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On Locally ϕ-Symmetric β-Kenmotsu Manifolds

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Abstract: The present paper deals with a study of locally ϕ-symmetric β-Kenmotsu manifold with the existence of β-Kenmotsu manifold by an interesting example.

Key words: Conformally flat, weakly locally ϕ-symmetric, strongly locally ϕ-symmetric, β-Kenmotsu, Einstein manifold, scalar curvature.

AMS *Subject Class.* (2000): 53C15, 53C25.

1. Introduction

In 1972 K. Kenmotsu [2] introduced a new class of almost contact Riemannian manifolds which are nowadays called Kenmotsu manifolds. It is well known that odd dimensional spheres admit Sasakian structures whereas odd dimensional hyperbolic spaces can not admit Sasakian structure, but have so-called Kenmotsu structure. Kenmotsu manifolds are normal (non-contact) almost contact Riemannian manifolds. Kenmotsu [2] investigated fundamental properties on local structure of such manifolds. Kenmotsu manifolds are locally isometric to warped product spaces with one dimensional base and Kähler fiber. As a generalization of both Sasakian and Kenmotsu manifolds, J. A. Oubiña [3] introduced the notion of trans-Sasakian manifolds, which are closely related to the locally conformal Kähler manifolds. A trans-Sasakian manifold of type (0,0), $(\alpha, 0)$ and $(0, \beta)$ are respectively called the cosympletic, α-Sasakian and β-Kenmotsu manifold, α, β being scalar functions. In particular, if $\alpha = 0, \beta = 1$; and $\alpha = 1, \beta = 0$ then a trans-Sasakian manifold will be a Kenmotsu and Sasakian manifold respectively. As β is a scalar function, β-Kenmotsu manifolds provide a large varieties of Kenmotsu manifolds.

The notion of locally ϕ-symmetric Sasakian manifolds was introduced by T. Takahashi [5]. In the context of Lorentzian geometry, the notions of weakly and strongly local ϕ-symmetry were introduced by Shaikh and Baishya [4] with several examples. The present paper deals with a study of conformally flat, weakly locally ϕ-symmetric and strongly locally ϕ-symmetric β-Kenmotsu manifolds. The class of weakly locally ϕ-symmetric β-Kenmotsu manifolds contains the class of strongly locally ϕ-symmetric β-Kenmotsu manifolds.

The paper is organised as follows. Section 2 is concerned with preliminaries and Section 3 is devoted to the study of conformally flat β-Kenmotsu manifolds. It is proved that a conformally flat β-Kenmotsu manifold is a generalized η-Einstein manifold. In Section 4, we investigate a necessary and sufficient condition for a β-Kenmotsu manifold to be of strongly locally ϕ-symmetric. It is shown that a strongly locally ϕ-symmetric 3-dimensional β-Kenmotsu manifold is an Einstein manifold. Section 5 consists of weakly locally ϕ-symmetric β-Kenmotsu manifolds and obtained a necessary and sufficient condition for a β-Kenmotsu manifold to be of weakly locally ϕ-symmetric and also in the last section the existence of β-Kenmotsu manifold is ensured by a non-trivial example.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be an *almost contact metric manifold* [1] if it admits a (1, 1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g which satisfy

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, Y) = -g(X, \phi Y), \qquad \eta(X) = g(X, \xi), \qquad \eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for all vector fields X, Y on M.

An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be *β-Kenmotsu manifold* if the following condition holds:

$$\nabla_X \xi = \beta\big(X - \eta(X)\xi\big) \tag{2.4}$$

and

$$(\nabla_X \phi)(Y) = \beta\big(g(\phi X, Y)\xi - \eta(Y)\phi X\big), \tag{2.5}$$

where ∇ denotes the Riemannian connection of g. If $\beta = 1$ then a β-Kenmotsu manifold is called *Kenmotsu manifold* and if β is constant then it is called *homothetic Kenmotsu manifold*.

In a β-Kenmotsu manifold [2], the following relations hold:

$$(\nabla_X \eta)(Y) = \beta\big(g(X,Y) - \eta(X)\eta(Y)\big), \tag{2.6}$$

$$\begin{aligned} R(X,Y)\xi = &-\beta^2\big(\eta(Y)X - \eta(X)Y\big) + (X\beta)\big(Y - \eta(Y)\xi\big) \\ &- (Y\beta)\big(X - \eta(X)\xi\big), \end{aligned} \tag{2.7}$$

$$R(\xi,X)Y = \big(\beta^2 + \xi\beta\big)\big(\eta(Y)X - g(X,Y)\xi\big), \tag{2.8}$$

$$\begin{aligned} \eta(R(X,Y)Z) = &\beta^2\big(\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\big) \\ &- (X\beta)\big(g(Y,Z) - \eta(Y)\eta(Z)\big) \\ &+ (Y\beta)\big(g(X,Z) - \eta(Z)\eta(X)\big), \end{aligned} \tag{2.9}$$

$$S(X,\xi) = -\big(2n\beta^2 + \xi\beta\big)\eta(X) - (2n-1)(X\beta), \tag{2.10}$$

$$S(\xi,\xi) = -\big(2n\beta^2 + \xi\beta\big), \tag{2.11}$$

for any vector field X, Y, Z on M and R is the Riemannian curvature tensor and S is the Ricci tensor of type (0, 2).

