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Unbounded Operators: Functional Calculus, Generation, Perturbations†

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Abstract: These lectures review the H^∞-functional calculus of sectorial operators and related classes of unbounded operators. Their theory is related to the well established theory of C_0-semigroups and cosine functions. In most cases the existence of a bounded H^∞-calculus is equivalent to certain quadratic estimates arising from harmonic analysis. In addition to describing the topic in general, the lectures include recent results in perturbation theory for functional calculus and for differentiable semigroups.

Key words: Sectorial operator, functional calculus, quadratic estimates, strip type, operator logarithm, semigroup, cosine function, perturbation, triangular operator.

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Introduction

The operators which arise naturally in models of the physical, biological or economic world are almost always unbounded operators, that is, they do not act continuously from one Banach space X to itself. For example, this applies to the operator of differentiation on functions of a single variable: if one takes X small enough that the operator is defined on the whole of X (for example, if X consists of C^1-functions, or X is a Sobolev space) then X is not invariant under differentiation.

Nevertheless mathematicians learn operator theory first in the context of bounded operators. Since the theory of unbounded operators is usually taught only at graduate level, many students learn about bounded operators in some detail but never encounter the general theory of unbounded operators. This arrangement may be anomalous but there are some good pedagogic reasons for it. Firstly, the theory of bounded operators is relatively neat, because one avoids the intricacies concerning the domains of unbounded operators.

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Secondly, there are various techniques for converting unbounded operators into bounded operators, and then one can apply the bounded theory.

The most basic way to turn an operator A into a bounded operator is to invert A, but often this does not solve the problem at hand. In applications, we usually have several operators involved, acting in space variables and time variables, and with derivatives of different orders; solving the model requires more than simple inversion of a single operator. Our hand is greatly strengthened if we can interpret the operator $f(A)$ for more complicated functions than the reciprocal $f(z) = z^{-1}$. Sometimes this process actually solves the problem for us, and sometimes it is a helpful step towards finding a solution.

Consider the abstract Cauchy problem

$$u'(t) = Au(t) \quad (t \geq 0), \qquad u(0) = x, \tag{0.1}$$

where $u : \mathbb{R}_+ \to X$, $x \in X$. In very simple examples it may be possible to give an explicit solution, but usually that is not possible. So the task is to find out information about the solutions from knowledge of A.

Abstractly and formally the solution of (0.1) should be

$$u(t) = \exp(tA)x.$$

Now we see that the relevant function of A is not a resolvent but an exponential. This raises some questions: What does $\exp(tA)$ mean? Is it a bounded operator? What knowledge of $\exp(tA)$ as a function of t can be inferred from knowledge of A?

Similarly the second-order problem

$$u''(t) = Au(t) \quad (t \geq 0), \qquad u(0) = x, u'(0) = y$$

should have the solution

$$u(t) = \cos\left(t\sqrt{-A}\right) x + \sin\left(t\sqrt{-A}\right) \left(\sqrt{-A}\right)^{-1} y.$$

Again, one needs to know whether these functions of A make sense, and what their properties are.

These questions are about functional calculus, concerning $f(A)$ for specific functions f. The theories of semigroups and groups (exponential functions), and cosine and sine functions, of operators were developed to answer these questions for those specific functions, and that occurred historically well before the introduction of general functional calculus. The latter has now found many other applications, and we describe it here, placing the theory of semigroups and groups within it.

The two most classical forms of functional calculus are as follows:

1. *Functional calculus of self-adjoint operators on Hilbert space* [19, Section XII.2]: $f(A)$ is defined by spectral theory when A is an (unbounded) self-adjoint operator and f is a bounded measurable function on $\mathbb{R}$ (or just on $\sigma(A)$). This calculus applies only to rather special operators and it is different in nature from the functional calculus of this article, although they agree whenever both make sense.

2. *Riesz-Dunford functional calculus of bounded operators* [19, Section VII.3]:
$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, A)\, d\lambda,$$
where f is holomorphic in a neighbourhood U of $\sigma(A)$ and Γ is a contour in U around $\sigma(A)$.

The functional calculus under consideration in this article is of Riesz-Dunford type, but extended to unbounded operators. Since Γ will not go around $\sigma(A)$ if $\sigma(A)$ is unbounded, it is necessary to make some assumptions on A. The first such functional calculus was defined by Bade [6] for operators with spectrum in a strip. However it was not until work of McIntosh [43] in the 1980s that the functional calculus for sectorial operators was introduced. With a view to parabolic differential equations, sectorial operators have become the standard context for the subject, but there are now several other classes of operators with similar functional calculi. Haase [25] has given an axiomatic approach to functional calculi which saves repetition of similar arguments in each different case. Some work has to be done in order to ensure that each functional calculus has reasonable analytic properties (the Convergence Lemma and Composition Rules) but then much follows more or less automatically. It is typical of these theories that there are some operators A for which $f(A)$ is not necessarily a bounded operator even if f is a bounded holomorphic function on a suitable domain Ω containing $\sigma(A)$, but many operators A have bounded H^∞-calculus in the sense that $f(A)$ is always a bounded operator for all $f \in H^\infty(\Omega)$.

In Section 2 we briefly describe the functional calculus for sectorial operators, as in [25]. In Section 3, we show how the theory of C_0-semigroups and groups, and cosine functions, fits within similar functional calculi for half-plane, strip-type and parabolic-type operators. The basic results about semigroups and groups go back to Hille, Yosida and Phillips in the 1950s (see

[30]), but our presentation of the material relies on the functional calculus which barely existed until the 1980s, and it is heavily influenced by the recent work of Haase [25], [26]. In Section 4, we describe some of the many results showing that certain types of perturbations of operators preserve certain properties of operators such as generation of semigroups or boundedness of the functional calculus.

In these lectures, we do not give any details of the numerous applications to differential equations, such as maximal regularity of abstract Cauchy problems which is discussed in detail in the survey article [40]. Our content overlaps to some extent with some other survey articles [1], [3], [10]. We do not reproduce any proofs which are easily available in books such as [4], [20], [25], [30], [49], but we give at least outlines of some proofs which are less easily available and not too technical, notably Iley's theorem on bounded perturbations of differentiable semigroups (Theorem 4.4) and Kalton's theorem on triangular perturbations of operators with bounded functional calculus (Theorem 4.5). Accordingly we have tried to give the most accessible reference for each main result. In many cases we have also given at least one of the main original references, but we have not systematically identified all the contributors to the theory.

1. Preliminaries

In this article, an *operator* A will be a linear operator $A : D(A) \to X$, where X is a complex Banach space and $D(A)$ is a dense subspace of X. The assumption that the domain $D(A)$ is dense in X is for simplicity of presentation—some results are true without it and some are not.

