

## Browder and Semi-Browder Operators and Perturbation Function

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*Abstract:* This paper is devoted to the investigation of the stability of closed densely defined semi-Browder and Browder operators on Banach spaces. Our approach consists to introduce the concepts of a perturbation function and a coperturbation function in order to deduce the stability under strictly singular and cosingular operator perturbations. Further, our results are used to show the invariance of Browder's spectrum.

*Key words:* Browder and semi-Browder operators, perturbation and coperturbation function, Browder's spectrum.

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### 1. INTRODUCTION

Let  $X$  be an infinite-dimensional Banach space. We denote by  $\mathcal{C}(X)$  (resp.  $\mathcal{L}(X)$ ) be the set of all closed densely defined linear operators on  $X$  (resp. the set of all bounded linear operators on  $X$ ). The subspace of all compact operators of  $\mathcal{L}(X)$  is designed by  $\mathcal{K}(X)$ . For  $T \in \mathcal{C}(X)$ , we write  $\mathcal{D}(T) \subset X$  for the domain,  $\mathcal{N}(T) = \{x \in \mathcal{D}(T) : Tx = 0\} \subset X$  for the null space and  $\mathcal{R}(T) \subset X$  for the range of  $T$ . The nullity,  $\alpha(T)$ , of  $T$  is defined as the dimension of  $\mathcal{N}(T)$  and the deficiency,  $\beta(T)$ , of  $T$  is defined as the codimension of  $\mathcal{R}(T)$  in  $X$ . The spectrum of  $T$  will be denoted by  $\sigma(T)$ . The resolvent set of  $T$ ,  $\rho(T)$ , is the complement of  $\sigma(T)$  in the complex plane. For a linear subspace  $M$  of  $X$  we denote by  $i_M$  the canonical injection of  $M$  into  $X$  and  $q_M$  the quotient map from  $X$  onto  $X/M$ . We write  $Ti_M$  for the restriction of  $T$  to  $M$ ; by the usual convention  $Ti_M = Ti_{(M \cap \mathcal{D}(T))}$ . The families of infinite dimensional, closed infinite codimensional subspaces of  $X$  are respectively denoted by  $I(X)$  and  $I_c(X)$ .

An operator  $T \in \mathcal{C}(X)$  is semi-Fredholm if  $\mathcal{R}(T)$  is closed and at least one of  $\alpha(T)$  and  $\beta(T)$  is finite. For such an operator we define an index  $i(T)$  by

$i(T) = \alpha(T) - \beta(T)$ . Let  $\Phi_+(X)$  (resp.  $\Phi_-(X)$ ) denote the set of upper (resp. lower) semi-Fredholm operators, i.e., the set of semi-Fredholm operators with  $\alpha(T) < \infty$  (resp.  $\beta(T) < \infty$ ). An operator  $T$  is Fredholm if it is both upper semi-Fredholm and lower semi-Fredholm. Let  $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$  denote the set of Fredholm operators on  $X$ .

To define ascent and descent we consider the case in which  $\mathcal{D}(T)$  and  $\mathcal{R}(T)$  are in the same linear space  $X$ . We can then define the iterates  $T^2, T^3, \dots$  of  $T$ . If  $n > 1$ ,  $\mathcal{D}(T^n)$  is the set  $\{x : x, Tx, \dots, T^{n-1}x \in \mathcal{D}(T)\}$ , and  $T^n x = T(T^{n-1}x)$ . We can then consider  $\mathcal{N}(T^n)$  and  $\mathcal{R}(T^n)$ . It is well known that  $\mathcal{N}(T^n) \subset \mathcal{N}(T^{n+1})$  and  $\mathcal{R}(T^{n+1}) \subset \mathcal{R}(T^n)$  for  $n \geq 0$ . We follow the convention that  $T^0 = I$  (the identity operator on  $X$ , with  $\mathcal{D}(I) = X$ ). Thus  $\mathcal{N}(T^0) = \{0\}$  and  $\mathcal{R}(T^0) = X$ . It is also well known that if  $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1})$ , then  $\mathcal{N}(T^n) = \mathcal{N}(T^k)$  when  $n \geq k$ . In this case the smallest nonnegative integer  $k$  such that  $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1})$  is called the ascent of  $T$ ; it is denoted by  $a(T)$ . If no such  $k$  exists we define  $a(T) = \infty$ . Similarly, if  $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$ , then  $\mathcal{R}(T^n) = \mathcal{R}(T^k)$  when  $n \geq k$ . If there is such a  $k$ , the smallest such  $k$  is called the descent of  $T$ , and denoted by  $d(T)$ . If no such integer exists, we shall say that  $T$  has infinite descent. For  $T \in \mathcal{C}(X)$  we define the generalized kernel of  $T$  by

$$\mathcal{N}^\infty(T) = \bigcup_{n=1}^{\infty} \mathcal{N}(T^n)$$

and the generalized range of  $T$  by

$$\mathcal{R}^\infty(T) = \bigcap_{n=1}^{\infty} \mathcal{R}(T^n).$$

If  $\mathcal{N}^\infty(T) = \mathcal{N}(T^k)$  for some  $k$ , then  $a(T) < \infty$  and the ascending sequence  $\mathcal{N}(T^n)$  terminates. If  $\mathcal{R}^\infty(T) = \mathcal{R}(T^k)$  for some  $k$ , then  $d(T) < \infty$  and the descending sequence  $\mathcal{R}(T^n)$  terminates.

An operator  $T \in \mathcal{C}(X)$  is called upper semi-Browder if  $T \in \Phi_+(X)$ ,  $i(T) \leq 0$  and  $a(T) < \infty$ .  $T$  is called lower semi-Browder if  $T \in \Phi_-(X)$ ,  $i(T) \geq 0$  and  $d(T) < \infty$ . Let  $\mathcal{B}_+(X)$  (resp.  $\mathcal{B}_-(X)$ ) denote the set of upper (resp. lower) semi-Browder operators. An operator  $T \in \mathcal{C}(X)$  is called Browder if it is both upper semi-Browder and lower semi-Browder, i.e.,  $T \in \Phi(X)$ ,  $i(T) = 0$ ,  $a(T) < \infty$  and  $d(T) < \infty$ . Let  $\mathcal{B}(X)$  the set of Browder operators, i.e.,  $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$ .

The set of lower (upper) semi-Browder operators and Browder operators define, respectively, the corresponding spectra, i.e., for  $T \in \mathcal{C}(X)$  set

$$\sigma_{db}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{B}_-(X)\},$$

$$\sigma_{ab}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{B}_+(X)\},$$

$$\sigma_{eb}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{B}(X)\}.$$

It is clear that  $\sigma_{eb}(T) = \sigma_{ab}(T) \cup \sigma_{db}(T)$ . The sets  $\sigma_{eb}(T)$ ,  $\sigma_{db}(T)$  and  $\sigma_{ab}(T)$  are called Browder's essential spectrum of  $T$ , Browder's essential defect spectrum of  $T$  and Browder's essential approximate point spectrum of  $T$ , respectively.

We denote by  $\Phi_+^b(X) = \Phi_+(X) \cap \mathcal{L}(X)$  (resp.  $\Phi_-^b(X) = \Phi_-(X) \cap \mathcal{L}(X)$ ) the set of upper (resp. lower) bounded semi-Fredholm operators and by  $\mathcal{B}_+^b(X) = \mathcal{B}_+(X) \cap \mathcal{L}(X)$  (resp.  $\mathcal{B}_-^b(X) = \mathcal{B}_-(X) \cap \mathcal{L}(X)$ ) the set of upper (resp. lower) bounded semi-Browder operators. Let  $\Phi^b(X) = \Phi_+^b(X) \cap \Phi_-^b(X)$  the set of bounded Fredholm operators on  $X$  and  $\mathcal{B}^b(X) = \mathcal{B}_+^b(X) \cap \mathcal{B}_-^b(X)$  the set of bounded Browder operators.

*Remark 1.1.* From [16, Chapter 3, Section 20, Corollary 11], we have

- (i) if  $T \in \Phi_+^b(X)$  and  $a(T) < \infty$ , then  $i(T) \leq 0$ ;
- (ii) if  $T \in \Phi_-^b(X)$  and  $d(T) < \infty$ , then  $i(T) \geq 0$ ;
- (iii) if  $T \in \Phi^b(X)$ ,  $a(T) < \infty$  and  $d(T) < \infty$ , then  $i(T) = 0$ .