If the Ricci tensor of an almost contact Riemannian manifold M is of the form

$$S = ag + b\eta \otimes \eta \tag{2.12}$$

for some functions a and b on M, then M is said to be an *η-Einstein manifold.*

We now state and prove some basic results in a β-Kenmotsu manifold which will be frequently used later on.

LEMMA 2.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a β-Kenmotsu manifold. Then for X, Y, W the following relation holds:*

$$\begin{aligned} (\nabla_W R)(X,Y)\xi = &-2\beta(W\beta)\big(\eta(Y)X - \eta(X)Y\big) \\ &- \beta^3\big(g(Y,W)X - g(X,W)Y\big) - \beta R(X,Y)W \\ &+ \beta(X\beta)\big(-g(Y,W)\xi + \eta(Y)\eta(W)\xi \\ &\qquad - \eta(Y)W + \eta(W)Y\big) \\ &- \beta(Y\beta)\big(-g(X,W)\xi + \eta(X)\eta(W)\xi \\ &\qquad - \eta(X)W + \eta(W)X\big). \end{aligned} \tag{2.13}$$

Proof. By virtue of (2.4), (2.6) and (2.7) we can easily get (2.13). ∎

LEMMA 2.2. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a β-Kenmotsu manifold. Then the following relation holds:*

$$\begin{aligned}(\nabla_W R)(X,\xi)Z = {}&\big(2\beta(W\beta)+W(\xi\beta)\big)\big(g(X,Z)\xi-\eta(Z)X\big)\\&+\beta\big(\beta^2+\xi\beta\big)\big(g(X,Z)W-g(W,Z)X\big)\\&-\beta R(X,W)Z\end{aligned} \tag{2.14}$$

for any vector field X, Z, W on M.

Proof. The relation (2.14) follows from (2.6) and (2.8). ∎

LEMMA 2.3. *In a Riemannian manifold, for any vector field X, Y, Z, the following relation holds:*

$$g\big((\nabla_W R)(X,Y)Z,U\big) = -g\big((\nabla_W R)(X,Y)U,Z\big). \tag{2.15}$$

Proof. It is easy to prove (2.15) and hence we omit it. ∎

3. Conformally flat β-Kenmotsu manifolds

Let us consider a β-Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$ $(n>1)$, which is conformally flat. Then we have

$$\begin{aligned}R(X,Y)Z = {}&\frac{1}{2n-1}\big(S(Y,Z)X-S(X,Z)Y\\&+g(Y,Z)QX-g(X,Z)QY\big)\\&-\frac{r}{2n(2n-1)}\big(g(Y,Z)X-g(X,Z)Y\big),\end{aligned} \tag{3.1}$$

where Q is the Ricci operator, i.e., $g(QX,Y)=S(X,Y)$ and r is the scalar curvature of the manifold.

Setting $Z=\xi$ in (3.1) and using (2.7) and (2.10) we obtain

$$\begin{aligned}\eta(Y)QX-\eta(X)QY = {}&\Big(\beta^2+\xi\beta+\frac{r}{2n}\Big)\big(\eta(Y)X-\eta(X)Y\big)\\&-(2n-1)\big((X\beta)\eta(Y)-(Y\beta)\eta(X)\big)\xi.\end{aligned} \tag{3.2}$$

Again plugging $Y=\xi$ in (3.2) we obtain by virtue of (2.11) that

$$\begin{aligned}QX = {}&\Big(\beta^2+\xi\beta+\frac{r}{2n}\Big)X\\&-\Big((2n+1)\beta^2-(2n-3)\xi\beta+\frac{r}{2n}\Big)\eta(X)\xi\\&-(2n-1)\big((X\beta)\xi+\eta(X)\ \mathrm{grad}\beta\big),\end{aligned} \tag{3.3}$$

which can be written as

$$\begin{aligned} S(X,Y) = {} & \Big(\beta^2 + \xi\beta + \frac{r}{2n}\Big)\, g(X,Y) \\ & - (2n-1)\big((X\beta)\eta(Y) + (Y\beta)\eta(X)\big) \\ & - \Big((2n+1)\beta^2 - (2n-3)\xi\beta + \frac{r}{2n}\Big)\, \eta(X)\eta(Y). \end{aligned} \tag{3.4}$$

Generalizing the notion of η-Einstein manifold we define the notion of *generalized η-Einstein manifold* as follows:

Definition 3.1. An almost contact Riemannian manifold is said to be a generalized η-Einstein manifold if its Ricci tensor S of type (0, 2) is of the form

$$S = ag + b\eta \otimes \eta + c[\eta \otimes \omega + \omega \otimes \eta], \tag{3.5}$$

where ω is a 1-form defined by $\omega(X) = g(X, \rho)$ for all X such that ρ and ξ are mutually orthogonal to each other.