It is essential for analysis that A should be closed, i.e., the graph $G(A) := \{(x, Ax) : x \in D(A)\}$ is closed in $X \times X$. In practical examples where A is a differential operator, it may be hard to identify the appropriate domain $D(A)$ precisely. However one may be able to show that A is closable on some dense domain, and then one can take the closure of A without identifying the domain explicitly [34, Section 3.5]. Alternatively, A may be defined initially on an L^2-space by means of a quadratic form, and then on corresponding L^p-spaces [34, Chapter 6], [47]. Again it may be difficult to identify the precise domains.

We will now summarise some basic properties of closed operators—for further details see [34, Section 3.5]. An operator A is *invertible* if there exists $A^{-1} \in \mathcal{B}(X)$ (the space of all bounded linear operators on X) such that

- For each $x \in D(A)$, $A^{-1}Ax = x$, and
- For each $y \in X$, $A^{-1}y \in D(A)$ and $AA^{-1}y = y$.

Any such operator is closed. Conversely if A is closed and bijective from $D(A)$ to X, then $A^{-1} \in \mathcal{B}(X)$, by the Closed Graph Theorem.

For a complex number λ, the *resolvent* of A is $R(\lambda, A) := (\lambda I - A)^{-1}$ if this exists; the *resolvent set* $\rho(A)$ is the set of all $\lambda \in \mathbb{C}$ for which $R(\lambda, A)$ exists, and the *spectrum* is $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

The spectrum and resolvent of unbounded operators have similar properties to the case of bounded operators, except that $\sigma(A)$ may be empty or it may be unbounded. In particular, $\rho(A)$ is an open subset of $\mathbb{C}$ and $R(\cdot, A)$ is a holomorphic function from $\rho(A)$ to $\mathcal{B}(X)$.

More specifically, if $\lambda \in \rho(A)$ and $|\lambda - \mu| < \|R(\lambda, A)\|^{-1}$, then $\mu \in \rho(A)$ and

$$R(\mu, A) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda, A)^{n+1} \tag{1.1}$$

This series expansion is called the *Neumann series*. Note also that

$$\sigma(R(\lambda, A)) = \left\{ \frac{1}{\lambda - \mu} : \mu \in \sigma(A) \right\} \cup \{0\}$$

if A is unbounded.

Example 1.1.

1. *Multiplication operators.* Let $X = \ell^p$ $(1 \le p < \infty)$ or $X = c_0$, and (α_n) be a sequence in $\mathbb{C}$. Let
$$Ax = (\alpha_n x_n),$$
with $D(A)$ consisting of all $x \in X$ such that $Ax \in X$. Then
$$\sigma(A) = \overline{\{\alpha_n\}}, \quad R(\lambda, A)x = \left((\lambda - \alpha_n)^{-1} x_n\right).$$
Similarly, if $X = L^p(\Omega, \mu)$ for some measure space (Ω, μ) and $h : \Omega \to \mathbb{C}$ is measurable, one can define
$$(Af)(\omega) = h(\omega) f(\omega).$$
Then $\sigma(A)$ is the essential range of h.

2. *Differential operators.*

(a) Let $X = L^p(\mathbb{R})$ where $1 \le p < \infty$. Let $D(\mathcal{D})$ be the first-order Sobolev space $W^{1,p}(\mathbb{R})$, and $\mathcal{D}f = f'$. Then $\sigma(\mathcal{D}) = i\mathbb{R}$ [25, Section 8.4].

(b) Let Y be a Banach space and $X = L^p(0,1;Y)$ be the usual Lebesgue-Bochner space of functions $f : (0,1) \to Y$, where $1 \le p < \infty$ (see [4, Section 1.1]). Let $\mathcal{D}_Y$ be the derivative operator with domain

$$D(\mathcal{D}_Y) = W_0^{1,p}(0,1;Y) := \{f \in W^{1,p}(0,1;Y) : f(0) = 0\},$$

where $W^{1,p}(0,1;Y)$ is the vector-valued Sobolev space. Then $\sigma(\mathcal{D}_Y) = \emptyset$ [25, Section 8.5].

(c) Let $X = L^p(\Omega)$ where $1 \le p < \infty$ and Ω is an open subset of $\mathbb{R}^n$ (equipped with n-dimensional Lebesgue measure unless otherwise specified). Let A be the Laplacian Δ with domain chosen to incorporate appropriate boundary conditions. Then $\sigma(\Delta)$ depends on Ω and on the boundary conditions, but it is often independent of p. See [47] for a general treatment of second-order elliptic operators.

In most of Examples 1.1, the solution of (0.1) can be written down explicitly (see Examples 3.1). We emphasise that this is rarely the case.

2. Functional calculus of sectorial operators

We shall describe the functional calculus here for "sectorial" operators in a way which will generalise easily to other classes of operators considered in Section 3. The theory was initiated by McIntosh but the presentation here follows Haase [25, Chapter 2].

An operator A on X is *sectorial* if there exists $\theta \in (0,\pi)$ such that

(1) $\sigma(A) \subseteq \Sigma_\theta \cup \{0\}$, where $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg\lambda| < \theta\}$, and

(2) there exists c_θ such that $\|\lambda R(\lambda, A)\| \le c_\theta$ whenever $\lambda \in \mathbb{C} \setminus \Sigma_\theta, \lambda \ne 0$.

The *sectorial angle* $\theta_{sect}(A)$ of A is the infimum of the set of all such θ. Note that this infimum is never attained, by (1.1).

EXAMPLE 2.1.

1. A multiplication operator A is sectorial if it satisfies the condition (1), because
$$\|R(\lambda, A)\| = d(\lambda, \sigma(A))^{-1}.$$
In particular, any non-negative self-adjoint operator on a Hilbert space has sectorial angle 0.

2. (a) The operators $\mathcal{D}$ and $-\mathcal{D}$ on $L^p(\mathbb{R})$ $(1 \le p < \infty)$ are both sectorial of angle $\pi/2$.

   (b) $\mathcal{D}_Y$ is sectorial of angle $\pi/2$ on $L^p(0,1;Y)$, but $-\mathcal{D}_Y$ is not sectorial.

   (c) For Dirichlet, Neumann and various other boundary conditions, $-\Delta$ is sectorial of angle 0 on $L^p(\Omega)$; it is self-adjoint and non-negative when $p = 2$.

Let $H_0^\infty(\Sigma_\theta)$ be the set of all bounded holomorphic functions $f : \Sigma_\theta \to \mathbb{C}$ for which there exist constants c and $\varepsilon > 0$ such that
$$|f(z)| \le c\,\frac{|z|^\varepsilon}{1+|z|^{2\varepsilon}} \qquad (z \in \Sigma_\theta). \tag{2.1}$$

For $\mu \in \mathbb{C} \setminus \overline{\Sigma_\theta}$, let $e_\mu(z) = z(\mu - z)^{-2}$. Then $e_\mu \in H_0^\infty(\Sigma_\theta)$.