The study of the problem of the stability of bounded semi-Browder operators under commuting operator perturbations was done by different authors [9, 17, 18]. In [9, Theorem 2], S. Grabiner proved that if  $T \in \Phi_-^b(X)$  (resp.  $\Phi_+^b(X)$ ) then  $d(V) < \infty$  (resp.  $a(V) < \infty$ ) if and only if  $d(T) < \infty$  (resp.  $a(T) < \infty$ ), where  $T - V = K \in \mathcal{K}(X)$  and  $TV = VT$ . The result of stability of bounded semi-Browder operators under commuting compact operator perturbations was improved by V. Rakočević in [18] for  $K \in R(X)$ , where  $R(X)$  designates the set of Riesz operators. Indeed, in [18, Corollary 2], V. Rakočević showed that if  $T \in \Phi_+^b(X)$  and  $a(T) < \infty$  (resp.  $T \in \Phi_-^b(X)$  and  $d(T) < \infty$ ), then  $T + K \in \Phi_+^b(X)$  and  $a(T + K) < \infty$  (resp.  $T + K \in \Phi_-^b(X)$  and  $d(T + K) < \infty$ ), where  $K$  satisfies  $TK = KT$ .

One of the main problems in studying the semi-Browder and Browder operators is the stability of the Browder's essential approximate point spectrum, Browder's essential defect spectrum and Browder's essential spectrum. In this way, V. Rakočević proved in [18, Theorem 7, Corollary 8] that:

(i) an operator  $K \in \mathcal{L}(X)$  satisfies

$$\sigma_{ab}(T + K) = \sigma_{ab}(T)$$

for all  $T \in \mathcal{L}(X)$  which commutes with  $K$  if and only if  $K \in R(X)$ ;

(ii) an operator  $K \in \mathcal{L}(X)$  satisfies

$$\sigma_{db}(T + K) = \sigma_{db}(T)$$

for all  $T \in \mathcal{L}(X)$  which commutes with  $K$  if and only if  $K \in R(X)$ ;

(iii) an operator  $K \in \mathcal{L}(X)$  satisfies

$$\sigma_{eb}(T + K) = \sigma_{eb}(T)$$

for all  $T \in \mathcal{L}(X)$  which commutes with  $K$  if and only if  $K \in R(X)$ .

The analysis of the stability of semi-Browder operators under compact operator perturbation, described above, was extended by F. Fakhfakh and M. Mnif in [6] to the case of closed densely defined linear operators. More precisely, F. Fakhfakh and M. Mnif proved in [6] under some assumptions that  $T \in \mathcal{B}_+(X)$  if and only if  $T + K \in \mathcal{B}_+(X)$  for all  $K \in \mathcal{K}(X)$ . On the other hand, in reflexive Banach space, we show that  $T \in \mathcal{B}_-(X)$  if and only if  $T + K \in \mathcal{B}_-(X)$  for all  $K \in \mathcal{K}(X)$  satisfying some hypotheses. Furthermore, as an application we infer  $\sigma_{ab}(T) = \sigma_{ab}(T + K)$  and  $\sigma_{db}(T) = \sigma_{db}(T + K)$  for all  $K \in \mathcal{K}(X)$  satisfying some hypotheses.

The purpose of this work is to pursue the investigation started in [6] and to extend it to general Banach spaces under less hypotheses. The study uses the concept of measure of non strict-singularity  $\Delta_\psi$  (see Definition 3.5) and non strict-cosingularity  $\Delta'_\varphi$  (see Definition 4.3) in order to deduce the stability under perturbations belonging to  $SS(X)$  and  $SC(X)$  (see Definition 3.1 and Definition 4.1).

By means of a perturbation function (see Definition 3.4) we show in Theorem 3.1 and Theorem 3.2 (see Section 3) under the hypotheses:

- (i)  $K$  commutes with  $T$  (see Definition 3.6),
- (ii) either  $\rho(T)$  or  $\rho(T + K) \neq \emptyset$ ,
- (iii)  $\Delta_\psi(K) < \Gamma_\psi(T)$  that

$$T \in \mathcal{B}_+(X) \quad \text{if and only if} \quad T + K \in \mathcal{B}_+(X)$$

and

$$T \in \mathcal{B}(X) \quad \text{if and only if} \quad T + K \in \mathcal{B}(X).$$

After that, using the properties of the functions  $\Delta_\psi$  and  $\Gamma_\psi$ , we derive the stability of upper semi-Browder operators under the class of strictly singular operator perturbations. This result generalizes Corollary 3.2 in [6]. Concerning the stability of lower semi-Browder operators, the analysis uses the concept of a coperturbation function. More precisely, let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$  and  $K \in \mathcal{L}(X)$ . Under the hypotheses:

- (i)  $K$  commutes with  $T$ ,
- (ii)  $\rho(T) \neq \emptyset$  and  $\rho(T + K) \neq \emptyset$ ,
- (iii)  $\Delta'_\varphi(K) < \Gamma'_\varphi(T)$ ,

we prove in Theorem 4.1 (see Section 4) that

$$T \in \mathcal{B}_-(X) \quad \text{if and only if} \quad T + K \in \mathcal{B}_-(X).$$

Next, using the properties of the functions  $\Delta'_\varphi$  and  $\Gamma'_\varphi$ , we derive the stability of lower semi-Browder operators by strictly cosingular operator perturbations.

In the last section of the paper we shall apply the results described above to study the invariance of the Browder's essential approximate point spectrum, Browder's essential spectrum and Browder's essential defect spectrum on Banach spaces.

The structure of this work is as follows. In Section 2, we establish some auxiliary results concerning the properties of lower semi-Fredholm operators and the relation between  $a(T)$  (resp.  $d(T)$ ) and  $d(T^*)$  (resp.  $a(T^*)$ ) for closed densely defined linear operators. In Section 3, we introduce the notion of a perturbation function to study the stability of Browder and upper semi-Browder operators. The main results of this section are Theorem 3.1 and Theorem 3.2. In Section 4, we use the concept of a coperturbation function to deduce the stability of lower semi-Browder operators under strictly cosingular operator perturbations. The main result of this section is Theorem 4.1. Finally, in Section 5 we apply the results of Section 3 and Section 4 to investigate the Browder's essential approximate point spectrum, Browder's essential spectrum and Browder's essential defect spectrum.

## 2. PRELIMINARIES RESULTS

Throughout this section our interest concentrates to gather some auxiliary results that we will need in the sequel.

We begin with the following result given in [19, Theorem 7.29] for closed densely defined linear operators. We remark that the hypothesis  $\overline{\mathcal{D}(T)} = X$  is unnecessary in [19, Theorem 7.29].

**LEMMA 2.1.** *Let  $X$  be a Banach space and  $T : \mathcal{D}(T) \subset X \rightarrow X$  a closed operator. If there is a seminorm  $|\cdot|$  defined on  $\mathcal{D}(T)$ , which is compact relative to the graph norm of  $T$  such that  $\|x\| \leq C\|Tx\| + |x|$  for all  $x \in \mathcal{D}(T)$ , then  $\alpha(T) < \infty$  and  $\mathcal{R}(T)$  is a closed subspace of  $X$ .*

*Proof.* Proceeding as the proof of [19, Theorem 7.29]. ■

The results of the two next lemmas were established in [19, Chapter 9, Theorem 9.41, Theorem 9.53 ] for bounded linear operators. Similar statements can be made for closed linear operators.

**LEMMA 2.2.** *Let  $X$  be a Banach space and  $T : \mathcal{D}(T) \subset X \rightarrow X$  a closed operator. Then,  $\alpha(T) < \infty$  and  $\mathcal{R}(T)$  is closed if and only if there are a projection  $P \in \mathcal{L}(X, X_T)$  with  $\dim \mathcal{R}(P) < \infty$ ,  $\mathcal{R}(P) \subset \mathcal{D}(T)$  and a constant  $C > 0$  such that*

$$d(x, \mathcal{R}(P)) \leq C\|Tx\| \quad \forall x \in \mathcal{N}(P) \cap \mathcal{D}(T). \quad (2.1)$$

*Proof.* Assume that  $\alpha(T) < \infty$  and  $\mathcal{R}(T)$  is closed. From [19, Chapter 5, Lemma 5.1, Lemma 5.2], there is a projection  $P \in \mathcal{L}(X)$  such that  $\mathcal{R}(P) = \mathcal{N}(T)$ . It is clear that  $P \in \mathcal{L}(X, X_T)$ , where  $X_T$  denotes the linear space  $\mathcal{D}(T)$  normed by  $\|x\|_T = \|x\| + \|Tx\|$  ( $x \in \mathcal{D}(T)$ ). Since  $\mathcal{R}(T)$  is closed, then by [19, Chapter 3, Theorem 3.14] there is a constant  $C$  such that  $d(x, \mathcal{N}(T)) \leq C\|Tx\|$  for all  $x \in \mathcal{D}(T)$ . Hence,

$$d(x, \mathcal{R}(P)) = d(x, \mathcal{N}(T)) \leq C\|Tx\| \quad \forall x \in \mathcal{N}(P) \cap \mathcal{D}(T).$$