We now suppose that $\omega(X) = g(X, \rho) = (X\beta) = g(grad\beta, X)$ for all X. If ρ and ξ are orthogonal then we have $(\xi\beta) = 0$ and hence (3.4) takes the form (3.5), where

$$a = \Big(\frac{r}{2n} + \beta^2\Big), \; b = -\Big(\frac{r}{2n} + (2n+1)\beta^2\Big) \text{ and } c = -(2n-1).$$

This leads to the following:

Theorem 3.1. *A conformally flat β-Kenmotsu manifold (M^{2n+1}, g) $(n > 1)$ is a generalized η-Einstein manifold.*

Let us consider a β-Kenmotsu manifold (M^{2n+1}, g) $(n > 1)$ which is Ricci semi-symmetric, that is, it satisfies the relation $R(X,Y)\cdot S = 0$, where $R(X,Y)$ is considered as the derivation of the tensor algebra at each point of the manifold for tangent vectors X and Y. Therefore we get

$$S\big(R(X,Y)Z, U\big) + S\big(Z, R(X,Y)U\big) = 0. \tag{3.6}$$

Setting $X = U = \xi$ in (3.6) and using (2.8), (2.9), (2.10) and (2.11) we get

$$\begin{aligned} S(Y,Z) = {} & -(2n\beta^2 + \xi\beta)g(Y,Z) \\ & - (2n-1)\big(\eta(Y)(Z\beta) - \eta(Z)(Y\beta)\big). \end{aligned} \tag{3.7}$$

If $\omega(X) = g(X,\rho) = (X\beta) = g(grad\,\beta, X)$ for all X then (3.7) yields

$$\begin{aligned} S(Y,Z) = {}& -(2n\beta^2 + \xi\beta)g(Y,Z) \\ & - (2n-1)\big(\eta(Y)\omega(Z) - \eta(Z)\omega(Y)\big). \end{aligned} \tag{3.8}$$

From (3.8) it follows that a Ricci-semisymmetric β-Kenmotsu manifold is an Einstein manifold if and only if

$$\eta(Y)\omega(Z) = \eta(Z)\omega(Y), \tag{3.9}$$

that is the vector fields ξ and $\rho = grad\beta$ are codirectional. This leads to the following:

Theorem 3.2. *A Ricci-semisymmetric β-Kenmotsu manifold (M^{2n+1}, g) $(n > 1)$ is an Einstein manifold if and only if the structure vector field ξ and the scalar potential of the structure function β are co-directional.*

We now suppose that a conformally flat β-Kenmotsu manifold (M^{2n+1}, g) $(n > 1)$ is Ricci semi-symmetric. Then we get the relation (3.6).

Using (3.1) in (3.6) we get

$$\begin{aligned} & g(Y,Z)S(QX,U) - g(X,Z)S(QY,U) \\ & \quad + g(Y,U)S(QX,Z) - g(X,U)S(QY,Z) \\ & = \frac{r}{2n-1}\big(g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + g(Y,U)S(X,Z) \\ & \qquad - g(X,U)S(Y,Z)\big). \end{aligned} \tag{3.10}$$

Let λ be the eigenvalue of the endomorphism Q corresponding to an eigenvector X. Then $QX = \lambda X$, i.e., $S(X,U) = \lambda g(X,U)$ and hence

$$S(QX,U) = \lambda S(X,U). \tag{3.11}$$

By virtue of (3.11) it follows from (3.10) that

$$\begin{aligned} \Big(\lambda - \frac{r}{2n}\Big)&\big(g(Y,Z)S(X,U) - g(X,Z)S(Y,U) \\ & + g(Y,U)S(X,Z) - g(X,U)S(Y,Z)\big) = 0, \end{aligned}$$

which yields

$$\begin{aligned} & g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + g(Y,U)S(X,Z) \\ & - g(X,U)S(Y,Z) = 0, \end{aligned} \tag{3.12}$$

provided $\lambda \neq \frac{r}{2n}$.