Let A be a sectorial operator, and let $\theta_{sect}(A) < \theta < \pi$ and $f \in H_0^\infty(\Sigma_\theta)$. Define
$$f(A) = \frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}} f(\lambda)R(\lambda, A)\,d\lambda, \tag{2.2}$$
where $\theta_{sect}(A) < \theta' < \theta$. It follows from (2.1) that this integral is absolutely convergent in $\mathcal{B}(X)$.

The definition (2.2) has the following properties which are fundamental for a functional calculus:

- $f(A)$ is independent of the choice of θ' (from Cauchy's Theorem);
- $(f+g)(A) = f(A) + g(A)$;
- $(f.g)(A) = f(A)g(A)$ (from Fubini's Theorem);
- Let $\mu \in \mathbb{C} \setminus \overline{\Sigma_\theta}$. Then $e_\mu(A) = AR(\mu, A)^2$.

In order to cover many applications, we would like to extend this to define $f(A)$ for all $f \in H^\infty(\Sigma_\theta)$ (the space of bounded holomorphic functions on Σ_θ) and perhaps for even more functions. To describe this process, we shall assume for simplicity that A is injective. There is no great loss of generality in doing so, because if A is any sectorial operator, $\widetilde{X}$ is the closure of the range of A and $\widetilde{A}$ is the restriction of A to $D(A) \cap \widetilde{X}$, then $\widetilde{A}$ is sectorial and injective. If X is reflexive, one also has that $X = \ker A \oplus \widetilde{X}$.

Let $f : \Sigma_\theta \to \mathbb{C}$ be holomorphic, and suppose that there exists $e \in H_0^\infty(\Sigma_\theta)$ such that $e.f \in H_0^\infty(\Sigma_\theta)$ and $e(A)$ is injective. Define an operator $f(A)$ on X by means of its graph $G(f(A))$: for $x, y \in X$,

$$(x, y) \in G(f(A)) \quad \text{if and only if} \quad (e.f)(A)x = e(A)y.$$

Note that $f(A)$ is not necessarily everywhere defined or bounded, but it is closed and independent of the choice of the *regulariser* e.

When $f \in H^\infty(\Sigma_\theta)$, we can take as regulariser $e = e_\mu$, for any $\mu \in \mathbb{C} \setminus \overline{\Sigma_\theta}$ (assuming that A is injective). So we have defined an operator $f(A)$ but it may be unbounded. As we would hope, $r_\mu(A) = R(\mu, A)$ when $r_\mu(z) = (\mu - z)^{-1}$.

We have also defined $f(A)$ for many unbounded functions. Using $(e_\mu)^n$ as regulariser for suitable $n \in \mathbb{N}$, we recover definitions of the following important operators:

- Fractional powers A^α for $\alpha > 0$ (originally defined in [37], [7]; see [25, Section 3.1], [42]),
- Imaginary powers A^{it} for $t \in \mathbb{R}$ (see [25, Section 3.5]),
- The logarithm $\log A$ (originally defined in [46]; see [25, Section 3.5]).

For a functional calculus to be useful it is essential that it should behave reasonably well for limits of sequences of functions. Thus the following result, originally from [43], is very important. The assumption that A has dense range is slightly stronger than our earlier assumption of injectivity in general, but they are equivalent in reflexive spaces. Again the assumption is of little significance because one can restrict A to the closure of its range.

Theorem 2.1. (Convergence Lemma) [25, Proposition 5.1.4]. *Let A be sectorial with dense range, and let $\theta \in (\theta_{sect}(A), \pi)$. Suppose that $f_n \in H^\infty(\Sigma_\theta)$, $f_n(A) \in \mathcal{B}(X)$ for each $n \in \mathbb{N}$, $\sup_n \|f_n\|_{H^\infty} < \infty$ and $\sup_n \|f_n(A)\| < \infty$, and $f_n(z) \to f(z)$, uniformly on compact subsets of Σ_θ, as $n \to \infty$. Then $f(A) \in \mathcal{B}(X)$ and $f_n(A) \to f(A)$ in the strong operator topology.*

Consequently, the following are equivalent:

(1) *There is a constant c such that $\|f(A)\| \le c\|f\|_\infty$ for all $f \in H_0^\infty(\Sigma_\theta)$.*

(2) *$f(A) \in \mathcal{B}(X)$ for all $f \in H^\infty(\Sigma_\theta)$.*

When these properties hold, the map $f \mapsto f(A)$ is an algebra homomorphism of $H^\infty(\Sigma_\theta)$ into $\mathcal{B}(X)$.

We say that A has *bounded H^∞-calculus* (*on a sector*) if there exist $\theta \in (\theta_{sect}(A), \pi)$ and c_θ such that $f(A) \in \mathcal{B}(X)$ and $\|f(A)\| \le c_\theta \|f\|_{H^\infty}$ for all $f \in H^\infty(\Sigma_\theta)$. The infimum of all such θ is the *H^∞-angle* $\theta_{H^\infty}(A)$ of A.

If A has dense range, then A has bounded H^∞-calculus if and only if either of the conditions (1) and (2) in Theorem 2.1 holds.

Example 2.2.

1. Any multiplication operator which is sectorial has bounded H^∞-calculus.

2. The derivative $\mathcal{D}$ has bounded H^∞-calculus (of angle $\pi/2$) on $L^p(\mathbb{R})$ if and only if $p \in (1, \infty)$. Furthermore, $\mathcal{D}_Y$ has bounded H^∞-calculus (of angle $\pi/2$) on $L^p(0,1;Y)$ if and only if $p \in (1,\infty)$ and Y is a UMD-space, i.e., the Hilbert transform is bounded on $L^2(\mathbb{R};Y)$. For example, this occurs if Y is an L^q-space for $1 < q < \infty$, but not for $q = 1$. See [25, Theorem 8.5.8], [28], [40, Examples 10.2].

3. Many classes of differential operators on L^p-spaces $(1 < p < \infty)$ have bounded H^∞-calculus. See [17], [25, Chapter 8], [40, Section 14].

4. Let X be a Hilbert space and A be an operator on X with numerical range contained in a sector Σ_θ where $\theta < \pi/2$, and assume that $A + I$ is surjective. (We call such operators *Kato-sectorial*. They are called *m-sectorial* operators in [34], and they coincide with operators defined by quadratic forms [34, Theorem VI.2.7].) Then A is sectorial with bounded H^∞-calculus, and $\theta_{H^\infty}(A) = \theta_{sect}(A) \le \theta$ [41]. Conversely, let A be a sectorial operator on a Hilbert space with bounded H^∞-calculus and $\theta_{H^\infty}(A) < \pi/2$. Then there is an equivalent scalar product on X, with respect to which A is Kato-sectorial [5]. See [25, Corollary 7.3.10], [40, Section 11].

5. There is a sectorial operator A on a Hilbert space, with $\theta_{sect}(A) < \pi/2$, which does not have bounded H^∞-calculus [44].

The following is an important characterisation of bounded H^∞-calculus. It is a special case of a result from [14, Theorems 4.2 and 4.4].