Conversely, if (2.1) holds for some  $P \in \mathcal{L}(X, X_T)$ , then from [19, Chapter 9, Lemma 9.40] it follows that  $\|(I - P)x\| \leq cd((I - P)x, \mathcal{R}(P))$  where  $c$  is a constant. Let  $x \in \mathcal{D}(T)$ . It follows from the hypothesis that  $d((I - P)x, \mathcal{R}(P)) \leq C\|T(I - P)x\|$ . So,

$$\|x\| \leq C\|Tx\| + |x| \quad \forall x \in \mathcal{D}(T)$$

where  $|x| = \|Px\| + C\|TPx\| \leq C\|Px\|_T$ . Since  $P \in \mathcal{K}(X, X_T)$ , we can verify that  $|\cdot|$  is a seminorm defined on  $\mathcal{D}(T)$  compact relative to the graph norm of  $T$ . This implies by the use of Lemma 2.1 that  $\alpha(T) < \infty$  and  $\mathcal{R}(T)$  is a closed subspace of  $X$ . ■

LEMMA 2.3. *Let  $X$  be a Banach space and  $T \in \mathcal{C}(X)$ . If  $\mathcal{R}(T)$  is not closed, then there are sequences  $\{x_k\} \subset X$ ,  $\{x_k^*\} \subset \mathcal{D}(T^*)$ , such that*

$$x_j^*(x_k) = \delta_{jk}, \quad \|x_k^*\| = 1, \quad \|x_k\| \leq a_k, \quad \|x_k\| \|T^* x_k^*\| < \frac{1}{2^k}, \quad (2.2)$$

where the  $a_k$  are given by

$$a_1 = 2, \quad a_n = 2 \left( 1 + \sum_{k=1}^{n-1} a_k \right) \quad n = 2, 3, \dots$$

*Proof.* We proceed by induction. Since  $\mathcal{R}(T)$  is not closed, also  $\mathcal{R}(T^*)$  is not closed. Hence by [19, Chapter 3, Theorem 3.14], there exists  $x^* \in \mathcal{D}(T^*)$  such that

$$\|T^* x^*\| < \frac{1}{4} d(x^*, \mathcal{N}(T^*)) < \frac{1}{4} \|x^* - x_0^*\|$$

where  $x_0^* \in \mathcal{N}(T^*)$ . Let  $x_1^* = \frac{x^* - x_0^*}{\|x^* - x_0^*\|} \in \mathcal{D}(T^*)$ . It is easy to see that  $\|x_1^*\| = 1$  and  $\|T^* x_1^*\| < \frac{1}{4}$ . It remains to find  $x_1$  such that  $x_1^*(x_1) = 1$  and  $\|x_1\| \leq 2$ . Suppose that for all  $x \in X$  such that  $x_1^*(x) = 1$  we have  $\|x\| > 2$ . Let  $x \in X$  such that  $x_1^*(x) \neq 0$  and let  $x_0 = \frac{x}{x_1^*(x)}$ . It is clear that  $\|x_0\| > 2$  since  $x_1^*(x_0) = 1$ . So  $|x_1^*(x)| \leq \frac{1}{2} \|x\|$ . This contradicts the fact that  $\|x_1^*\| = 1$ . Now assume that  $x_1, x_2, \dots, x_{n-1}$  and  $x_1^*, x_2^*, \dots, x_{n-1}^*$  have been found satisfying equation (2.2). Define an operator  $P_n$  on  $X^*$  by

$$P_n x^* = \sum_{k=1}^{n-1} x^*(x_k) x_k^*.$$

Due to equation (2.2), it is easy to verify that  $P_n \in \mathcal{L}(X^*, X_{T^*}^*)$ . Since  $\mathcal{R}(T^*)$  is not closed, then by Lemma 2.2 there is  $x_n^* \in \mathcal{N}(P_n) \cap \mathcal{D}(T^*)$  such that

$$\|T^* x_n^*\| < \frac{d(x_n^*, \mathcal{R}(P_n))}{2^n a_n C_n},$$

where  $C_n$  is such that  $d(x^*, \mathcal{R}(P_n)) \leq \|x^* - P_n x^*\| \leq C_n \|x^*\|$ . Consequently,  $\|T^* x_n^*\| < \frac{\|x_n^*\|}{2^n a_n}$ . We can take  $\|x_n^*\| = 1$ . Let  $x \in X$  such that  $x_n^*(x) = 1$  and  $\|x\| \leq 2$ . Set

$$x_n = x - \sum_{k=1}^{n-1} x_k^*(x) x_k.$$

Therefore equation (2.2) holds for all  $n$ . ■

The following proposition is well known for bounded lower semi-Fredholm operators in [2, Theorem 4.4.10]. We will improve it for closed densely defined linear operators.

**PROPOSITION 2.1.** *Let  $X$  be a Banach space and  $T \in \mathcal{C}(X)$ . Then we have,  $T \in \Phi_-(X)$  if and only if  $\overline{\beta}(T - K) < \infty$  for all  $K \in \mathcal{K}(X)$ , where  $\overline{\beta}(T)$  denotes the codimension of  $\overline{\mathcal{R}(T)}$  in  $X$ .*

*Proof.* If  $T \in \Phi_-(X)$  and  $K \in \mathcal{K}(X)$ , then  $T - K \in \Phi_-(X)$  by [13, Chapter 4, Theorem 5.26]. In particular,  $\overline{\beta}(T - K) = \beta(T - K) < \infty$ . Suppose  $T \notin \Phi_-(X)$ . Then either  $\mathcal{R}(T)$  is closed and  $\beta(T) = \overline{\beta}(T) = \infty$  or  $\mathcal{R}(T)$  is not closed. In the first case taking  $K = 0$  we obtain  $\beta(T - K) = \overline{\beta}(T - K) = \infty$  and we are finished. So assume that  $\mathcal{R}(T)$  is not closed. Therefore, by Lemma 2.3 there are sequences  $\{x_k\}$  and  $\{x_k^*\}$  satisfying (2.2). We now define the finite rank operators

$$K_n(x) = \sum_{k=1}^n T^* x_k^*(x) x_k, \quad n = 1, 2, \dots$$

For  $n > m$  we obtain

$$\|K_n(x) - K_m(x)\| \leq \sum_{k=m+1}^n \|T^* x_k^*\| \|x\| \|x_k\| \leq \|x\| \sum_{k=m+1}^n 2^{-k}.$$

Therefore  $K_n$  converges to the compact operator

$$K(x) = \sum_{k=1}^{\infty} T^* x_k^*(x) x_k.$$

Now for each  $x \in \mathcal{D}(T)$  and each  $k$  we have  $x_k^*(K(x)) = T^* x_k^*(x) = x_k^*(Tx)$ . Consequently each of the  $x_k^*$  annihilates  $\overline{\mathcal{R}(T - K)}$ . Since the  $x_k^*$  are linearly independent, it follows that  $\overline{\beta}(T - K) = \infty$ . ■

It is well known for bounded semi-Fredholm operators that the ascent (resp. descent) of  $T$  equals the descent (resp. ascent) of  $T^*$ . The result holds true for closed densely defined lower semi-Fredholm operators.

We begin with the following useful result.

**LEMMA 2.4.** *Let  $X$  be a Banach space and  $T \in \mathcal{C}(X)$  such that  $\rho(T) \neq \emptyset$ . If  $T \in \Phi_-(X)$ , then  $T^k \in \Phi_-(X)$  for every  $k \in \mathbb{N}$ .*



*Proof.* Since  $\rho(T) \neq \emptyset$ , then by [5, Proposition 20, p.1181]  $T^k$  is a closed operator for every  $k \in \mathbb{N}$ . On the other hand, from [19, Chapter 7, Lemma 7.36], it follows that  $\overline{\mathcal{D}(T^k)} = X$  for all  $k \in \mathbb{N}$ . By [12, Lemma 544], we infer that  $\mathcal{R}(T^k)$  is a closed subspace of  $X$  and  $\beta(T^k) < \infty$  for all  $k \in \mathbb{N}$ . This completes the proof. ■

**PROPOSITION 2.2.** *Let  $X$  be a Banach space and  $T \in \mathcal{C}(X)$  such that  $\rho(T) \neq \emptyset$ . If  $T \in \Phi_-(X)$ , then*

- (i)  $a(T^*) = d(T)$ ,
- (ii)  $a(T) = d(T^*)$ .