Since a conformally flat β-Kenmotsu manifold (M^{2n+1}, g) $(n > 1)$ is generalized η-Einstein, therefore using (3.5) in (3.12) we obtain

$$\begin{aligned}
&b\Big(g(Y,Z)\eta(X)\eta(U) - g(X,Z)\eta(Y)\eta(U) + g(Y,U)\eta(X)\eta(Z)\\
&\quad - g(X,U)\eta(Y)\eta(Z)\Big)\\
&+ c\Big(g(Y,Z)\big(\eta(X)\omega(U) + \eta(U)\omega(X)\big)\\
&\quad - g(X,Z)\big(\eta(Y)\omega(U) + \eta(U)\omega(Y)\big)\\
&\quad + g(Y,U)\big(\eta(X)\omega(Z) + \eta(Z)\omega(X)\big)\\
&\quad - g(X,U)\big(\eta(Y)\omega(Z) + \eta(Z)\omega(Y)\big)\Big) = 0,
\end{aligned} \tag{3.13}$$

provided $\lambda \neq \frac{r}{2n}$. Setting $Z = \xi$ and $U = \rho$ we get

$$\Big(\frac{r}{2n} + (2n+1)\beta^2\Big)\big(\eta(X)\omega(Y) - \eta(Y)\omega(X)\big) = 0. \tag{3.14}$$

From (3.14) we get either $\beta^2 = -\frac{r}{2n(2n+1)}$ or

$$\eta(X)\omega(Y) = \eta(Y)\omega(X),$$

that is, the vector fields ξ and $\rho =$ gradβ are co-directional. Thus we can state the following:

THEOREM 3.3. *Let (M^{2n+1}, g) $(n > 1)$ be a conformally flat β-Kenmotsu and Ricci-semisymmetric manifold such that $\frac{r}{2n}$ is not an eigenvalue of the Ricci tensor and $\frac{r}{2n} + (2n+1)\beta^2 \neq 0$. Then the structure vector field ξ and the scalar potential of the structure function β are co-directional.*

4. Strongly locally ϕ-symmetric β-Kenmotsu Manifolds

DEFINITION 4.1. A β-Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be strongly locally ϕ-symmetric if the relation

$$\phi^2\big((\nabla_W R)(X,Y)Z\big) = 0 \tag{4.1}$$

holds for any vector X, Y, Z, W tangent to M.

In particular, if X, Y, Z, W are horizontal vector fields, then it is a weakly locally ϕ-symmetric β-Kenmotsu manifold.

Let us take a β-Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$ which is strongly locally ϕ-symmetric. Then for all $X,Y,Z,W\in T_pM$ we have from (4.1)

$$(\nabla_W R)(X,Y)Z=\eta\big((\nabla_W R)(X,Y)Z\big)\xi. \tag{4.2}$$

Using (2.15) we obtain from (4.2) that

$$(\nabla_W R)(X,Y)Z=-g\big((\nabla_W R)(X,Y)\xi,Z\big)\xi. \tag{4.3}$$

Hence from (2.13) and (4.3) we obtain

$$\begin{aligned}(\nabla_W R)(X,Y)Z=\beta\Big(&2(W\beta)\big(\eta(Y)g(X,Z)-\eta(X)g(Y,Z)\big)\\&+\beta^2\big(g(Y,W)g(X,Z)-g(X,W)g(Y,Z)\big)\\&-(X\beta)\big(\eta(Y)\eta(W)-g(Y,W)\big)\eta(Z)\\&+(Y\beta)\big(\eta(X)\eta(W)-g(X,W)\big)\eta(Z)\\&+g\big(R(X,Y)W,Z\big)\Big)\xi\end{aligned} \tag{4.4}$$

for any vector field X,Y,Z,W tangent to M. Thus in a strongly locally ϕ-symmetric β-Kenmotsu manifold the relation (4.4) holds. Conversely if in a β-Kenmotsu manifold the relation (4.4) holds, then applying ϕ^2 on both sides of (4.4) and using (2.1) we obtain the relation (4.1) and hence the manifold is strongly locally ϕ-symmetric. This leads to the following:

THEOREM 4.1. *A β-Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is strongly locally ϕ-symmetric if and only if the relation* (4.4) *holds.*

Replacing Z by ξ in (4.2) and then using (2.15), we have

$$(\nabla_W R)(X,Y)\xi=0 \tag{4.5}$$

for any vector field X,Y,Z,W on M. From (4.5) and (2.13) it follows that

$$\begin{aligned}R(X,Y)W=&-2(W\beta)\big(\eta(Y)X-\eta(X)Y\big)\\&-\beta^2\big(g(Y,W)X-g(X,W)Y\big)\\&+(X\beta)\big(\eta(Y)\eta(W)\xi-g(Y,W)\xi-\eta(Y)W+\eta(W)Y\big)\\&-(Y\beta)\big(\eta(X)\eta(W)\xi-g(X,W)\xi-\eta(X)W+\eta(W)X\big).\end{aligned} \tag{4.6}$$