Theorem 2.2. [40, Theorem 12.2]. *Suppose that A is sectorial with dense range, and $\theta_{sect}(A) < \theta < \pi$. The following are equivalent:*

(1) *A has bounded H^∞-calculus on a sector;*

(2) *There exist $\theta \in (\theta_{sect}(A), \pi)$ and a constant c_θ such that*

$$\int_{\partial\Sigma_\theta} \left|\langle AR(\lambda, A)^2 x, x^*\rangle\right| \, |d\lambda| < c_\theta \|x\| \, \|x^*\| \tag{2.3}$$

for all $x \in X$ and $x^* \in X^*$.

Moreover, (2.3) *holds whenever $\theta > \theta_{H^\infty}(A)$ and it fails whenever $\theta < \theta_{H^\infty}(A)$.*

By the Uniform Boundedness Principle, it suffices that each integral in (2.3) is finite. The condition (2.3), and other similar equivalent conditions, are called "quadratic estimates" or "square-function estimates" in harmonic analysis. They are sometimes much easier to verify than the definition of bounded H^∞-calculus, because they involve only resolvents, and not arbitrary functions, of A. For an example of this, see the proof of Theorem 4.5.

3. Generation

Now we turn to the more classical subject of generation of C_0-semigroups of operators, corresponding to applying a function e^{tz} to an operator, and similar families arising from abstract Cauchy problems. This theory goes back to work of Hille [29], [30] in the 1940s, but we present it in terms of the functional calculus described in Section 2 and other similar calculi.

3.1. Bounded holomorphic semigroups Let A be sectorial with $\theta_{sect}(A) < \pi/2$. Take $\theta \in (\theta_{sect}(A), \pi/2)$. For $t > 0$, the function $f_t : z \mapsto e^{-tz} - 1$ belongs to $H_0^\infty(\Sigma_\theta)$. Hence $T(t) := \exp(-tA) = f_t(A) + I \in \mathcal{B}(X)$. Let $T(0) = I$.

The function $T : \mathbb{R}_+ \to \mathcal{B}(X)$ has the following properties:

(i) $T(s)T(t) = T(s+t)$ $(s, t \geq 0)$ and $T(0) = I$,

(ii) T is strongly continuous,

(iii) For $x, y \in X$,

$$(x, -y) \in G(A) \quad \text{if and only if} \quad \lim_{t \downarrow 0} \frac{1}{t}(T(t)x - x) = y,$$

(iv) T extends to a holomorphic function on $\Sigma_{\theta'}$ where $\theta' = \pi/2 - \theta_{sect}(A)$, and T is bounded on Σ_ϕ for each $\phi \in (0, \theta')$.

By definition, the first two properties say that $T := \{T(t) : t \geq 0\}$ is a C_0-*semigroup*; the third says that $-A$ is the *generator* of T; the fourth that T is a *bounded holomorphic* semigroup. Conversely, if $-A$ generates a bounded holomorphic semigroup, then A is sectorial with $\theta_{sect}(A) < \pi/2$. Thus bounded holomorphic C_0-semigroups T correspond exactly to sectorial operators A with $\theta_{sect}(A) < \pi/2$. For further information, see [4, Section 3.7], [20, Section I.4a], [49, Section 2.5].

3.2. C_0-SEMIGROUPS AND HALF-PLANE OPERATORS Let T be an arbitrary C_0-semigroup with generator B so that (3.1) and (3.1) of Subsection 3.1 hold, with B replacing $-A$ in (3.1). In addition,

(v) There exist constants ω and M such that $\|T(t)\| \leq Me^{\omega t}$ for all $t > 0$ and
$$R(\lambda, B)x = \int_0^\infty e^{-\lambda t} T(t)x\, dt \qquad (x \in X, \operatorname{Re}\lambda > \omega). \tag{3.1}$$

(vi) $u(t) = T(t)x$ is the unique solution of the Cauchy problem
$$u'(t) = Bu(t)\ (t \geq 0), \quad u(0) = x,$$
in the classical sense if $x \in D(B)$ and in a mild sense if $x \in X$.

See [4, Section 3.1], [20, Chapter 2], [49, Chapter 1] for further information.

EXAMPLE 3.1.

1. A multiplication operator generates a C_0-semigroup if and only if its spectrum is contained in a left half-plane.
2. For $1 \leq p < \infty$, $\mathcal{D}$ generates the left-shift C_0-semigroup on $L^p(\mathbb{R})$, given by $(T(t)f)(s)f(s+t)$, and $-\mathcal{D}$ generates the corresponding right-shift C_0-semigroup. Similarly, $-\mathcal{D}_Y$ generates the right-shift C_0-semigroup on the space $L^p(0,1;Y)$.

In these examples, the C_0-semigroups can be written down explicitly, but that is unusual. In typical applications, there is no hope of giving the semigroup explicitly, and one tries to show that a given operator generates a C_0-semigroup by using criteria discussed in this section and in Section 4.

Numerous situations in which C_0-semigroups can be introduced are described in [20, Chapter VI].

An operator B is said to be of *strong half-plane type* if there exist constants ω and M_ω such that

$$\sigma(B) \subseteq L_\omega := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le \omega\}, \quad \|R(\lambda, B)\| \le \frac{M_\omega}{\operatorname{Re}\lambda - \omega} \quad (\operatorname{Re}\lambda > \omega).$$

The infimum of all such ω is the *strong half-plane type* $\omega_{shp}(B)$ of B.

It follows easily from (3.1) that the generator B of a C_0-semigroup is of strong half-plane type. However the converse is not true [20, Exercise II.3.12].

Let B be a strong half-plane type operator and $\omega > \omega_{shp}(B)$. Then $\omega I - B$ is sectorial of angle $\pi/2$. Using the functional calculus for sectorial operators described in Section 2, we can define $g(\omega I - B)$ when $g \in H^\infty(\Sigma_\theta)$ for some $\theta \in (0, \pi/2)$. If f is bounded and holomorphic on the reversed sector $\{z \in \mathbb{C} : |\arg(\omega - z)| > \theta\}$ of angle $\pi - \theta > \pi/2$, then we can define $f(B) = g(\omega I - B)$, where $g(z) = f(\omega - z)$. However a C_0-semigroup generated by B should correspond to $\exp(tB)$, and the function $f(z) = e^{tz}$ is not bounded on such a sector. On the other hand, it is bounded on every left half-plane. So, instead of using the sectorial functional calculus, a parallel notion of half-plane functional calculus is used.

