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Compact Hausdorff Pseudoradial Spaces and their Pseudoradial Order*

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Abstract: It is proved that there are compact Hausdorff spaces of any pseudoradial order up to ω_0 included.

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1. Introduction

Given an ordinal γ and a set X, a transfinite sequence in X of lenght γ is a map $S : \gamma \longrightarrow X$. It is usually denoted $(x_\alpha)_{\alpha<\gamma}$. A transfinite sequence $(x_\alpha)_{\alpha<\gamma}$ in a topological space X *converges* to a point $x \in X$ (written $x_\alpha \to x$ or $\lim_{\alpha\to\gamma} x_\alpha = x$) provided that for each neighborhood U of x there is some $\overline{\alpha} < \gamma$ such that $\{x_\alpha \mid \overline{\alpha} \leq \alpha < \gamma\} \subseteq U$.

A topological space X is called *pseudoradial* (see [5], [1] or [3]) provided that for each $A \subseteq X$, if A is not closed, then there are a point $x \in \overline{A} \setminus A$ and a transfinite sequence $(x_\alpha)_{\alpha<\lambda}$ in A such that $x_\alpha \to x$.

Following [2] and [6], we define the *pseudoradial closure of A in X* as the set

$$\widehat{A} = \{x \in X \mid \text{there is a transfinite sequence } (x_\alpha)_{\alpha<\lambda} \text{ in } A \text{ converging to } x\}.$$

By transfinite recursion define

$$\begin{array}{lcll} \widehat{A}^{(0)} & = & A; & \\ \widehat{A}^{(\alpha+1)} & = & \widehat{\left(\widehat{A}^{(\alpha)}\right)} & \text{for every ordinal } \alpha; \\ \widehat{A}^{(\beta)} & = & \bigcup_{\alpha<\beta} A^{(\alpha)} & \text{if } \beta \text{ is a limit ordinal.} \end{array}$$

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The *pseudoradial order* of a pseudoradial space X is the least ordinal number α such that for each $A \subseteq X$,

$$\widehat{A}^{(\alpha)} = \overline{A}.$$

The pseudoradial order of a pseudoradial space X is denoted by $\mathrm{pro}(X)$.

In a previous article ([6]) we proved that there are normal (T_4+T_1) pseudoradial spaces and compact T_1 ones of pseudoradial order given by any ordinal number. Here we exhibit the construction of Hausdorff compact pseudoradial spaces of any pseudoradial order less than or equal to ω_0.

2. The main construction

For each natural number $n \geq 1$, we construct a compact pseudoradial Hausdorff space G_n such that $\mathrm{pro}(G_n) = n$.

For each $j = 0, \ldots, n-1$, let $x(j), y(j)$ be ordinal numbers. For the sake of convenience $x(j)$ could also assume the value -1, so that we can use the notation $(-1, y(j)]$ for denoting the segment of ordinals $[0, y(j)]$. Let $x = (x(0), \ldots, x(n-1))$, $y = (y(0), \ldots, y(n-1))$. We say that $x < y$ if and only if $x(j) < y(j)$ for each $j = 0, \ldots, n-1$. If $x < y$, let

$$C(x, y) = (x(0), y(0)] \times \cdots \times (x(n-1), y(n-1)]$$

be the n-dimensional cube with vertexes x, y, where each $(x(j), y(j)]$ has the order topology and $C(x, y)$ has the product topology. If $x = (-1, \ldots, -1)$, we denote $C(x, y)$ by $C(y)$. If $x \leq x' < y' \leq y$, $C(x', y')$ is both an open and a closed subspace of $C(x, y)$. For each $j = 0, \ldots, n-1$ we denote by

$$E_j = \{y(0)\} \times \cdots \times \{y(j-1)\} \times (x(j), y(j)] \times \{y(j+1)\} \times \cdots \times \{y(n-1)\}$$

the *j-th edge* of the cube $C(x, y)$ and

$$H_j = (x(0), y(0)] \times \cdots \times (x(j-1), y(j-1)] \times \{y(j)\} \times \\ \times (x(j+1), y(j+1)] \times \cdots \times (x(n-1), y(n-1)]$$

the *j-th hyperface* of the cube $C(x, y)$ (we are interested only in the edges and hyperfaces which y belongs to). Finally let us observe that if $z \in C(x, y)$, then $z < y$ if and only if $z \notin H_j$ for each $j = 0, \ldots, n-1$.