Taking the inner product on both sides of (4.6) by U we obtain

$$\begin{aligned}\tilde{R}(X,Y,W,U) = &-2(W\beta)\big(\eta(Y)g(X,U)-\eta(X)g(Y,U)\big)\\ &-\beta^2\big(g(Y,W)g(X,U)-g(X,W)g(Y,U)\big)\\ &+(X\beta)\big(\eta(Y)\eta(W)\eta(U)-g(Y,W)\eta(U)\\ &\qquad -\eta(Y)g(W,U)+\eta(W)g(Y,U)\big)\\ &-(Y\beta)\big(\eta(X)\eta(W)\eta(U)-g(X,W)\eta(U)\\ &\qquad -\eta(X)g(W,U)+\eta(W)g(X,U)\big),\end{aligned} \tag{4.7}$$

where $\tilde{R}(X,Y,W,U) = g(R(X,Y)W,U)$ for all vector fields X,Y,W,U on M. Taking an orthonormal frame field at any point of the manifold and contracting over X and U in (4.7) we get

$$\begin{aligned}S(Y,W) = &-(2n\beta^2+\xi\beta)g(Y,W)+(\xi\beta)\eta(Y)\eta(W)\\ &-(4n+1)(W\beta)\eta(Y)-(2n-1)(Y\beta)\eta(W).\end{aligned} \tag{4.8}$$

Substituting Y by ξ in (4.8) and using (2.10) we obtain

$$(n+1)(W\beta) = -(n-1)(\xi\beta)\eta(W) \quad \text{for } n>1. \tag{4.9}$$

By virtue of (4.9), (4.8) yields

$$\begin{aligned}S(Y,W) = &-(2n\beta^2+\xi\beta)g(Y,W)+(2n-1)(\xi\beta)\eta(Y)\eta(W)\\ &-(2n-1)\big((W\beta)\eta(Y)+(Y\beta)\eta(W)\big).\end{aligned} \tag{4.10}$$

If $\omega(Y) = g(Y,\rho) = (Y\beta) = g(grad\,\beta, Y)$ for all Y and also ρ and ξ are orthogonal then we have $(\xi\beta)=0$ and hence (4.10) becomes

$$S(Y,W) = -2n\beta^2 g(Y,W)-(2n-1)\big(\eta(Y)\omega(W)+\eta(W)\omega(Y)\big). \tag{4.11}$$

Thus we can state the following:

THEOREM 4.2. *A strongly locally ϕ-symmetric β-Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$ $(n>1)$ is a special type of generalized η-Einstein manifold.*

If $\beta=1$ then (4.8) reduces to

$$S(Y,W) = -2ng(Y,W). \tag{4.12}$$

This leads to the following:

Corollary 4.1. *A strongly locally ϕ-symmetric Kenmotsu manifold is an Einstein manifold.*

If $n = 1$ then (4.9) yields $(W\beta) = 0$ for all W and hence (4.8) reduces to (4.12). Thus we can state the following:

Corollary 4.2. *A strongly locally ϕ-symmetric 3-dimensional β-Kenmotsu manifold is an Einstein manifold.*

Taking contraction over Y and W in (4.11) we obtain

$$r = -2n(2n+1)\beta^2. \tag{4.13}$$

This leads to the following:

Theorem 4.3. *In a strongly locally ϕ-symmetric β-Kenmotsu manifold the scalar curvature r is given by the relation* (4.13).

5. Weakly locally ϕ-symmetric β-Kenmotsu manifolds

Definition 5.1. A β-Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be weakly locally ϕ-symmetric if and only if

$$\phi^2\big((\nabla_W R)(X,Y)Z\big) = 0 \tag{5.1}$$

holds for any vector X, Y, Z, W orthogonal to ξ, that is for any horizontal vector field X, Y, Z, W.

We consider a β-Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$, which is weakly locally ϕ-symmetric. Then using (2.1) in (5.1) we have

$$(\nabla_W R)(X,Y)Z = \eta\big((\nabla_W R)(X,Y)Z\big)\xi \tag{5.2}$$

for any X, Y, Z, W orthogonal to ξ. In view of (2.15) it follows from (5.2) that

$$(\nabla_W R)(X,Y)Z = -g\big((\nabla_W R)(X,Y)\xi, Z\big)\xi. \tag{5.3}$$

Using (2.13) in (5.3) we obtain the relation

$$\begin{aligned}(\nabla_W R)(X,Y)Z = \beta^3\big(g(Y,W)g(X,Z) - g(X,W)g(Y,Z)\big)\xi \\ + \beta g\big(R(X,Y)W, Z\big)\xi\end{aligned} \tag{5.4}$$

for any horizontal vector field X, Y, Z, W orthogonal to ξ.

Also if in a β-Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$ the relation (5.4) holds, then applying ϕ^2 on both sides of (5.4) and using (2.1) we obtain the relation (5.1). Thus we can state the following:

THEOREM 5.1. *A β-Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is weakly locally ϕ-symmetric if and only if the relation* (5.4) *holds.*

We now suppose that a β-Kenmotsu manifold satisfies the relation

$$\phi^2\big((\nabla_W R)(X,Y)\xi\big) = 0 \tag{5.5}$$

for any horizontal vector fields X, Y, W.