The functional calculus for operators of strong half-plane type is constructed in a similar way to sectorial operators (see Section 2). Sectors Σ_θ are replaced by half-planes L_ω where $\omega > \omega_{shp}(B)$. Let $H_0^\infty(L_\omega)$ be the space of all holomorphic functions f on L_ω such that

$$|f(z)| \le \frac{c}{1 + |\operatorname{Im} z|^{1+\varepsilon}}$$

for some c and $\varepsilon > 0$. Then define

$$f(B) = \frac{1}{2\pi i} \int_{\partial L_{\omega'}} f(\lambda) R(\lambda, B)\, d\lambda, \tag{3.2}$$

where $\omega_{shp}(B) < \omega' < \omega$. One extends this definition to a larger class of functions exactly as for sectorial operators. In particular, $f(B)$ is defined as a closed operator for all $f \in H^\infty(L_\omega)$, using the function $r_\mu^2(z) = (\mu - z)^{-2}$ as regulariser where $\operatorname{Re}\mu > \omega$, so that

$$f(B)x = y \quad \text{if and only if} \quad (r_\mu^2.f)(B)x = R(\mu, B)^2 y.$$

This functional calculus has the same basic properties as the sectorial case, and in particular the Convergence Lemma (Theorem 2.1) holds for all operators of strong half-plane type. Thus one can define the notion of an operator of strong half-plane type having bounded H^∞-functional calculus in the same way as for sectorial operators. However there does not seem to be a natural analogue of Theorem 2.2.

Let B be an operator of strong half-plane type. Since the exponential function is bounded and holomorphic on each left half-plane, the functional calculus defines $\exp(tB)$ as a closed operator for any $t \geq 0$. Let $\operatorname{Re}\mu > \omega > \omega_{shp}(B)$ and $t \geq 0$. The function $z \mapsto e^{tz}/(\mu - z)^2$ belongs to $H_0^\infty(L_\omega)$. Using it as regulariser, we find that $D(B^2) \subseteq D(\exp(tB))$ and

$$\exp(tB)x = \frac{1}{2\pi i}\int_{\operatorname{Re} z=\omega} \frac{e^{tz}}{(\mu - z)^2} R(z,B)(\mu I - B)^2 x\, dz \quad (x \in D(B^2)).$$

By the Dominated Convergence Theorem, this function of t is continuous on $[0,\infty)$. It is also exponentially bounded, and by Fubini's Theorem, Cauchy's Residue Theorem and some algebraic manipulations, its Laplace transform is

$$\begin{aligned}\int_0^\infty e^{-t\lambda}\exp(tB)x\, dt &\frac{1}{2\pi i}\int_{\operatorname{Re} z=\omega} \frac{1}{(\lambda - z)(\mu - z)^2} R(z,B)(\mu I - B)^2 x\, dz \\ &= R(\lambda, B)x.\end{aligned} \tag{3.3}$$

If B generates a C_0-semigroup T, then it follows from (3.1), (3.3) and uniqueness of Laplace transforms that $\exp(tB)x = T(t)x$ for $x \in D(B^2)$. Since $D(B^2)$ is dense, $\exp(tB)$ is closed and $T(t)$ is bounded, it follows that $\exp(tB) = T(t) \in \mathcal{B}(X)$.

Conversely, suppose that $\exp(tB) \in \mathcal{B}(X)$ for each $t \geq 0$ and

$$\sup\{\|\exp(tB)\| : 0 \leq t \leq 1\} < \infty.$$

Let $T(t) = \exp(tB)$. Then $T(0) = I$ and $T(s)T(t) = T(s+t)$ $(s,t \geq 0)$ by general properties of functional calculus, and T is strongly continuous (by density of $D(B^2)$). So T is a C_0-semigroup. If B' is the generator of T, then $R(\lambda, B)$ and $R(\lambda, B')$ coincide on the dense subspace $D(B^2)$, so $B' = B$. Thus we have shown the following.

Proposition 3.1. [26, Proposition 2.4]. *Let B be an operator of strong half-plane type. Then B generates a C_0-semigroup if and only if*

(1) $\exp(tB) \in \mathcal{B}(X)$ *for all* $t \geq 0$, *and*

(2) $\sup\{\|\exp(tB)\| : 0 \leq t \leq 1\} < \infty$.

Let B be an operator of strong half-plane type, $\omega > \omega_{shp}(B)$ and

$$r_{n,t}(z) = \left(1 - \frac{tz}{n}\right)^{-n}.$$

For $\operatorname{Re} z < \omega$, $0 \le t \le 1$ and $n \ge \max(2\omega, 1)$,

$$|r_{n,t}(z)| \le 4^{\omega}.$$

Also, $r_{n,t}(z) \to e^{tz}$ as $n \to \infty$. By the Convergence Lemma, $\exp(tB)$ is bounded, uniformly for $0 \le t \le 1$, if $\|r_{n,t}(B)\|$ is bounded uniformly in n and t. From this and some use of (1.1), we obtain the classical Hille-Yosida Theorem, presented here in the general form due to Feller, Miyadera and Phillips [4, Theorem 3.3.4], [20, Theorem II.3.8], [49, Theorem 1.5.3], but we have used the method of [26, Theorem 3.2].

Theorem 3.1. (Hille-Yosida) *An operator B generates a C_0-semigroup T if and only if there exist ω and M such that $\{\lambda \in \mathbb{R} : \lambda > \omega\} \subseteq \rho(B)$ and*

$$\|R(\lambda, B)^n\| \le \frac{M}{(\lambda - \omega)^n} \tag{3.4}$$

whenever $\lambda > \omega$ and $n \ge 1$.

The condition (3.4) is not easy to verify for large values of n, so it is useful to have criteria involving only $n = 1$ or $n = 2$. Such criteria have long been known for holomorphic semigroups (see Subsection 3.1) and for contraction semigroups (when $M = 1$, so $n = 1$ suffices in (3.4)) [4, Corollary 3.3.5], [20, Theorem II.3.5], [49, Theorem 1.3.1]—indeed these cases were known to Hille and Yosida before the general form of Theorem 3.1. For general semigroups, no such criterion is known, but we will now give an integral criterion involving only squares of the resolvents, which is sufficient for generation of a C_0-semigroup and which is satisfied in many examples.

Let B be an operator of strong half-plane type. For $\operatorname{Re}\lambda = a > \omega_{shp}(B)$, $\mu > a$ and $x \in D(B^2)$,

$$\begin{aligned} R(\lambda, B)^2 x &= (R(\lambda, B)R(\mu, B))^2 (\mu I - B)^2 x \\ &= \left(\frac{R(\lambda, B) - R(\mu, B)}{\mu - \lambda}\right)^2 (\mu I - B)^2 x. \end{aligned}$$

It follows that

$$\int_{\operatorname{Re}\lambda = a} e^{t\lambda} R(\lambda, B)^2 x \, d\lambda$$

is absolutely convergent. Since $R(\lambda, B)x$ is the Laplace transform of $\exp(tB)x$ by (3.3), its negative derivative $R(\lambda, B)^2x$ is the Laplace transform of $t\exp(tB)x$. By complex inversion of Laplace transforms [4, Theorems 2.3.4, 4.2.21],

$$\exp(tB)x = \frac{1}{2\pi it}\int_{\operatorname{Re}\lambda=a} e^{t\lambda}R(\lambda, B)^2x\,d\lambda \quad \big(t>0, x\in D(B^2), a>\omega_{shp}(B)\big).$$

