *Let A be a subset of a metric space (X, d). Then any bounded metric ρ on A uniformly equivalent to d_A can be extended to a metric $\overline{\rho}$ on X which is uniformly equivalent to d on X.*

5. Chronological table and interrelations

The next diagram roughly corresponds to relations among various extension results. The arrows between boxes mean implications or equivalences, resp., proved by an "elementary" way.

We shall use the following abbreviations:

M	metric space
N	normal space
CN	collectionwise normal space
B	Banach space
U	uniform space
LCS	locally convex space
c	continuous mapping
uc	uniformly continuous mapping
c-p	continuous pseudometric
uc-p	uniformly continuous pseudometric
prefix s before a space	*separable*
prefix c before a space	*complete*



The next table briefly summarizes development of extension of maps $f : A \to Y$ from a closed subset A of a space X. The rows with empty cell for author correspond to our consideration how to get more general results from old ones.

AUTHOR	YEAR	Original result $X \to Y$	Original result f	METHOD	Proof gives $X \to Y$	Proof gives f	RESULT
Legesgue	1907	$\mathbb{R}^2 \to \mathbb{R}$	c	covers			Th.1
Tietze	1915	$\mathbb{R}^n \to \mathbb{R}$	c	covers			Th.2
Tietze	1915	$M \to \mathbb{R}$	c	formula		c+uc	Th.8
Vallé-Poussin	1916	$\mathbb{R}^n \to \mathbb{R}$	c	formula	$sM \to cLCS$	c+uc	Th.10, 10'
Bohr	1918	$\mathbb{R}^n \to \mathbb{R}$	c	formula	$M \to \mathbb{R}$	c+uc	Th.11
Brouwer	1919	$\mathbb{R}^n \to \mathbb{R}$	c	covers	$\mathbb{R}^n \to LCS$	c+uc	Th.3, 3'
Hausdorff	1919	$M \to \mathbb{R}$	c	insertion			Th.6
Hausdorff	1919	$M \to \mathbb{R}$	c	formula		c+uc	2.3.4
Hahn	1921	$M \to \mathbb{R}$	c	formula		c+uc	Th.12
Riesz	1923	$M \to \mathbb{R}$	c	formula		c+uc	Th.13
Urysohn	1925	$N \to \mathbb{R}$	c	Urysohn		c+uc	2.4
Katětov,Tong	1951	$N \to \mathbb{R}$	c	insertion			Th.7
Dugundji	1951	$M \to LCS$	c	covers			Th.4
		$M \to \mathbb{R}$	c	formulas	$M \to C(Z)$		2.6
		$M \to \mathbb{R}$	c	covers	$N \to \mathbb{R}$		2.5
		$M \to LCS$	c	covers	$N \to csLCS$		2.5
Katětov	1951	$U \to \mathbb{R}$	uc	Urysohn			Th.14
Preis,Vilímovský	1980	$U \to \mathbb{R}$	uc	insertion			Th.17
		$M \to \mathbb{R}$	uc	formulas	$U \to \mathbb{R}$		3.6
Hausdorff	1930	$M \to \mathbb{R}$	c-p	formula			Th.21, 23
Bing	1947	$M \to \mathbb{R}$	c-p	covers			Th.24
Isbell	1959	$U \to \mathbb{R}$	uc-p	covers			Th.27

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References

[1] P.S. Aleksandrov, Sur les ensembles de la première classe et les ensembles abstraits., *Compt. Rendus Acad. Sci. Paris* **178** (1924), 185 – 187.

[2] R. Arens, Extensions of functions on fully normal spaces, *Pacific J.Math* **2** (1952), 11 – 22.

[3] R. Arens, Extensions of coverings, of pseudometrics, and of linear-space-valued mappings, *Canad.J.Math.* **5** (1953), 211 – 215.

[4] R. Baire, Nouvelle démonstration d'un théorème sur les fonctions discontinues, *Bull. Soc .Math. France* **28** (1900), 173 – 179.

[5] R. Baire, Sur la séries à termes continues at tous de meme signe, *Bull. Soc. Math. France* **32** (1904), 125 – 128.

[6] R.H. Bing, Extending a metric, *Duke Math. J.* **14** (1947), 511 – 519.

[7] R.H. Bing, A convex metric for a locally connected continuum, *Bull. Amer. Math. Soc.* **55** (1949), 812 – 819.

[8] M. Boulingand, Sur les modes de continuité de certaines fonctionelles, *Bull. Sci. Math.* **47** (1923), 1 – 15.

[9] L.E.J. Brouwer, Über die Erweiterung des Definitionsbereichs einer stetigen Funktion, *Math. Ann.* **79** (1919), 209 – 211, 403.

[10] C. Carathéodory, "Vorlesungen über Reelle Funktionen", Teubner, Leipzig 1918.