Using (2.1) in (5.5) we obtain

$$(\nabla_W R)(X,Y)\xi = \eta\big((\nabla_W R)(X,Y)\xi\big)\xi. \tag{5.6}$$

In view of (2.15), (5.6) yields

$$(\nabla_W R)(X,Y)\xi = 0 \tag{5.7}$$

for any horizontal vector field X, Y, W. Also for any X, Y, Z, W orthogonal to ξ, the relation (2.13) reduces to

$$\begin{aligned}(\nabla_W R)(X,Y)\xi = -\beta^3\big(g(Y,W)X - g(X,W)Y\big) - \beta R(X,Y)W \\ - \beta(X\beta)g(Y,W)\xi + \beta(Y\beta)g(X,W)\xi.\end{aligned} \tag{5.8}$$

From (5.7) and (5.8), it follows that

$$\begin{aligned}R(X,Y)W = -\beta^2\big(g(Y,W)X - g(X,W)Y\big) \\ - (X\beta)g(Y,W)\xi + (Y\beta)g(X,W)\xi\end{aligned} \tag{5.9}$$

for any horizontal vector field X, Y, W.

Taking the inner product on both sides of (5.9) by U we obtain

$$\begin{aligned}\tilde{R}(X,Y,W,U) = -\beta^2\big(g(Y,W)g(X,U) - g(X,W)g(Y,U)\big) \\ - (X\beta)g(Y,W)\eta(U) + (Y\beta)g(X,W)\eta(U).\end{aligned} \tag{5.10}$$

Also since X, Y, W, U are orthogonal to ξ, therefore (5.10) yields

$$\tilde{R}(X,Y,W,U) = -\beta^2\{g(Y,W)g(X,U) - g(X,W)g(Y,U)\} \tag{5.11}$$

for any horizontal vector field X, Y, W, U. In view of Schur's theorem it can be shown that β is a constant. Thus we can state the following:

THEOREM 5.2. *If a weakly locally ϕ-symmetric β-Kenmotsu manifold satisfies the relation* (5.5) *then the manifold is of constant curvature* $-\beta^2$.

Corollary 5.1. *The weakly locally ϕ-symmetric Kenmotsu manifold satisfying the relation* (5.5) *is a manifold of constant curvature* -1.

We now consider a weakly locally ϕ-symmetric β-Kenmotsu manifold. Then the relation (5.4) holds for any horizontal vector field X, Y, Z, W.

Let X, Y, Z, W be arbitrary vector fields on T_pM. We now compute

$$(\nabla_{\phi^2 W} R)(\phi^2 X, \phi^2 Y)\phi^2 Z$$

in two different ways. Firstly from (5.4) it follows by virtue of (2.1) and (2.2) that

$$\begin{aligned}(\nabla_{\phi^2 W} R)(\phi^2 X, \phi^2 Y)\phi^2 Z = \beta^3\big(&g(Y,W)g(X,Z) - g(X,W)g(Y,Z)\\ &- g(Y,W)\eta(X)\eta(Z)\\ &- g(X,Z)\eta(Y)\eta(W)\\ &+ g(X,W)\eta(Y)\eta(Z)\\ &+ g(Y,Z)\eta(X)\eta(W)\big)\xi\\ + \beta g\big(&R(\phi^2 X, \phi^2 Y)\phi^2 W, \phi^2 Z\big)\xi.\end{aligned} \tag{5.12}$$

From (2.1), (2.2), (2.7) and (2.8) it follows that

$$\begin{aligned}R(\phi^2 X, \phi^2 Y)\phi^2 W = -R(X,Y)W&\\ + \eta(W)\Big(&-\beta^2\big(\eta(Y)X - \eta(X)Y\big)\\ &+ (X\beta)\big(Y - \eta(Y)\xi\big)\\ &- (Y\beta)\big(X - \eta(X)\xi\big)\Big)\\ + (\beta^2 + \xi\beta)&\big(\eta(Y)g(X,W) - \eta(X)g(Y,W)\big)\xi.\end{aligned} \tag{5.13}$$

By virtue of (5.13) we obtain from (5.12) that

$$\begin{aligned}(\nabla_{\phi^2 W} R)(\phi^2 X, \phi^2 Y)\phi^2 Z = \beta\Big(&\tilde{R}(X,Y,Z,W) - \eta(Z)\tilde{R}(X,Y,W,\xi)\\ &+ \big(\beta^2 g(Y,W) + \eta(W)(Y\beta)\big)\\ &\quad\big(g(X,Z) - \eta(X)\eta(Z)\big)\\ &- \big(\beta^2 g(X,W) + \eta(W)(X\beta)\big)\\ &\quad\big(g(Y,Z) - \eta(Y)\eta(Z)\big)\Big)\xi.\end{aligned} \tag{5.14}$$

Again from (2.1) and (2.2) it follows that

$$g(\phi^2 X, \xi) = g(\phi^2 Y, \xi) = g(\phi^2 Z, \xi) = 0$$

and hence $\phi^2X, \phi^2Y, \phi^2Z$ are horizontal vector fields on M. Then by virtue of (2.1) and (2.2) we have

$$\begin{aligned}(\nabla_{\phi^2W}R)(\phi^2X,\phi^2Y)\phi^2Z &= -(\nabla_W R)(\phi^2X,\phi^2Y)\phi^2Z\\ &\quad + \eta(W)(\nabla_\xi R)(\phi^2X,\phi^2Y)\phi^2Z.\end{aligned} \tag{5.15}$$