Now suppose that, for some $\omega \ge \omega_{shp}(B)$ and some K,

$$\int_{\operatorname{Re}\lambda=a} \big|\langle R(\lambda, B)^2x, x^*\rangle\big|\;|d\lambda| \le \frac{K\|x\|\,\|x^*\|}{a-\omega} \qquad (x\in D(B^2), x^*\in X^*, a>\omega). \tag{3.5}$$

Then

$$\|\exp(tB)x\| \le \frac{Ke^{ta}\|x\|}{2\pi(a-\omega)t} \qquad (t>0, x\in D(B^2), a>\omega).$$

Since $D(B^2)$ is dense in X and $\exp(tB)$ is closed, it follows that $\exp(tB)\in\mathcal{B}(X)$ and

$$\|\exp(tB)\| \le \frac{Ke^{at}}{2\pi(a-\omega)t}$$

for any $a>\omega$. Choosing $a=\omega+t^{-1}$ gives $\|\exp(tB)\|\le Me^{\omega t}$, where $M=Ke/2\pi$. So B generates a C_0-semigroup.

Conversely, suppose that B generates a C_0-semigroup T with $\|T(t)\|\le Me^{\omega t}$. Since the resolvent of B is the Laplace transform of T, $s\mapsto R(a+is, B)x$ is the Fourier transform of the function

$$t\mapsto \begin{cases} e^{-at}T(t)x & (t\ge 0),\\ 0 & (t<0).\end{cases}$$

If X is a Hilbert space, then Plancherel's Theorem gives that

$$\int_{-\infty}^{\infty}\|R(a+is, B)x\|^2\;ds = 2\pi\int_0^\infty e^{-2at}\|T(t)x\|^2\,dt \le \frac{\pi M^2\|x\|^2}{a-\omega}\,.$$

A similar estimate holds for the dual semigroup generated by B^*. Since

$$\langle R(a+is, B)^2x, x^*\rangle = \langle R(a+is, B)x, R(a-is, B^*)x^*\rangle,$$

the Cauchy-Schwarz inequality leads to (3.5) with $K=\pi M^2$.

The following result summarises these facts, in a slightly more general form obtained independently by Gomilko [23], and Shi and Feng [54].

THEOREM 3.2. *Let B be an operator on X and suppose that there exist K and ω such that $\sigma(B) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le \omega\}$ and*

$$\int_{-\infty}^{\infty} \left|\langle R(a+is,B)^2x,x^*\rangle\right| \, ds \le \frac{K\|x\|\,\|x^*\|}{a-\omega} \tag{3.6}$$

whenever $a > \omega$, $x \in X$ and $x^* \in X^*$. Then B generates a C_0-semigroup on X.

Conversely, if X is a Hilbert space and B generates a C_0-semigroup, then (3.6) holds for some K and ω.

We shall see in Example 3.3 that the derivative operator $\mathcal{D}$ on $L^p(\mathbb{R})$ does not satisfy (3.6) when $p \ne 2$.

3.3. C_0-GROUPS AND STRIP-TYPE OPERATORS A *C_0-group* is a strongly continuous function $T : \mathbb{R} \to \mathcal{B}(X)$ such that $T(0) = I$ and $T(s)T(t) = T(s+t)$ for all $s, t \in \mathbb{R}$. In other words, $T|_{\mathbb{R}_+}$ is a C_0-semigroup, each operator $T(t)$ is invertible, and $t \mapsto T(t)^{-1} = T(-t)$ is also a C_0-semigroup. Thus, B generates a C_0-group if and only if B and $-B$ both generate C_0-semigroups, equivalently B and $-B$ both satisfy the Hille-Yosida condition (3.4).

EXAMPLE 3.2.

1. The translation, or shift, group on $L^p(\mathbb{R})$ $(1 \le p < \infty)$ is generated by $\mathcal{D}$ (see Example 3.1(2)).
2. If H is a self-adjoint operator on a Hilbert space, then $\{\exp(itH) : t \in \mathbb{R}\}$, defined by the functional calculus of self-adjoint operators, is a C_0-group. Moreover, every C_0-group of unitaries on a Hilbert space is of this form [20, Theorem II.3.24].

If B generates a C_0-group then the Hille-Yosida condition (3.4) for $n = 1$, applied to both B and $-B$, shows that B is an operator *of strong vertical strip type* in the sense that there exist $\omega > 0$ and M_ω such that

- $\sigma(B) \subseteq S_\omega := \{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| < \omega\}$;
- $\|R(\lambda,B)\| \le \dfrac{M_\omega}{|\operatorname{Re}\lambda| - \omega}$ whenever $\lambda \in \mathbb{C} \setminus \overline{S_\omega}$.

The infimum of such ω is the *strong vertical strip type* $\omega_{svst}(B)$ of B.

The functional calculus for an operator B of strong vertical strip type is defined in a similar way to sectorial operators and operators of strong half-plane type. Now the sector Σ_θ or half-plane L_ω is replaced by the strip S_ω. Thus in (3.2), f is holomorphic on a strip S_ω, and the integral is now taken over $\partial S_{\omega'}$, two vertical lines in opposite directions, instead of over a single line, where $\omega_{svst}(B) < \omega' < \omega$.

One may then define the notion of a strong vertical strip type operator having *bounded H^∞-calculus* on a strip. The following analogue of Theorem 2.2 characterizes this notion in terms of square-function estimates. This contrasts with the case of Subsection 3.2 and exhibits the advantages of integrals being taken over two lines.

THEOREM 3.3. [55, Corollary 4.27]. *Let B be an operator of strong vertical strip type on a Banach space X. Then B has bounded H^∞-calculus on some strip if and only if there exist $\omega > \omega_{svst}(B)$ and c such that*

$$\int_{\partial S_\omega} \left|\langle R(\lambda, B)^2 x, x^*\rangle\right| \, |d\lambda| \le c\|x\| \, \|x^*\|$$

for all $x \in X$ and $x^* \in X^*$.

If B is a strong vertical strip type operator with bounded H^∞-calculus, then B generates a C_0-group given by $T(t) = \exp(tB)$ defined by the functional calculus. In the light of Example 2.2(5), one might expect the converse to be false even on Hilbert spaces. Actually it is true and it follows directly from the final statement of Theorem 3.2 together with Theorem 3.3. The result was first obtained by a different method in [11], and other proofs are given in [25, Section 7.2].

THEOREM 3.4. *Let B be the generator of a C_0-group on a Hilbert space. Then B has bounded H^∞-calculus on a strip.*

EXAMPLE 3.3. Theorem 3.4 is not true on L^p-spaces for $1 \le p < \infty$, $p \ne 2$. For example, the generator $\mathcal{D}$ of the translation group on $L^p(\mathbb{R})$ does not have bounded H^∞-calculus on any strip when $p \ne 2$ [25, p.240]. It follows from Theorem 3.3 that $\pm\mathcal{D}$ does not satisfy (3.6). Since $-\mathcal{D}$ and $\mathcal{D}$ are similar operators, $\mathcal{D}$ does not satisfy (3.6).