*Proof.* (i) It is well known that

$$a(T^*) = \inf \left\{ k : \mathcal{N}(T^{*k}) = \mathcal{N}(T^{**k+1}) \right\}.$$

The use of Lemma 2.4 and [19, Chapter 7, Theorem 7.35] leads to

$$a(T^*) = \inf \left\{ k : \mathcal{N}((T^k)^*) = \mathcal{N}((T^{k+1})^*) \right\}. \tag{2.3}$$

Using the fact that  $T^k \in \Phi_-(X)$ , we deduce that

$$\mathcal{R}(T^k)^\perp = \mathcal{N}((T^k)^*). \tag{2.4}$$

Let  $p = a(T^*)$ . Due to equations (2.3) and (2.4), we obtain  $\mathcal{R}(T^p)^\perp = \mathcal{R}(T^{p+1})^\perp$ . So  $d(T) \leq a(T^*)$ . On the other hand, for  $k = d(T)$  we have  $\mathcal{R}(T^k)^\perp = \mathcal{R}(T^{k+1})^\perp$ . Equations (2.3) and (2.4) allowed us to write  $a(T^*) \leq d(T)$ . This ends the proof of (i).

(ii) Applying Lemma 2.4 together with [19, Chapter 7, Theorem 7.15, Theorem 7.35], we get

$$\mathcal{R}(T^{*k}) = \mathcal{R}((T^k)^*) = \mathcal{N}(T^k)^\perp. \tag{2.5}$$

For  $p = a(T)$  we have  $\mathcal{N}(T^p)^\perp = \mathcal{N}(T^{p+1})^\perp$ . Equation (2.5) gives  $d(T^*) \leq a(T)$ . On the other hand, from equation (2.5) we have  $\mathcal{N}(T^k)^\perp = \mathcal{N}(T^{k+1})^\perp$  where  $k = d(T^*)$ . So  $a(T) \leq d(T^*)$ . This completes the proof of (ii). ■

As a consequence of Proposition 2.2, we obtain the following results.

**COROLLARY 2.1.** *Let  $X$  be a Banach space and  $T \in \mathcal{C}(X)$ .*

- (i) If  $T \in \Phi_+(X)$  and  $a(T) < \infty$ , then  $i(T) \leq 0$ .
- (ii) Suppose moreover  $\rho(T) \neq \emptyset$ . If  $T \in \Phi_-(X)$  and  $d(T) < \infty$ , then  $i(T) \geq 0$ .
- (iii) Suppose moreover  $\rho(T) \neq \emptyset$ . If  $T \in \Phi(X)$ ,  $a(T) < \infty$  and  $d(T) < \infty$ , then we have  $i(T) = 0$ .

*Proof.* (i) The statement follows from [20, Theorem 4.2].

(ii) From [13, Chapter 4, Theorem 5.13] we have  $\alpha(T^*) = \beta(T) < \infty$ . On the other hand, by Proposition 2.2 we infer  $a(T^*) = d(T) < \infty$ . Hence applying [20, Theorem 4.2], we get  $\alpha(T^*) \leq \beta(T^*)$ . So  $i(T) = -i(T^*) \geq 0$ .

(iii) This assertion is obvious from (i) and (ii). ■

*Remark 2.1.* Corollary 2.1 allows us to write:

- (i)  $T \in \mathcal{B}_+(X)$  if and only if  $T \in \Phi_+(X)$  and  $a(T) < \infty$ .
- (ii) If  $\rho(T) \neq \emptyset$ , then  $T \in \mathcal{B}_-(X)$  if and only if  $T \in \Phi_-(X)$  and  $d(T) < \infty$ .
- (iii) If  $\rho(T) \neq \emptyset$ , then  $T \in \mathcal{B}(X)$  if and only if  $T \in \Phi(X)$ ,  $a(T) < \infty$  and  $d(T) < \infty$ .

### 3. STABILITY OF UPPER SEMI-BROWDER AND BROWDER OPERATORS

The purpose of this section is to introduce the concept of a perturbation function in order to deduce the stability of upper semi-Browder and Browder operators under strictly singular operator perturbations.

At the beginning of this section let us recall some useful definitions.

**DEFINITION 3.1.** ([19]) Let  $X$  be a Banach space and  $T \in \mathcal{L}(X)$ . We say that  $T$  is strictly singular if there is no  $M \in I(X)$  such that  $Ti_M$  has a continuous inverse.

Let  $SS(X)$  denote the set of strictly singular operators on  $X$ . It forms a closed two-sided ideal of  $\mathcal{L}(X)$  containing  $\mathcal{K}(X)$ .

**DEFINITION 3.2.** ([13]) Let  $X$  be a Banach space and  $T : \mathcal{D}(T) \subset X \rightarrow X$  a linear operator. We define the minimum modulus of  $T$  by

$$\gamma(T) = \sup \{ \alpha : \alpha d(x, \mathcal{N}(T)) \leq \|Tx\| \ \forall x \in \mathcal{D}(T) \},$$

where  $d(x, \mathcal{N}(T))$  is the distance of  $x$  from  $\mathcal{N}(T)$ .

DEFINITION 3.3. ([3]) Let  $X$  be a Banach space and  $T : \mathcal{D}(T) \subset X \rightarrow X$  a linear operator. We say that  $T$  is somewhere continuous if  $Ti_M$  is continuous for some infinite dimensional subspace  $M$  of  $\mathcal{D}(T)$ ; otherwise we call  $T$  nowhere continuous.

EXAMPLE 3.1. Let  $X$  be a Banach space,  $A \in \mathcal{L}(X)$  and  $B \in \mathcal{C}(X)$  such that  $\alpha(B) = \infty$ . It is clear that the closed densely defined linear operator  $T = A + B$  is somewhere continuous.

DEFINITION 3.4. ([1]) We define a perturbation function to be a function  $\psi$  assigning to each pair of normed spaces  $X, Y$  and a linear operator  $T : \mathcal{D}(T) \subset X \rightarrow Y$  a number  $\psi(T) \in [0, \infty]$ , verifying the following properties:

- (i)  $\psi(\lambda T) = |\lambda|\psi(T)$ , where  $\lambda \in \mathbb{C}$ ;
- (ii)  $\psi(T + S) = \psi(T)$ , for all  $S \in \mathcal{PK}(X, Y)$ , where  $\mathcal{PK}(X, Y)$  designates the set of precompact operators;
- (iii)  $\psi(T) \leq \|T\|$  whenever  $T$  is a bounded operator. Otherwise  $\|T\| = \infty$ ;
- (iv)  $\gamma(T) \leq \psi(T)$  whenever  $\dim \mathcal{D}(T) = \infty$  and  $\alpha(T) < \infty$ ;
- (v)  $\psi(Ti_M) \leq \psi(T)$ ,  $M \in I(\mathcal{D}(T))$ .

DEFINITION 3.5. ([1]) Given a perturbation function  $\psi$ . We define the functions  $\Gamma_\psi$  and  $\Delta_\psi$  as follows: for a linear operator  $T : \mathcal{D}(T) \subset X \rightarrow Y$  where  $X$  and  $Y$  denote two normed spaces, if  $\dim \mathcal{D}(T) < \infty$ , then  $\Gamma_\psi(T) = 0$ . If  $\dim \mathcal{D}(T) = \infty$ , then

$$\Gamma_\psi(T) = \inf \{ \psi(Ti_M) : M \in I(\mathcal{D}(T)) \}$$

and

$$\Delta_\psi(T) = \sup \{ \Gamma_\psi(Ti_M) : M \in I(\mathcal{D}(T)) \}.$$

$\Delta_\psi$  is said measure of non strict-singularity.

*Remark 3.1.* From the definitions given above, we see that there is an unbounded operator  $T$  satisfying  $\Gamma_\psi(T) < \infty$ . More precisely, if  $T$  is somewhere continuous, then  $\Gamma_\psi(T) < \infty$ . Then by Example 3.1, it is easy to conclude that the condition  $T \in \mathcal{C}(X)$  and  $\Gamma_\psi(T) < \infty$  does not imply that  $T \in \mathcal{L}(X)$ .

We start our analysis by the following lemmas needed in the proof of the main result of this section.