Also from (5.4) it follows that for any horizontal vector field X, Y, Z, we have

$$(\nabla_\xi R)(X,Y)Z = \beta^3\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\},$$

which implies that

$$(\nabla_\xi R)(\phi^2X,\phi^2Y)\phi^2Z = 0. \tag{5.16}$$

Using (5.16) in (5.15) we obtain

$$(\nabla_{\phi^2W}R)(\phi^2X,\phi^2Y)\phi^2Z = -(\nabla_W R)(\phi^2X,\phi^2Y)\phi^2Z. \tag{5.17}$$

In view of (2.1) and (2.2) we have

$$\begin{aligned}(\nabla_W R)(\phi^2X,\phi^2Y)\phi^2Z &= -(\nabla_W R)(X,Y)Z + \eta(Z)(\nabla_W R)(X,Y)\xi\\ &\quad + \eta(Y)(\nabla_W R)(X,\xi)Z\\ &\quad - \eta(Y)\eta(Z)(\nabla_W R)(X,\xi)\xi\\ &\quad + \eta(X)(\nabla_W R)(\xi,Y)Z\\ &\quad - \eta(X)\eta(Z)(\nabla_W R)(\xi,Y)\xi.\end{aligned} \tag{5.18}$$

By virtue of (2.13) and (2.14) it follows from (5.18) that

$$\begin{aligned}&(\nabla_W R)(\phi^2X,\phi^2Y)\phi^2Z\\ &= -(\nabla_W R)(X,Y)Z\\ &\quad - \beta\big(\eta(Z)R(X,Y)W + \eta(Y)R(X,W)Z - \eta(X)R(Y,W)Z\big)\\ &\quad + \big(2\beta(W\beta) + W(\xi\beta)\big)\big(\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\big)\xi\\ &\quad - 3\beta(W\beta)\eta(Z)\big(\eta(Y)X - \eta(X)Y\big)\\ &\quad - \beta(\beta^2 + \xi\beta)\Big(\eta(X)\big(g(Y,Z)W - g(W,Z)Y\big)\\ &\qquad\qquad + \eta(Y)\big(g(W,Z)X - g(X,Z)W\big)\\ &\qquad\qquad + \eta(Z)\eta(W)\big(\eta(X)Y - \eta(Y)X\big)\Big)\\ &\quad - \beta\eta(Z)\Big(\beta^2\big(g(Y,W)X - g(X,W)Y\\ &\qquad\qquad + \eta(W)\eta(Y)X - \eta(W)\eta(X)Y\big)\\ &\qquad\quad - (X\beta)\big(\eta(W)Y - g(Y,W)\xi\big)\\ &\qquad\quad + (Y\beta)\big(\eta(W)X - g(X,W)\xi\big)\Big).\end{aligned} \tag{5.19}$$

From (5.17) and (5.19) we have

$$
\begin{aligned}
&(\nabla_{\phi^2 W}R)(\phi^2X,\phi^2Y)\phi^2Z\\
&=(\nabla_W R)(X,Y)Z\\
&\quad+\beta\big(\eta(Z)R(X,Y)W+\eta(Y)R(X,W)Z-\eta(X)R(Y,W)Z\big)\\
&\quad-\big(2\beta(W\beta)+W(\xi\beta)\big)\big(\eta(Y)g(X,Z)-\eta(X)g(Y,Z)\big)\xi\\
&\quad+3\beta(W\beta)\eta(Z)\big(\eta(Y)X-\eta(X)Y\big)\\
&\quad+\beta(\beta^2+\xi\beta)\Big(\eta(X)\big(g(Y,Z)W-g(W,Z)Y\big)\\
&\qquad\qquad+\eta(Y)\big(g(W,Z)X-g(X,Z)W\big)\\
&\qquad\qquad+\eta(Z)\eta(W)\big(\eta(X)Y-\eta(Y)X\big)\Big)\\
&\quad+\beta\eta(Z)\Big(\beta^2\big(g(Y,W)X-g(X,W)Y+\eta(W)\eta(Y)X\\
&\qquad\qquad-\eta(W)\eta(X)Y\big)-(X\beta)\big(\eta(W)Y-g(Y,W)\xi\big)\\
&\qquad+(Y\beta)\big(\eta(W)X-g(X,W)\xi\big)\Big).
\end{aligned}
\tag{5.20}
$$