Nevertheless there is a version of Theorem 3.4 which holds in UMD-spaces, in particular in L^p-spaces for $p \in (1, \infty)$. If B generates a C_0-group T on a UMD-space X and there exists ω such that $\left\{e^{-\omega|t|}T(t) : t \in \mathbb{R}\right\}$ is R-bounded in the sense of [40, Section 2], then B has bounded H^∞-calculus on a strip [27].

It will be convenient in the next subsection to consider operators associated with horizontal strips instead of vertical strips. Clearly there is a corresponding, essentially identical, theory of operators *of strong horizontal strip type* and their functional calculus. In fact, B is of strong horizontal strip type if and only if iB is of strong vertical strip type.

3.4. Operator logarithms The logarithm is a conformal mapping of the sector Σ_θ onto the horizontal strip of half-width θ. As observed in Section 2, if A is an injective sectorial operator, the operator logarithm $\log A$ is a closed operator. The following two results relate the functional calculi of A and $\log A$.

Theorem 3.5. [25, Theorem 4.3.1]. *Let A be an injective sectorial operator. Then $\log A$ is of strong horizontal strip type and $\omega_{shst}(\log A) = \theta_{sect}(A)$.*

Theorem 3.6. (Composition Rule) [25, Corollary 4.2.5]. *Let A be an injective sectorial operator, and let f be a holomorphic function on $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \omega\}$ for some $\omega \in (\theta_{sect}(A), \pi)$. Then $f(\log A)$ is defined by the functional calculus for operators of strong horizontal strip type if and only if $(f \circ \log)(A)$ is defined by the functional calculus for sectorial operators. In that case,*

$$f(\log A) = (f \circ \log)(A).$$

Composition rules of this type are an important feature of this subject, but they do not come for free—they have to be proved in each context in which they occur. Then consequences can be read off quickly. For example, it follows immediately from Theorem 3.6 that A has bounded H^∞-calculus on a sector if and only if $\log A$ has bounded H^∞-calculus on a strip.

The extended definition of functional calculus for operators of strong horizontal strip type includes the case of the exponential function, so we can define $\exp B$ if B is of strong horizontal strip type. The Composition Rule provides the expected result that $\exp(\log A) = A$. One would also hope that if B is of strong horizontal strip type with $\omega_{shst}(B) < \pi$, then $\exp B$ is sectorial. If B has bounded H^∞-calculus on a strip of half-width less than π, then $\exp B$ is sectorial with bounded H^∞-calculus on the sector, and $\log(\exp B) = B$. Nevertheless there is a surprise.

Example 3.4. Let $B = i\mathcal{D}$ on $L^1(\mathbb{R})$. Then B is of strong horizontal strip type with $\omega_{shst}(B) = 0$. However $\exp B$ has empty resolvent set [25, Example 4.4.1].

On the other hand there are some positive results, under supplementary assumptions. The first is originally due to Monniaux [45].

Theorem 3.7. [25, Theorem 4.4.3]. *If B generates a C_0-group on a UMD-space, then $\exp(iB)$ is sectorial.*

Theorem 3.8. [13, Proposition 5.2.9]. *Suppose that B is of strong horizontal strip type with $\omega_{shst}(B) = \omega < \pi$, and $\rho(\exp B) \cap (\mathbb{C} \setminus \Sigma_\omega)$ is non-empty. Then*

(1) $\sigma(\exp B) \subseteq \overline{\Sigma_\omega}$,

(2) *for each $\theta \in (\omega, \pi)$, there exists c_θ such that*

$$\|\lambda R(\lambda, \exp B)\| \leq c_\theta \log(|\log(|\lambda|)| + 2) \tag{3.7}$$

whenever $\lambda \in \mathbb{C} \setminus \Sigma_\theta$.

Theorem 3.8 indicates that $\exp B$ is very close to being sectorial (provided that $\sigma(\exp B)$ does not contain $\mathbb{C} \setminus \Sigma_\omega$), but it is not known whether it can be improved to give sectoriality. Having an iterated logarithm in (3.7) is significantly better than having an ordinary logarithm, because the basic method of [46] extends to the former class of operators, but not the latter. In other words, if an operator A satisfies (1) and (2) in Theorem 3.8, then $\log A$ can be defined and it is very close to being of strong horizontal strip type [13, Chapter 6].

3.5. Cosine functions and operators of parabolic type

Let B be a strong vertical strip type operator, and $t \in \mathbb{R}$. Then $\cos(itB)$ is defined as a closed operator by the functional calculus. If B generates a C_0-group, then

$$C(t) := \cos(itB) = \frac{1}{2} (\exp(tB) + \exp(-tB)) \in \mathcal{B}(X).$$

Then

(i) C is strongly continuous,

(ii) $C(0) = I$ and

$$2C(t)C(s) = C(t+s) + C(t-s) \qquad (t, s \in \mathbb{R}),$$

(iii) For any $x, y \in X$,

$$B^2 x = y \quad \text{if and only if} \quad \lim_{t\downarrow 0} \frac{2}{t^2}(C(t)x - x)y.$$

The first two properties say that C is a *cosine function*, and the third says that B^2 is the *generator* of C.

An arbitrary cosine function C with generator A has further properties as follows:

(iv) There exist real constants ω and M such that $\|C(t)\| \le Me^{\omega|t|}$ for all $t \in \mathbb{R}$, and

$$\lambda R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t} C(t)x \, dt \qquad (x \in X, \operatorname{Re}\lambda > \omega).$$

(v) $u(t) = C(t)x$ is a solution of the Cauchy problem

$$u''(t) = Au(t) \quad (t \ge 0), \qquad u(0) = x, \ u'(0) = 0,$$

in the classical sense if $x \in D(A)$ and in a mild sense if $x \in X$.

In particular A is an *operator of parabolic type* in the sense that there exist $\omega > 0$ and M such that

$$\sigma(A) \subseteq \Pi_\omega := \left\{\lambda^2 : \lambda \in S_\omega\right\} = \left\{\xi + i\eta : \xi < \omega^2 - \eta^2/4\omega^2\right\},$$

$$\|R(\mu, A)\| \le \frac{M}{\sqrt{|\mu|}(|\operatorname{Re}\sqrt{\mu}| - \omega)} \quad (\mu \in \mathbb{C} \setminus \overline{\Pi_\omega}).$$

In fact, if B is an operator of strong vertical strip type, then $B^2 + \omega I$ is of parabolic type for every $\omega \in \mathbb{R}$. On the other hand if A is of parabolic type, then for sufficiently large ω, $\omega I - A$ is a sectorial operator and its square root is of strong horizontal strip type. Thus an operator A is of parabolic type if and only if $A = B^2 + \omega I$ for some $\omega > 0$ and some operator B of strong vertical strip type.