LEMMA 3.1. *Let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$  and  $K \in \mathcal{L}(X)$ . If  $\Delta_\psi(K) < \Gamma_\psi(T)$ , then  $T + K \in \Phi_+(X)$ ,  $T \in \Phi_+(X)$  and  $i(T + K) = i(T)$ .*

*Proof.* First we will prove that  $T + K \in \Phi_+(X)$ . Suppose this is not so. Then, by [4, Proposition 6] and [3, Corollary 3.7] there is  $S \in \mathcal{K}(X)$  such that  $\alpha(T + K - S) = \infty$ . Take  $M = \mathcal{N}(T + K - S) \subset \mathcal{D}(T)$ . Thus,  $(T + K)i_M = Si_M$ . Let  $N$  be a subspace  $\subset M$  such that  $\dim N = \infty$ . So  $(Ti_M)i_N = -(Ki_M)i_N + (Si_M)i_N$ . Using the properties of  $\psi$ , we get  $\psi(Ti_N) = \psi(Ki_N)$ . Hence,  $\Gamma_\psi(Ti_M) = \Gamma_\psi(Ki_M)$ . On the other hand, it is easy to show that  $\Gamma_\psi(T) \leq \Gamma_\psi(Ti_M)$ . Therefore,

$$\Gamma_\psi(T) \leq \Gamma_\psi(Ti_M) = \Gamma_\psi(Ki_M) \leq \Delta_\psi(K).$$

This contradiction shows that  $T + K \in \Phi_+(X)$ . Let  $0 \leq t_0 \leq 1$ . Then  $\Delta_\psi(t_0K) = t_0\Delta_\psi(K) < \Gamma_\psi(T)$  and so, by what we have just proved,  $T + t_0K \in \Phi_+(X)$ . Therefore,  $T \in \Phi_+(X)$ . In order to complete the proof we will check that the index is constant. Let  $t_0 \in [0, 1]$ . There is an  $\alpha > 0$  such that for all  $t \in [0, 1]$  satisfying  $|t - t_0| < \alpha$  we have by [13, Chapter 4, Theorem 5.22]

$$i(T + t_0K) = i(T + t_0K + (t - t_0)K) = i(T + tK).$$

From the Heine-Borel theorem, there is a finite number of sets which cover  $[0, 1]$ . Since each of these sets overlaps with at least one other and the index is constant on each one, we see that  $i(T + K) = i(T)$ . ■

DEFINITION 3.6. ([11]) Let  $X$  be a Banach space,  $T : \mathcal{D}(T) \subset X \rightarrow X$  and  $K : \mathcal{D}(K) \subset X \rightarrow X$  two linear operators. We say that  $K$  commutes with  $T$  if

- (i)  $\mathcal{D}(T) \subset \mathcal{D}(K)$ ,
- (ii)  $Kx \in \mathcal{D}(T)$  whenever  $x \in \mathcal{D}(T)$ ,
- (iii)  $KTx = TKx$  for  $x \in \mathcal{D}(T^2)$ .

LEMMA 3.2. ([11, LEMMA 1.1]) *Let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$  and  $K \in \mathcal{L}(X)$ . If  $K$  commutes with  $T$ , then  $K$  commutes with  $T + K$ .*

*Remark 3.2.* Let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$  and  $K \in \mathcal{L}(X)$ . If  $K$  commutes with  $T$ , then  $K$  commutes with  $T - \lambda$  for every  $\lambda \in \mathbb{C}$ .

LEMMA 3.3. *Let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$  and  $K \in \mathcal{L}(X)$ . If  $K$  commutes with  $T$  and either  $\rho(T)$  or  $\rho(T + K) \neq \emptyset$ , then  $KTx = TKx$  for all  $x \in \mathcal{D}(T)$ .*

*Proof.* First, we suppose  $\rho(T) \neq \emptyset$ . Let  $x \in \mathcal{D}(T)$  and  $\lambda \in \rho(T)$ . Since  $K : \mathcal{D}(T) \rightarrow \mathcal{D}(T)$ , then  $K(\lambda - T)^{-1}x \in \mathcal{D}(T)$ . Take  $y = (\lambda - T)^{-1}x$ . Thus,  $(\lambda - T)y = (\lambda - T)(\lambda - T)^{-1}x \in \mathcal{D}(T)$ . So  $y \in \mathcal{D}(T^2)$ . Using Remark 3.2, we get  $K(\lambda - T)y = (\lambda - T)Ky$ . Therefore,  $(\lambda - T)^{-1}Kx = K(\lambda - T)^{-1}x$ . By continuity, we infer that  $K(\lambda - T)^{-1}x = (\lambda - T)^{-1}Kx$  for all  $x \in X$ . Let  $u = (\lambda - T)^{-1}x \in \mathcal{D}(T)$ . So that,

$$\begin{aligned} (\lambda - T)Ku &= (\lambda - T)K(\lambda - T)^{-1}x \\ &= (\lambda - T)(\lambda - T)^{-1}Kx = Kx = K(\lambda - T)u \end{aligned}$$

which implies that  $KTu = TKu$  for all  $u \in \mathcal{D}(T)$ . Now, we assume  $\rho(T+K) \neq \emptyset$ . Hence, applying the reasoning above together with Lemma 3.2, we get  $K(T + K)x = (T + K)Kx$  for all  $x \in \mathcal{D}(T) = \mathcal{D}(T + K)$ . Thus,  $KTx = TKx$  for all  $x \in \mathcal{D}(T)$ . ■

We recall the following result owing to T.T. West.

LEMMA 3.4. ([21, PROPOSITION 1.6]) *Let  $X$  be a linear space and  $T : \mathcal{D}(T) \subset X \rightarrow X$  a linear operator. If  $\alpha(T) < \infty$ , then  $a(T) < \infty$  if and only if  $\mathcal{N}^\infty(T) \cap \mathcal{R}^\infty(T) = \{0\}$ .*

Now we are ready to state our first main result.

THEOREM 3.1. *Let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$  and  $K \in \mathcal{L}(X)$  such that either  $\rho(T)$  or  $\rho(T + K) \neq \emptyset$ . Assume that*

- (i)  $K$  commutes with  $T$ ,
- (ii)  $\Delta_\psi(K) < \Gamma_\psi(T)$ , where  $\psi$  is a perturbation function.

Then

$$T \in \mathcal{B}_+(X) \quad \text{if and only if} \quad T + K \in \mathcal{B}_+(X).$$

*Proof.* We first claim that if  $T \in \mathcal{B}_+(X)$ , then  $T + K \in \mathcal{B}_+(X)$ . Indeed, let  $T \in \mathcal{B}_+(X)$  thus  $T \in \Phi_+(X)$  and  $a(T) < \infty$ . From Lemma 3.1 we conclude that  $T + K \in \Phi_+(X)$ . Since  $T$  is a closed operator such that  $\alpha(T) < \infty$  and  $\mathcal{R}(T)$  is a closed subspace of  $X$ , then we can deduce from [7, Chapter 17,

Proposition 3.2], that  $T^p$  is a closed operator for all  $p \in \mathbb{N}$ . Therefore, by [13, Chapter 3, Problem 5.9]  $\mathcal{N}(T^p)$  is a closed subspace. Thus,  $\mathcal{N}^\infty(T) = \overline{\mathcal{N}^\infty(T)}$  since  $a(T) = p < \infty$ . Using Lemma 3.4 we infer that

$$\overline{\mathcal{N}^\infty(T)} \cap \mathcal{R}^\infty(T) = \mathcal{N}^\infty(T) \cap \mathcal{R}^\infty(T) = \{0\}.$$

Set  $T_\lambda = T + \lambda K$ , where  $\lambda \in [0, 1]$ . It is clear that  $T_\lambda \in \Phi_+(X)$  for each  $\lambda \in [0, 1]$ , since  $\Delta_\psi(\lambda K) = \lambda \Delta_\psi(K) < \Gamma_\psi(T)$ . Therefore, by Lemma 3.3 and [8, Theorem 3], there exists  $\varepsilon = \varepsilon(\lambda)$  such that

$$\overline{\mathcal{N}^\infty(T_\lambda)} \cap \mathcal{R}^\infty(T_\lambda) = \overline{\mathcal{N}^\infty(T_\mu)} \cap \mathcal{R}^\infty(T_\mu) \tag{3.1}$$

for all  $\mu$  in the open disc  $S(\lambda)$  with center  $\lambda$  and radius  $\varepsilon$ . Equation (3.1) proves that  $\overline{\mathcal{N}^\infty(T_\lambda)} \cap \mathcal{R}^\infty(T_\lambda)$  is a locally constant function of  $\lambda$  on the interval  $[0, 1]$ . Or every locally constant function on a connected set like  $[0, 1]$  is constant, then we conclude that  $\overline{\mathcal{N}^\infty(T + K)} \cap \mathcal{R}^\infty(T + K) = \{0\}$ . Thus,  $\mathcal{N}^\infty(T + K) \cap \mathcal{R}^\infty(T + K) = \{0\}$  and again by Lemma 3.4 it follows that  $a(T + K) < \infty$ . Conversely, let  $T + K \in \mathcal{B}_+(X)$ . Lemma 3.1 proves that  $T \in \Phi_+(X)$ . It remains to show that  $a(T) < \infty$ . To do this, we consider

$$(T + K)_\lambda = T + K + \lambda K = T + (\lambda + 1)K,$$

where  $\lambda \in [-1, 0]$  and we reasone in the same way as above. ■

As a consequence of Theorem 3.1, we obtain the stability of upper semi-Browder operators under strictly singular operator perturbations.