Let $G_n = [0, \omega_0] \times [0, \omega_1] \times \cdots \times [0, \omega_{n-1}]$. G_n is a T_2 compact space since it is product of T_2 compact spaces. It was proved in [4] that the product of two pseudoradial T_2 compact spaces is pseudoradial if one of them is radial (i.e. its pseudoradial order is 1). Since for each natural number k the segment of ordinals $[0, \omega_k]$ with the order topology is a compact T_2 radial space, it is easy to see that G_n is a pseudoradial space.

By the next three lemmas we prove that $\mathrm{pro}(G_n) \leq n$, i.e. that for each subspace A of G_n, $\widehat{A}^{(n)} = \overline{A}$.

Lemma 2.1. *As earlier, let n be a natural number, $n \geq 1$ and let x, y be two n-tuples of ordinals, $x < y$. Let A be a subspace of $C(x, y)$. Assume that for each $j = 0, \ldots, n-1$, $\widehat{A} \cap H_j = \emptyset$. Then $y \notin \overline{A}$.*

Proof. If $A = \emptyset$, the proof is trivial. Assume $A \neq \emptyset$. By transfinite recursion we determine an ordinal γ and a sequence $(z_\alpha)_{\alpha<\gamma}$ in A of lenght γ in the following way. Let $z_0 \in A$. Assume that we have defined $z_\alpha \in A$. Since for each $j = 0, \ldots, n-1$, $z_\alpha \notin H_j$, $z_\alpha < y$, so we can consider $C(z_\alpha, y)$. If $C(z_\alpha, y) \cap A = \emptyset$, let $\gamma = \alpha + 1$ and break the recursion. If not, choose $z_{\alpha+1} \in C(z_\alpha, y) \cap A$. Assume now that we have defined z_α for each $\alpha < \beta$, β a limit ordinal, and for each $j = 0, \ldots, n-1$, let $\widetilde{z_\beta}(j) = \sup\{z_\alpha(j) \mid \alpha < \beta\}$. Let $\widetilde{z_\beta} = (\widetilde{z_\beta}(0), \ldots, \widetilde{z_\beta}(n-1))$. It is easy to prove that $\widetilde{z_\beta} = \lim_{\alpha\to\beta} z_\alpha$; then $\widetilde{z_\beta} \in \widehat{A}$, so $\widetilde{z_\beta} \notin H_0 \cup \cdots \cup H_{n-1}$, and so $\widetilde{z_\beta} < y$. Thus we can consider $C(\widetilde{z_\beta}, y)$. If $C(\widetilde{z_\beta}, y) \cap A = \emptyset$, let $\gamma = \beta$ and break the recursion. If not, choose $z_\beta \in C(\widetilde{z_\beta}, y) \cap A$. Then

$$U = \begin{cases} C(z_{\gamma-1}, y) & \text{if } \gamma \text{ is a successor ordinal} \\ C(\widetilde{z_\gamma}, y) & \text{if } \gamma \text{ is a limit ordinal} \end{cases}$$

is a neighborhood of y in which there are no points of A, so $y \notin \overline{A}$. ∎

Lemma 2.2. *Let x, y be two n-tuples of ordinals. Let A be a subspace of $C(x, y)$. Assume that for each $j = 0, \ldots, n-1$, $\widehat{A}^{(n-1)} \cap E_j = \emptyset$. Then $y \notin \overline{A}$.*

Proof. By induction on n. If $n = 1$ the proof is trivial. If $n = 2$, then $E_0 = H_0$ and $E_1 = H_1$, so by Lemma 2.1 $y \notin \overline{A}$.

Now let $n \geq 3$ and assume that the lemma is proved for $n - 1$ and let us prove it for n. First let us observe that for each $j = 0, \ldots, n-1$, H_j is homeomorphic to an $(n-1)$-dimensional cube, whose edges are the E_k, $k \neq j$. Furthermore E_j, H_j are closed subspaces of $C(x, y)$ and so we can use the closure and pseudoradial closure operators in $C(x, y)$, in E_j and in H_j without ambiguity.