[11] E. Čech, On bicompact spaces, *Ann. of Math. (2)* **38** (4) (1937), 823 – 844.

[12] E. Čech, "Topological Spaces", (rewritten by Z.Frolík and M.Katětov), Publishing House of the Czechoslovak Academy of Sciences, Prague; Interscience Publishers John Wiley & Sons, London-New York-Sydney 1966.

[13] J. Dieudonné, Une généralization des espaces compacts, *J. Math. Pure Appl.* **23** (1944), 65 – 76.

[14] C.H. Dowker, On a theorem of Hanner, *Ark. Mat.* **2** (1952), 307 – 313.

[15] J. Dugundji, An extension of Tietze's theorem, *Pacific J.math.* **1** (1951), 353 – 367.

[16] M. Fréchet, Prolongement des fonctionelles continues sur un ensemble abstrait, *Bull. Sci. Math.* **48** (1924), 171 – 183.

[17] T.E. Gantner, Extensions of uniformly continuous pseudometrics, *Trans. Amer. Math. Soc.* **132** (1968), 147 – 157.

[18] H. Hahn, Über halbstetige und unstetige Funktionen, *Wiener Akad. Ber.* **126** (1917), 91 – 110.

[19] H. Hahn, "Theorie der Reellen Funktionen", Springer, Berlin 1921.

[20] O. Hanner, Solid spaces and absolute retracts, *Ark. Mat.* **1** (1951), 375 – 382.

[21] F. Hausdorff, Mengenlehre, Teubner, Berlin 1914.

[22] F. Hausdorff, Über halbstetige Funtionen und deren Verallgemeinerung, *Math. Z.* **5** (3-4) (1919), 292 – 309.

[23] F. Hausdorff, Erweiterung einer Homöomorphie, *Fund. Math.* **16** (1930), 353 – 360.

[24] F. Hausdorff, Erweiterung einer stetigen Abbildung, *Fund. Math.* **30** (1938), 40 – 47.

[25] M. Hušek, History and development of Hausdorff's work in extension in metric spaces, in Recent Development of General Topology and its Applications, Math. Research 67, Akademie-Verlag, Berlin 1992), 160 – 169.

[26] J. Isbell, On finite-dimensional uniform spaces, *Pacific J. Math.* **9** (1959), 107 – 121.

[27] M. Katětov, On real-valued functions in topological spaces, *Fund. Math.* **38** (1951), 85 – 91, Correction **40** (1953), 203 – 205.

[28] B. Kerékjartó, "Vorlesungen über Topologie", Springer, Berlin 1923.

[29] H. Lebesgue, Sur le problème de Dirichlet, *Rend. Circ. Mat. Palermo* **24** (1907), 371 – 402.

[30] M. Mandelkern, On the uniform continuity of Tietze extension, *Arch. Math. (Basel)* **55** (4) (1990), 387 – 388.

[31] I. Mardones-Pérez, M.A. de Prada Vicente, Monotone insertion of lattice-valued functions, *Acta Math. Hungar.* **117** (1-2) (2007), 187 – 200.

[32] N.T. Nhu, On the extensions of uniformly continuous mappings, *Colloq. Math.* **41** (2) (1979), 243 – 252.

[33] N.T. Nhu, Extending metrics uniformly, *Colloq. Math.* **43** (1) (1980), 91 – 97.

[34] V. Novotný, "Extension of Mappings into Banach Spaces" (in Czech), Thesis Charlees Univ., Prague 2010.

[35] D. Preiss, J. Vilimovský, In-Between Theorems In Uniform Spaces, *Trans. Amer. Math. Soc.* **261** (2) (1980), 483 – 501.

[36] H.L. Shapiro, Extensions of pseudometrics, *Canad. J. Math.* **18** (1966), 981 – 998.

[37] A.H. Stone, Paracompactness and product spaces, *Bull. Amer. Math. Soc.* **54** (1948), 977 – 982.

[38] H. Tietze, Über Funktionen, die auf einer abgeschlossenen menge stetig sind, *J. Reine Angew. Math* **145** (1915), 9 – 14.

[39] H. Tietze, Bemerkungen zu den Sätzen von J.Pál über ebene und räumliche Jordansche Kurven, *J. Reine Angew. Math* **145** (1915), 15 – 25.

[40] H. Tong, Some characterizations of normal and perfectly normal spaces (announcement), *Bull. Amer. Math. Soc.* **54** (1948), 65.

[41] H. Tong, Some characterizations of normal and perfectly normal spaces, *Duke Math. J.* **19** (1952), 289 – 292.

[42] P.S. Urysohn, Über die Mächtigkeit der zusammen hängenden Mengen, *Math. Ann.* **94** (1) (1925), 262 – 295.

[43] D.L. Vallé-Poussin, Intégrales de Lebesgue, Fonctions d'Ensemble, Classes de Baire, Gauthier-Villars, Paris 1916.