**COROLLARY 3.1.** *Let  $X$  be a Banach space and  $T \in \mathcal{C}(X)$  such that  $\rho(T) \neq \emptyset$ . Let  $K \in SS(X)$  such that  $K$  commutes with  $T$ . Then,*

$$T \in \mathcal{B}_+(X) \quad \text{if and only if} \quad T + K \in \mathcal{B}_+(X).$$

*Proof.* We first claim that if  $T \in \mathcal{B}_+(X)$ , then  $T + K \in \mathcal{B}_+(X)$ . If  $\dim \mathcal{D}(T) < \infty$ , then  $T \in SS(X)$ . This contradicts the hypothesis  $T \in \Phi_+(X)$ . So, we may assume  $\dim \mathcal{D}(T) = \infty$  throughout. From [1, Theorem 2.7 (i)-(ii)], it follows that  $\Gamma_\psi(T) > 0 = \Delta_\psi(K)$ . This implies by the use of Theorem 3.1 that  $T + K \in \mathcal{B}_+(X)$ . Conversely, by [1, Theorem 2.7 (i)-(ii)] we infer that

$$\Gamma_\psi(T + K) > 0 = \Delta_\psi(-K) = \Delta_\psi(K).$$

From Lemma 3.2 together with Theorem 3.1, we deduce that  $T + K - K = T \in \mathcal{B}_+(X)$ . ■

Now as a corollary, we get the following result which represents an improvement of [6, Corollary 3.2].

**COROLLARY 3.2.** *Let  $X$  be a Banach space and  $T \in \mathcal{C}(X)$  such that  $\rho(T) \neq \emptyset$ . Let  $K \in \mathcal{K}(X)$  such that  $K$  commutes with  $T$ . Then*

$$T \in \mathcal{B}_+(X) \quad \text{if and only if} \quad T + K \in \mathcal{B}_+(X).$$

*Proof.* Since  $\mathcal{K}(X) \subset SS(X)$ , the assertion follows from Corollary 3.1. ■

The second main result of this section concerns the stability of Browder operators.

**THEOREM 3.2.** *Let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$  and  $K \in \mathcal{L}(X)$  such that either  $\rho(T)$  or  $\rho(T + K) \neq \emptyset$ . Assume that*

- (i)  $K$  commutes with  $T$ ,
- (ii)  $\Delta_\psi(K) < \Gamma_\psi(T)$ , where  $\psi$  is a perturbation function.

Then,

$$T \in \mathcal{B}(X) \quad \text{if and only if} \quad T + K \in \mathcal{B}(X).$$

*Proof.* We first claim that if  $T \in \mathcal{B}(X)$ , then  $T + K \in \mathcal{B}(X)$ . Indeed, let  $T \in \mathcal{B}(X)$  thus  $T \in \Phi(X)$ ,  $i(T) = 0$ ,  $a(T) < \infty$  and  $d(T) < \infty$ . The use of Lemma 3.1 shows that  $T + K \in \Phi_+(X)$  and  $i(T + K) = i(T) = 0$ . So  $T + K \in \Phi(X)$  and  $i(T + K) = 0$ . By Theorem 3.1, we deduce that  $a(T + K) < \infty$ . From [10, Theorem 4.3], we have  $d(T + K) = a(T + K) < \infty$ . Conversely, the proof is very similar as above. ■

As an immediate consequence of Theorem 3.2 we have:

**COROLLARY 3.3.** *Let  $X$  be a Banach space and  $T \in \mathcal{C}(X)$  such that  $\rho(T) \neq \emptyset$ . Let  $K \in SS(X)$  such that  $K$  commutes with  $T$ . Then*

$$T \in \mathcal{B}(X) \quad \text{if and only if} \quad T + K \in \mathcal{B}(X).$$

#### 4. STABILITY OF LOWER SEMI-BROWDER OPERATORS

This section is devoted to introduce the notion of a coperturbation function in order to deduce the stability of lower semi-Browder operators by strictly cosingular operator perturbations.

Let us recall the following definitions.

DEFINITION 4.1. ([1]) Let  $X$  be a Banach space and  $T \in \mathcal{L}(X)$ . We say that  $T$  is strictly cosingular if there is no  $M \in I_c(X)$  such that  $(q_M T)^*$  has a continuous inverse.

Let  $SC(X)$  denote the set of strictly cosingular operators on  $X$ . It forms a closed two-sided ideal of  $\mathcal{L}(X)$  containing  $\mathcal{K}(X)$ .

DEFINITION 4.2. ([1]) A coperturbation function will be a function  $\varphi$  which determines, for each pair of normed spaces  $X, Y$  and a linear operator  $T : \mathcal{D}(T) \subset X \rightarrow Y$ , a number  $\varphi(T) \in [0, \infty]$  with the following properties:

- (i)  $\varphi(\lambda T) = |\lambda|\varphi(T)$ , where  $\lambda \in \mathbb{C}$ ;
- (ii)  $\varphi(T + S) = \varphi(T)$ , for all  $S \in \mathcal{PK}(X, Y)$ ;
- (iii)  $\varphi(T) \leq \|T\|$  whenever  $T$  is a bounded operator; otherwise  $\|T\| = \infty$ ;
- (iv)  $\gamma(T^*) \leq \varphi(T)$  whenever  $\dim \mathcal{D}(T^*) = \infty$  and  $\alpha(T^*) < \infty$ ;
- (v)  $\varphi(q_M T) \leq \varphi(T)$ ,  $M \in I_c(Y)$ .

DEFINITION 4.3. ([1]) Given a coperturbation function  $\varphi$ . We define the functions  $\Gamma'_\varphi$  and  $\Delta'_\varphi$  as follows: for a linear operator  $T : \mathcal{D}(T) \subset X \rightarrow Y$  where  $X$  and  $Y$  denote two Banach spaces,

$$\Gamma'_\varphi(T) = \inf \{ \varphi(q_M T) : M \in I_c(Y) \}$$

and

$$\Delta'_\varphi(T) = \sup \{ \Gamma'_\varphi(q_M T) : M \in I_c(Y) \}.$$

$\Delta'_\varphi$  is said measure of non strict-cosingularity.

*Remark 4.1.* We give two examples of an unbounded operator  $T$  satisfying  $\Gamma'_\varphi(T) < \infty$ :

(i) Let  $X, Y$  be two Banach spaces and a linear operator  $T : \mathcal{D}(T) \subset X \rightarrow Y$ . By [14, Theorem 4.1], if  $\mathcal{D}(T^*)$  contains an infinite dimensional  $\sigma(Y^*, Y)$  closed subspaces, then  $\Gamma'(T) := \inf_{M \in I_c(Y)} \|q_M T\| < \infty$ . And so,  $\Gamma'_\varphi(T) \leq \Gamma'(T) < \infty$  whenever  $\varphi$  is a coperturbation function.

(ii) Let  $X$  be a Banach space,  $A \in \mathcal{L}(X)$  and  $B \in \mathcal{C}(X)$  such that  $\mathcal{R}(B)$  is a closed subspace of  $X$  and  $\dim X/\mathcal{R}(B) = \infty$ . Clearly,  $T = A + B \in \mathcal{C}(X)$  and  $\Gamma'_\varphi(T) = \Gamma'_\varphi(A + B) < \infty$ . So the condition  $T \in \mathcal{C}(X)$  and  $\Gamma'_\varphi(T) < \infty$  does not imply that  $T \in \mathcal{L}(X)$ .

We start our study with the following useful lemmas.