Now, for each $j = 0, \dots, n-1$, let $B_j = \widehat{A} \cap H_j$. First we prove that for each $j = 0, \dots, n-1$ and for each $k \neq j$, $\widehat{B_j}^{(n-2)} \cap E_k = \emptyset$. If not, $\emptyset \neq \widehat{B_j}^{(n-2)} \cap E_k = \widehat{(\widehat{A} \cap H_j)}^{(n-2)} \cap E_k \subseteq \widehat{A}^{(n-1)} \cap \widehat{H_j}^{(n-2)} \cap E_k = \widehat{A}^{(n-1)} \cap H_j \cap E_k$, but this contradicts the hypothesis. So for each $j = 0, \dots, n-1$ the hyperface H_j of $C(x,y)$ is homeomorphic to an $(n-1)$-dimensional hypercube such that in each of its edges there are no points of $\widehat{B_j}^{(n-2)}$. So by inductive assumption, $y \notin \overline{B_j}$. Thus for each $j = 0, \dots, n-1$, and for each $k \neq j$, there is an ordinal $w_j(k) < y(k)$ such that in

$$\begin{aligned}&(w_j(0), y(0)] \times \cdots \times (w_j(j-1), y(j-1)] \times \{y(j)\} \times \\ &\quad \times (w_j(j+1), y(j+1)] \times \cdots \times (w_j(n-1), y(n-1)]\end{aligned}$$

there are no points of $B_j = \widehat{A} \cap H_j$. Let

$$\begin{array}{lcll} w(0) & = & \max\{w_j(0) \mid j = 0, \dots, n-1\} & < y(0) \\ & \cdots & & \\ w(n-1) & = & \max\{w_j(n-1) \mid j = 0, \dots, n-1\} & < y(n-1) \end{array}$$

and let $w = (w(0), \dots, w(n-1))$. Thus $C(w,y)$ is an n-dimensional hypercube such that in each of its hyperfaces there are no points of $\widehat{A}$. So by Lemma 2.1 $y \notin \overline{A}$. ∎

LEMMA 2.3. *Let y be an n-tuple of ordinals. Let A be a subspace of $C(y)$ and $y \in \overline{A}$. Then $y \in \widehat{A}^{(n)}$.*

Proof. By contradiction assume that $y \notin \widehat{A}^{(n)}$. Then there is $x = (x(0), \dots, x(n-1))$ such that in each edge E_j of the cube $C(x,y)$ there are no points of $\widehat{A}^{(n-1)}$. By Lemma 2.2, $y \notin \overline{A \cap C(x,y)}$ and so $y \notin \overline{A}$. ∎

By the next lemma we prove that $\mathrm{pro}(G_n) \geq n$, i.e. that there is a subspace A of G_n such that $\widehat{A}^{(k)} \subsetneq \overline{A}$ for each $k = 0, \dots, n-1$.

LEMMA 2.4. *Let $A = [0, \omega_0) \times \cdots \times [0, \omega_{n-1}) \subseteq G_n$. Then for each $k = 0, \dots, n$,*

$$\widehat{A}^{(k)} = \{(x(0), \dots, x(n-1)) \mid x(j) = \omega_j \textit{ for at most } k \textit{ indices}\}.$$

Proof. By induction on k. For $k = 0$ the proof is trivial. Assume that the lemma is proved for $k-1$ and let us prove it for k.

"$\subseteq$" Let $x \in \widehat{A}^{(k)}$. Assume $x(j) = \omega_j$ for more than k indices. We can assume without restriction $x = (\omega_0, \dots, \omega_{k-1}, \omega_k, x(k+1), \dots, x(n-1))$. Since $x \in \widehat{A}^{(k)}$, there is a sequence $(x_\alpha)_{\alpha<\lambda}$ of lenght λ in $\widehat{A}^{(k-1)}$ such that $x_\alpha \to x$. First assume $\lambda \leq \omega_{k-1}$. Let $\overline{\gamma} = \sup\{x_\alpha(k) \mid \alpha < \lambda\}$. Since $\lambda \leq \omega_{k-1}$, then $\overline{\gamma}$ is strictly less than ω_k and so x_α cannot converge to x. Now assume $\lambda \geq \omega_k$. Let $h \in \{0, \dots, k-1\}$. Since $x_\alpha \to x$, for each $\gamma < \omega_h$ there is $\alpha(h,\gamma) < \lambda$ such that for each $\alpha > \alpha(h,\gamma)$, $x_\alpha(h) > \gamma$. Let $\overline{\alpha}_h = \sup\{\alpha(h,\gamma) \mid \gamma < \omega_h\}$ and $\overline{\alpha} = \max\{\alpha_h \mid h = 0, \dots, k-1\}$. Since $\lambda \geq \omega_k$, $\overline{\alpha}_h < \omega_k$ for each h and so $\overline{\alpha} < \omega_k$. Then for each $\alpha > \overline{\alpha}$, $x_\alpha(h) = \omega_h$ for each $h = 0, \dots, k-1$. Then by inductive assumption $x_\alpha \notin \widehat{A}^{(k-1)}$, a contradiction.