Comparing (5.14) and (5.20) we obtain

$$
\begin{aligned}
&(\nabla_W R)(X,Y)Z\\
&=\big(2\beta(W\beta)+W(\xi\beta)\big)\big(\eta(Y)g(X,Z)-\eta(X)g(Y,Z)\big)\xi\\
&\quad-3\beta(W\beta)\eta(Z)\big(\eta(Y)X-\eta(X)Y\big)\\
&\quad+\beta\Big(\tilde{R}(X,Y,Z,W)-\eta(Z)\tilde{R}(X,Y,W,\xi)\\
&\qquad+\big(\beta^2g(Y,W)+\eta(W)(Y\beta)\big)\big(g(X,Z)-\eta(X)\eta(Z)\big)\\
&\qquad-\big(\beta^2g(X,W)+\eta(W)(X\beta)\big)\big(g(Y,Z)-\eta(Y)\eta(Z)\big)\Big)\xi\\
&\quad-\beta\big(\eta(Z)R(X,Y)W+\eta(Y)R(X,W)Z-\eta(X)R(Y,W)Z\big)\\
&\quad-\beta(\beta^2+\xi\beta)\Big(\eta(X)\big(g(Y,Z)W-g(W,Z)Y\big)\\
&\qquad\qquad+\eta(Y)\big(g(W,Z)X-g(X,Z)W\big)\\
&\qquad\qquad+\eta(Z)\eta(W)\big(\eta(X)Y-\eta(Y)X\big)\Big)\\
&\quad-\beta\eta(Z)\Big(\beta^2\big(g(Y,W)X-g(X,W)Y+\eta(W)\eta(Y)X-\eta(W)\eta(X)Y\big)\\
&\qquad-(X\beta)\big(\eta(W)Y-g(Y,W)\xi\big)\\
&\qquad+(Y\beta)\big(\eta(W)X-g(X,W)\xi\big)\Big).
\end{aligned}
\tag{5.21}
$$

Thus in a weakly locally ϕ-symmetric β-Kenmotsu manifold, the relation (5.21) holds for any $X, Y, Z, W \in T_pM$.

Next, if the relation (5.21) holds in a β-Kenmotsu manifold, then for any horizontal vector field X, Y, Z, W we obtain the relation (5.4) and hence the manifold is weakly locally ϕ-symmetric. Thus we can state the following:

THEOREM 5.3. *A β-Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is weakly locally ϕ-symmetric if and only if the relation* (5.21) *holds for any vector fields X, Y, Z, W tangent to M.*

Taking the inner product on both sides of (5.4) by an arbitrary vector field U tangent to M, we obtain

$$\begin{aligned} g\big((\nabla_W R)(X,Y)Z, U\big) = \beta^3\big(g(Y,W)g(X,Z) - g(X,W)g(Y,Z)\big)\eta(U) \\ + \beta g\big(R(X,Y)W, Z\big)\eta(U). \end{aligned} \tag{5.22}$$

In a Riemannian manifold it is known that

$$(divR)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \tag{5.23}$$

for any vector field X, Y, Z on M, where "*div*" denotes the divergence.

Taking an orthonormal frame field at any point of the manifold and contracting (5.22) over U and W we get by virtue of (5.23) that

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = 0 \tag{5.24}$$

for any X, Y, Z orthogonal to ξ.

Again contracting (5.24) over Y and Z we obtain $dr(X) = 0$ for any X orthogonal to ξ. This leads to the following:

THEOREM 5.4. *In a weakly locally ϕ-symmetric β-Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$, the scalar curvature is constant along the orthogonal direction of the vector field ξ.*

6. EXAMPLE OF β-KENMOTSU MANIFOLD

EXAMPLE 6.1. We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be a linearly independent global frame on M given by

$$E_1 = z^2\frac{\partial}{\partial x}, \qquad E_2 = z^2\frac{\partial}{\partial y}, \qquad E_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0,$$
$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1.$$

Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$. Let ϕ be the (1, 1) tensor field defined by $\phi E_1 = -E_2$, $\phi E_2 = E_1$ and $\phi E_3 = 0$. Then using the linearity of ϕ and g we have $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Riemannian connection of g. Then we have

$$[E_1, E_2] = 0, \qquad [E_1, E_3] = -\frac{2}{z}E_1, \qquad [E_2, E_3] = -\frac{2}{z}E_2.$$

Using Koszul formula for the Riemannian metric g, we can easily calculate

$$\begin{aligned}
&\nabla_{E_1}E_1 = \frac{2}{z}E_3, && \nabla_{E_1}E_2 = 0, && \nabla_{E_1}E_3 = -\frac{2}{z}E_1,\\
&\nabla_{E_2}E_1 = 0, && \nabla_{E_2}E_2 = \frac{2}{z}E_3, && \nabla_{E_2}E_3 = -\frac{2}{z}E_2,\\
&\nabla_{E_3}E_1 = 0, && \nabla_{E_3}E_2 = 0, && \nabla_{E_3}E_3 = 0.
\end{aligned}$$

From the above it can be easily seen that (ϕ, ξ, η, g) is a β-Kenmotsu structure on M. Consequently $M^3(\phi, \xi, \eta, g)$ is a β-Kenmotsu manifold with $\beta = -\frac{2}{z}$.

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