LEMMA 4.1. *Let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$  and  $K \in \mathcal{L}(X)$ . If  $\Delta'_\varphi(K) < \Gamma'_\varphi(T)$ , then  $T + K \in \Phi_-(X)$ ,  $T \in \Phi_-(X)$  and  $i(T + K) = i(T)$ .*

*Proof.* Assume that  $T + K \notin \Phi_-(X)$ . By Proposition 2.1, it follows that there is  $S \in \mathcal{K}(X)$  such that  $\overline{\beta}(T + K - S) = \infty$ . Set  $M = \overline{\mathcal{R}(T + K - S)}$ . Thus  $q_M(T + K - S) = 0$ . Let  $N \in I_c(X/M)$ . From  $q_N q_M T = -q_N q_M K + q_N q_M S$  and the properties of  $\varphi$ , we get  $\varphi(q_N q_M T) = \varphi(q_N q_M K)$ . So  $\Gamma'_\varphi(q_M T) = \Gamma'_\varphi(q_M K)$ . On the other hand, [1, Lemma 3.5] gives  $\Gamma'_\varphi(T) \leq \Gamma'_\varphi(q_M T)$ . Hence

$$\Gamma'_\varphi(T) \leq \Gamma'_\varphi(q_M T) = \Gamma'_\varphi(q_M K) \leq \Delta'_\varphi(K),$$

this contradicts the hypothesis. Let  $0 \leq t_0 \leq 1$ . Then  $\Delta'_\varphi(t_0 K) < \Gamma'_\varphi(T)$  and so, by what we have just proved,  $T + t_0 K \in \Phi_-(X)$ . Thus  $T \in \Phi_-(X)$ . For the remainder of the proof it is very similar to the proof of Lemma 3.1. ■

LEMMA 4.2. *Let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$  such that  $\rho(T) \neq \emptyset$  and  $K \in \mathcal{L}(X)$ . If  $K$  commutes with  $T$ , then*

- (i)  $K^* : \mathcal{D}(T^*) \rightarrow \mathcal{D}(T^*)$ ,
- (ii)  $K^* T^* f = T^* K^* f$  for all  $f \in \mathcal{D}(T^*)$ ,
- (iii)  $K^*$  commutes with  $T^*$ .

*Proof.* (i) Let  $f \in \mathcal{D}(T^*) \subset \mathcal{D}(K^*)$  and  $u \in \mathcal{D}(T)$ . So  $K^* f \circ T(u) = f \circ K(T(u)) = f \circ K \circ T(u)$ . By Lemma 3.3 it follows that

$$K^* f \circ T(u) = f \circ T \circ K(u) = f \circ T(K(u)) = g \circ K(u)$$

where  $g = T^* f$ . From the fact that  $g = T^* f \in \mathcal{D}(K^*) = X^*$ , we infer that there exists  $h \in X^*$  such that  $K^* f \circ T(u) = g \circ K(u) = h(u)$  for all  $u \in \mathcal{D}(T)$ .

(ii) Let  $f \in \mathcal{D}(T^*) \subset \mathcal{D}(K^*)$  and  $u \in \mathcal{D}(T)$ . We denote by  $g = T^* f$ ,  $h = K^*(g)$ ,  $\tilde{g} = K^* f$  and  $\tilde{h} = T^*(\tilde{g})$ . Hence,  $h(u) = K^*(g)(u) = g \circ K(u) = f \circ T \circ K(u)$ . On the other hand,

$$\tilde{h}(u) = T^*(\tilde{g})(u) = \tilde{g} \circ T(u) = f \circ K \circ T(u) = f \circ T \circ K(u).$$

Thus  $h = \tilde{h}$ .

- (iii) This assertion is a consequence from (i) and (ii). ■

Now we are in the position to give the fundamental result of this section.

THEOREM 4.1. *Let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$  and  $K \in \mathcal{L}(X)$ . Assume that*

- (i)  $K$  commutes with  $T$ ,
- (ii)  $\rho(T) \neq \emptyset$  and  $\rho(T + K) \neq \emptyset$ ,
- (iii)  $\Delta'_\varphi(K) < \Gamma'_\varphi(T)$ , where  $\varphi$  is a coperturbation function.

Then,

$$T \in \mathcal{B}_-(X) \quad \text{if and only if} \quad T + K \in \mathcal{B}_-(X).$$

*Proof.* We first claim that if  $T \in \mathcal{B}_-(X)$ , then  $T + K \in \mathcal{B}_-(X)$ . Indeed, let  $T \in \mathcal{B}_-(X)$  thus  $T \in \Phi_-(X)$  and  $d(T) = p < \infty$ . By Lemma 4.1, we deduce that  $T + K \in \Phi_-(X)$ . From [13, Chapter 4, Theorem 5.13] it is easy to see that  $\mathcal{R}(T^*)$  is a closed subspace of  $X^*$  and  $\alpha(T^*) = \beta(T) < \infty$ . This together with [7, Chapter 17, Proposition 3.2] proves that  $T^{*p}$  is a closed operator. Thus,  $\mathcal{N}^\infty(T^*) = \overline{\mathcal{N}^\infty(T^*)}$  since by Proposition 2.2  $a(T^*) = d(T) = p < \infty$ . On the other hand, Lemma 3.4 shows

$$\mathcal{N}^\infty(T^*) \cap \mathcal{R}^\infty(T^*) = \overline{\mathcal{N}^\infty(T^*)} \cap \mathcal{R}^\infty(T^*) = \{0\}.$$

Set  $T_\lambda^* = (T + \lambda K)^* = T^* + \lambda K^*$ , where  $\lambda \in [0, 1]$ . Since  $\Delta'_\varphi(\lambda K) < \Gamma'_\varphi(T)$ ,  $T_\lambda = T + \lambda K \in \Phi_-(X)$  for each  $\lambda \in [0, 1]$ . By [13, Chapter 4, Theorem 5.13],  $\mathcal{R}(T_\lambda^*)$  is a closed subspace of  $X^*$  and  $\alpha(T_\lambda^*) = \beta(T_\lambda) < \infty$ . Now, using Lemma 4.2 (i)-(ii) together with [8, Theorem 3] and proceeding as in the proof of Theorem 3.1, we get  $a(T^* + K^*) = a((T + K)^*) < \infty$  since  $\alpha((T + K)^*) = \beta(T + K) < \infty$ . Then by Proposition 2.2 we infer  $d(T + K) < \infty$ . Conversely, let  $T + K \in \mathcal{B}_-(X)$ . By Lemma 4.1, it follows that  $T \in \Phi_-(X)$ . Now, from Proposition 2.2 it suffices to prove  $a(T^*) < \infty$ . To do this, we consider

$$(T + K)_\lambda^* = T^* + K^* + \lambda K^* = T^* + (\lambda + 1)K^*,$$

where  $\lambda \in [-1, 0]$  and we reason in the same way as above. ■

As a consequence we infer the stability of lower semi-Browder operators under strictly cosingular operator perturbations.

COROLLARY 4.1. *Let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$  and  $K \in SC(X)$ . Assume that  $K$  commutes with  $T$ ,  $\rho(T) \neq \emptyset$  and  $\rho(T + K) \neq \emptyset$ . Then,*

$$T \in \mathcal{B}_-(X) \quad \text{if and only if} \quad T + K \in \mathcal{B}_-(X).$$

*Proof.* Let  $T \in \mathcal{B}_-(X)$ . From [1, Theorem 3.7 (i)-(ii)], we deduce that  $\Gamma'_\varphi(T) > 0 = \Delta'_\varphi(K)$ . Hence applying Theorem 4.1, we get  $T + K \in \mathcal{B}_-(X)$ . Conversely, let  $T + K \in \mathcal{B}_-(X)$ . By [1, Theorem 3.7 (i)-(ii)], we obtain that

$$\Gamma'_\varphi(T + K) > 0 = \Delta'_\varphi(-K) = \Delta'_\varphi(K).$$

This implies by the use of Theorem 4.1 together with Lemma 3.2 that  $T + K - K = T \in \mathcal{B}_-(X)$ . ■

Further, as an application of Corollary 4.1, we have:

**COROLLARY 4.2.** *Let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$  and  $K \in \mathcal{K}(X)$ . Assume that  $K$  commutes with  $T$ ,  $\rho(T) \neq \emptyset$  and  $\rho(T + K) \neq \emptyset$ . Then,*

$$T \in \mathcal{B}_-(X) \quad \text{if and only if} \quad T + K \in \mathcal{B}_-(X).$$

This result represents an improvement of [6, Corollary 3.1] on Banach spaces.

*Remark 4.2.* The result of Theorem 3.2 remains valid if we replace the hypotheses either  $\rho(T)$  or  $\rho(T + K) \neq \emptyset$  by  $\rho(T)$  and  $\rho(T + K) \neq \emptyset$ ,  $\Delta_\psi(K)$  by  $\Delta'_\varphi(K)$  and  $\Gamma_\psi(T)$  by  $\Gamma'_\varphi(T)$ . We just replace in the proof [10, Theorem 4.3] by [10, Theorem 4.6] and [15, Lemma 1.1]). Therefore, Corollary 3.3 holds true if we replace  $SS(X)$  by  $SC(X)$ .