"$\supseteq$" Let $x = (x(0), \dots, x(n-1))$ such that $x(j) = \omega_j$ for at most k indices. If $x(j) = \omega_j$ for at most $k-1$ indices, by inductive assumption $x \in \widehat{A}^{(k-1)}$. So assume $x(j) = \omega_j$ for exactly k indices. We can assume without restriction that $x = (\omega_0, \dots, \omega_{k-1}, x(k), \dots, x(n-1))$ and $x(k) \neq \omega_k, \dots, x(n-1) \neq \omega_{n-1}$. For each $\alpha < \omega_{k-1}$, let $x_\alpha = (\omega_0, \dots, \omega_{k-2}, \alpha, x(k), \dots, x(n-1))$. By inductive assumption $x_\alpha \in \widehat{A}^{(k-1)}$. Clearly $x_\alpha \to x$ and so $x \in \widehat{A}^{(k)}$. ∎

Theorem 2.5. *$G_n = [0,\omega_0] \times [0,\omega_1] \times \cdots \times [0,\omega_{n-1}]$ is a compact pseudoradial Hausdorff space and* $\mathrm{pro}(G_n) = n$.

Proof. Clearly G_n is a T_2 compact space since it is product of T_2 compact spaces. We have already observed that G_n is a pseudoradial space. In order to prove that $\mathrm{pro}(G_n) = n$ it suffices to prove that:

(i) for each $A \subseteq G_n$, $\widehat{A}^{(n)} = \overline{A}$;

(ii) there exists $A \subseteq G_n$ such that for each $k < n$, $\widehat{A}^{(k)} \subsetneqq \overline{A}$.

Let us prove the first claim. Let $A \subseteq G_n$. Let $y \in \overline{A}$. Since $C(y)$ is both an open and a closed subspace of G_n, $x \in \overline{A \cap C(y)}$. Thus, by Lemma 2.3, $x \in \widehat{A \cap C(y)}^{(n)}$ and so $x \in \widehat{A}^{(n)}$.

Let us prove the second claim. Let A be as in Lemma 2.4 and let $x = (\omega_0, \dots, \omega_{n-1})$. Clearly $x \in \overline{A}$, but by Lemma 2.4, $x \notin \widehat{A}^{(k)}$, for each $k = 0, \dots, n-1$. ∎

3. A space of order ω_0

Let X be the disjoint topological sum of the spaces G_n, $n < \omega_0$, constructed in the previous section. Let G_ω be the one-point compactification of X, i.e. $G_\omega = X \cup \{\infty\}$.

Remark 3.1. Let us observe that:

(i) $\infty \notin X$;

(ii) a basic neighborhood of ∞ has the form $G_\omega \setminus K$, where K is a compact subspace of X;

(iii) if K is a compact subspace of X, then there is $n < \omega_0$ such that $K \subseteq \bigcup_{1 \le k \le n} G_k$.

Theorem 3.2. *G_ω is a compact Hausdorff pseudoradial space and its pseudoradial order is ω_0.*

Proof. Clearly G_ω is a compact Hausdorff space. In order to prove that G_ω is pseudoradial and $\mathrm{pro}(G_\omega) = \omega_0$ it suffices to prove that:

(i) for each $A \subseteq G_\omega$, $\widehat{A}^{(\omega_0)} = \overline{A}$;

(ii) for each $n < \omega_0$, there exists $A \subseteq G_\omega$ such that $\widehat{A}^{(n)} \subsetneqq \overline{A}$.

Let us prove the first claim. Let $A \subseteq G_\omega$ and let $x \in \overline{A} \setminus A$. If $x = \infty$, then for each $n < \omega_0$,

$$U_n = (G_\omega \setminus \bigcup_{1 \le k \le n} G_k)$$

is a neighborhood of ∞ and so there is $x_n \in A \cap U_n$. It follows immediately from Remark 3.1 that $x_n \to \infty$. So $\infty \in \widehat{A} \subseteq \widehat{A}^{(\omega_0)}$. If $x \neq \infty$, then there is $n < \omega_0$ such that $x \in G_n$. Since G_n is a compact open subspace of G_ω and $\mathrm{pro}(G_n) = n$, then $x \in \widehat{A}^{(n)} \subseteq \widehat{A}^{(\omega_0)}$.

The second claim is an easy consequence of the fact that for each $n < \omega_0$ the space G_n is a compact open subspace of G_ω and its pseudoradial order is n. ∎

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