### 5. STABILITY OF BROWDER'S SPECTRUM

In this section, we apply the results obtained in Section 3 and Section 4 to investigate the invariance of the Browder's essential approximate point spectrum, Browder's essential spectrum and Browder's essential defect spectrum.

**THEOREM 5.1.** *Let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$ ,  $\psi$  a perturbation function and  $K \in \mathcal{L}(X)$ . Assume that  $K$  commutes with  $T$  and either  $\rho(T)$  or  $\rho(T + K) \neq \emptyset$ .*

(i) *If there exists  $\varepsilon \geq 0$  such that  $\Delta_\psi(K) < \Gamma_\psi(T - \mu)$  for all  $\mu \in \rho_{ab}(T)$ , with  $\text{dist}(\mu, \sigma_{ab}(T)) > \varepsilon$ , then*

$$\sigma_{ab}(T + K) \subset \sigma_{ab}(T) \cup \{\mu : \text{dist}(\mu, \sigma_{ab}(T)) \leq \varepsilon\}.$$

(ii) *If there exists  $\varepsilon \geq 0$  such that  $\Delta_\psi(K) < \Gamma_\psi(T - \mu)$  for all  $\mu \in \rho_{ab}(T + K)$ , with  $\text{dist}(\mu, \sigma_{ab}(T + K)) > \varepsilon$ , then*

$$\sigma_{ab}(T) \subset \sigma_{ab}(T + K) \cup \{\mu : \text{dist}(\mu, \sigma_{ab}(T + K)) \leq \varepsilon\}.$$

*Proof.* (i) If  $\lambda \notin \sigma_{ab}(T) \cup \{\mu : \text{dist}(\mu, \sigma_{ab}(T)) \leq \varepsilon\}$ , then  $T - \lambda \in \mathcal{B}_+(X)$ . Hence, applying Theorem 3.1 and Remark 3.2, we get  $T + K - \lambda \in \mathcal{B}_+(X)$ . Thus  $\lambda \notin \sigma_{ab}(T + K)$ .

(ii) The proof of this assertion may be checked in the same way as the proof of (i). ■

As a consequence we obtain

**COROLLARY 5.1.** *Let  $X$  be a Banach space and  $T \in \mathcal{C}(X)$  such that  $\rho(T) \neq \emptyset$ . Let  $K \in SS(X)$  such that  $K$  commutes with  $T$ . Then,*

$$\sigma_{ab}(T) = \sigma_{ab}(T + K).$$

*Proof.* By [1, Theorem 2.7 (i)-(ii)], it follows that for all  $\mu \in \rho_{ab}(T)$ ,  $\Gamma_\psi(T - \mu) > \Delta_\psi(K) = 0$ . The use of Theorem 5.1 (i) gives  $\sigma_{ab}(T + K) \subset \sigma_{ab}(T)$ . Similarly, by [1, Theorem 2.7 (i)-(ii)] we conclude that for all  $\mu \in \rho_{ab}(T + K)$ ,  $\Gamma_\psi(T + K - \mu) > \Delta_\psi(K) = \Delta_\psi(-K) = 0$ . Hence, applying Lemma 3.2 together with Theorem 5.1 (i), we get  $\sigma_{ab}(T + K - K) = \sigma_{ab}(T) \subset \sigma_{ab}(T + K)$ . ■

For the stability of Browder's essential spectrum, as for Browder's essential approximate point spectrum, by the use of Theorem 3.2 and Remark 3.2, we get

**THEOREM 5.2.** *Let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$ ,  $\psi$  a perturbation function and  $K \in \mathcal{L}(X)$ . Assume that  $K$  commutes with  $T$  and either  $\rho(T)$  or  $\rho(T + K) \neq \emptyset$ .*

(i) *If there exists  $\varepsilon \geq 0$  such that  $\Delta_\psi(K) < \Gamma_\psi(T - \mu)$  for all  $\mu \in \rho_{eb}(T)$ , with  $\text{dist}(\mu, \sigma_{eb}(T)) > \varepsilon$ , then*

$$\sigma_{eb}(T + K) \subset \sigma_{eb}(T) \cup \{\mu : \text{dist}(\mu, \sigma_{eb}(T)) \leq \varepsilon\}.$$

(ii) *If there exists  $\varepsilon \geq 0$  such that  $\Delta_\psi(K) < \Gamma_\psi(T - \mu)$  for all  $\mu \in \rho_{eb}(T + K)$ , with  $\text{dist}(\mu, \sigma_{eb}(T + K)) > \varepsilon$ , then*

$$\sigma_{eb}(T) \subset \sigma_{eb}(T + K) \cup \{\mu : \text{dist}(\mu, \sigma_{eb}(T + K)) \leq \varepsilon\}.$$

**COROLLARY 5.2.** *Let  $X$  be a Banach space and  $T \in \mathcal{C}(X)$  such that  $\rho(T) \neq \emptyset$ . Let  $K \in SS(X)$  such that  $K$  commutes with  $T$ . Then,*

$$\sigma_{eb}(T) = \sigma_{eb}(T + K).$$

*Remark 5.1.* The results of Theorem 5.2 remains valid if we replace the hypotheses either  $\rho(T)$  or  $\rho(T + K) \neq \emptyset$  by  $\rho(T)$  and  $\rho(T + K) \neq \emptyset$ ,  $\Delta_\psi$  by  $\Delta'_\varphi$  and  $\Gamma_\psi$  by  $\Gamma'_\varphi$ . Therefore, Corollary 5.2 holds true if we replace  $SS(X)$  by  $SC(X)$ .

Finally, we close this section by the stability of Browder’s essential defect spectrum.

**THEOREM 5.3.** *Let  $X$  be a Banach space and  $T \in \mathcal{C}(X)$  such that  $\rho(T) \neq \emptyset$ . Let  $\varphi$  be a coperturbation function and  $K \in \mathcal{L}(X)$  such that  $K$  commutes with  $T$  and  $\rho(T + K) \neq \emptyset$ .*

(i) *If there exists  $\varepsilon \geq 0$  such that  $\Delta'_\varphi(K) < \Gamma'_\varphi(T - \mu)$  for all  $\mu \in \rho_{db}(T)$ , with  $\text{dist}(\mu, \sigma_{db}(T)) > \varepsilon$ , then*

$$\sigma_{db}(T + K) \subset \sigma_{db}(T) \cup \{\mu : \text{dist}(\mu, \sigma_{db}(T)) \leq \varepsilon\}.$$

(ii) *If there exists  $\varepsilon \geq 0$  such that  $\Delta'_\varphi(K) < \Gamma'_\varphi(T - \mu)$  for all  $\mu \in \rho_{db}(T + K)$ , with  $\text{dist}(\mu, \sigma_{db}(T + K)) > \varepsilon$ , then*

$$\sigma_{db}(T) \subset \sigma_{db}(T + K) \cup \{\mu : \text{dist}(\mu, \sigma_{db}(T + K)) \leq \varepsilon\}.$$

*Proof.* The proof is as the proof of Theorem 5.1. We just replace Theorem 3.1 by Theorem 4.1. ■

**COROLLARY 5.3.** *Let  $X$  be a Banach space and  $T \in \mathcal{C}(X)$  such that  $\rho(T) \neq \emptyset$ . Let  $K \in SC(X)$  such that  $K$  commutes with  $T$  and  $\rho(T + K) \neq \emptyset$ . Then,*

$$\sigma_{db}(T) = \sigma_{db}(T + K).$$

*Proof.* From [1, Theorem 3.7 (i)-(ii)], we deduce that for all  $\mu \in \rho_{db}(T)$ ,  $\Gamma'_\varphi(T - \mu) > \Delta'_\varphi(K) = 0$ . Hence applying Theorem 5.3 (i), we get  $\sigma_{db}(T + K) \subset \sigma_{db}(T)$ . Similarly, by [1, Theorem 3.7 (i)-(ii)] we have

$$\Gamma'_\varphi(T + K - \mu) > \Delta'_\varphi(K) = \Delta'_\varphi(-K) = 0$$

for all  $\mu \in \rho_{db}(T + K)$ . The use, again, of Theorem 5.3 (i) and Lemma 3.2 allowed us to conclude that  $\sigma_{db}(T + K - K) = \sigma_{db}(T) \subset \sigma_{db}(T + K)$ . ■

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