It is possible to define functional calculus for operators of parabolic type, in a similar way to sectorial and strip-type operators. The function $z \mapsto \cos(t\sqrt{-z})$ is well-defined, entire, and bounded on each Π_ω. An operator A of parabolic type generates a cosine function if and only if $\cos(t\sqrt{-A})$ (defined by this functional calculus) is bounded, uniformly for $0 \le t \le 1$ [26].

If B generates a C_0-group and $A = B^2 + \omega I$, then A generates a cosine function. The converse is not true in general [35] but Fattorini [21] proved that it is true on UMD-spaces. This indicates that there are few examples of cosine functions which are not associated with C_0-groups.

THEOREM 3.9. [4, Theorem 3.16.7]. *If X is a UMD-space and A generates a cosine function, then $A = B^2 + \omega I$ where B generates a C_0-group and $\omega \in \mathbb{R}$.*

In the case of Hilbert spaces, generators of cosine functions can be described precisely in terms of the numerical range of the operators, up to equivalence of norms.

THEOREM 3.10. (Crouzeix [15]). *Let A be an operator on a Hilbert space such that the numerical range and spectrum of A are both contained in Π_ω for some $\omega > 0$. Then A generates a cosine function.*

THEOREM 3.11. [25, Corollary 7.4.8]. *Let A be the generator of a cosine function on a Hilbert space X. Then there is an equivalent scalar product on X with respect to which the numerical range of A is contained in Π_ω for some $\omega > 0$.*

Theorem 3.10 is part of a remarkable discovery by Crouzeix [16] for matrices. If A is an $n \times n$ matrix with numerical range $W(A)$ and $p(x)$ is any complex polynomial, then

$$\|p(A)\|_{\mathcal{B}(\mathbb{C}^n)} \le 12 \sup\{|p(z)| : z \in W(A)\}.$$

Once this result is established for matrices, then standard approximation methods allow one to pass from matrices to operators on Hilbert space and from polynomials to holomorphic functions. Thus the assumptions of Theorem 3.10 imply that A has bounded H^∞-calculus on Π_ω.

One further result, due to Kisyński [36], links general cosine functions to C_0-semigroups.

THEOREM 3.12. [4, Theorem 3.14.11]. *Let A be an operator on X. If A generates a cosine function on X then there exists a unique Banach space V such that*

(1) $D(A) \hookrightarrow V \hookrightarrow X$, *and*

(2) *the operator*

$$\mathcal{B} := \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix},$$

with $D(\mathcal{B}) = D(A) \times V$, *generates a* C_0*-semigroup on* $V \times X$.

Conversely, if there is a Banach space V *such that* (1) *and* (2) *hold, then* A *generates a cosine function on* X.

The space V in Theorem 3.12 is sometimes called the *Kisyński space* associated with the cosine function or its generator, and sometimes the *phase space*, but the latter terminology is sometimes used for $V \times X$. If $A = B^2 + \omega I$ where B generates a C_0-group, then $V = D(B)$.

For further information on cosine functions, see [4, Sections 3.14-3.16], [22].

4. Perturbations

Perturbation theory is important for applications because many models involve an operator $\tilde{A}$ which can be written as

$$\tilde{A} = A + Q$$

where A is well understood (say, a differential operator which is purely second order) and Q is a perturbation which is small in some sense (say, lower order terms only). One wants to establish that $\tilde{A}$ has similar properties to A. Actually it is often sufficient if we can show that $\omega I + \tilde{A}$ has those properties, and we shall be satisfied with that.

4.1. Sectorial operators, C_0-semigroups and cosine functions

Let $\lambda \in \rho(A)$ and $Q \in \mathcal{B}(X)$. Then

$$\lambda I - (A + Q) = (I - QR(\lambda, A))(\lambda - A).$$

If $\|QR(\lambda, A)\| < 1/2$, then

$$\lambda \in \rho(A + Q) \quad \text{and} \quad \|R(\lambda, A + Q)\| \leq 2\|R(\lambda, A)\|. \tag{4.1}$$

If A is sectorial and $\theta > \theta_{sect}(A)$, then $\|\lambda R(\lambda, A + Q)\| \leq 2c_\theta$ if $\lambda \notin \Sigma_\theta$ and $|\lambda|$ is large enough. Then $\omega I + A + Q$ is sectorial for some $\omega \in \mathbb{R}$, and we can make $\theta_{sect}(\omega I + A + Q)$ as close as we like to $\theta_{sect}(A)$ by choosing ω large enough.

This argument can be adapted to some situations where Q is unbounded. Let $Q : D(A) \to X$ be relatively bounded, i.e., bounded when $D(A)$ has the graph norm. Assuming (without loss) that A is invertible, this means that $Q = SA$ where $S \in \mathcal{B}(X)$. Now $QR(\lambda, A) = S(\lambda R(\lambda, A) - I)$ and it follows from sectoriality of A that $\|QR(\lambda, A)\| < 1/2$ whenever $\lambda \notin \Sigma_\theta$, if $\|S\|$ is sufficiently small. Then $A + Q$ is sectorial.

Now assume that $Q : D(A) \to X$ is relatively compact, and write

$$\lambda I - (A + Q) = (\lambda I - A)(I_{D(A)} - R(\lambda, A)Q).$$

When A is sectorial, $R(\lambda, A) : X \to D(A)$ is bounded uniformly for $\lambda \in \mathbb{C} \setminus \Sigma_\theta$ and strongly convergent to 0 as $|\lambda| \to \infty$. Since $Q : D(A) \to X$ is compact, it follows that $\|R(\lambda, A)Q\|_{\mathcal{B}(D(A))} \to 0$ as $|\lambda| \to \infty$. In a similar way to before, it follows that $\omega I + A + Q$ is sectorial with angle arbitrarily close to $\theta_{sect}(A)$.

Considering the case when the angle is less than $\pi/2$ and applying facts from Subsection 3.1 provides the following result.

Theorem 4.1. *Let $-A$ be the generator of a holomorphic semigroup and $Q : D(A) \to X$ be relatively bounded. Assume that either*

(1) $\|Q\|_{\mathcal{B}(D(A),X)}$ *is sufficiently small, or*

(2) Q *is relatively compact.*

Then $-(A + Q)$ generates a holomorphic semigroup.

The case (1) of Theorem 4.1 was known to Hille [29]. Case (2) is due to Desch and Schappacher [18], and they also showed that no result of this type holds for semigroups which are not holomorphic (see also [2]). Another type of perturbation theorem for holomorphic semigroups may be found in [39].

For general C_0-semigroups, Phillips [50] showed the following for bounded perturbations.

Theorem 4.2. *If B generates a C_0-semigroup and $Q \in \mathcal{B}(X)$ then $B + Q$ generates a C_0-semigroup.*

For unbounded perturbations Q of generators B of C_0-semigroups which are not holomorphic, there are many results saying that $B + Q$, or some extension of it, generates a C_0-semigroup under various assumptions on B and Q, separately or in combination. We refer the reader to [20, Sections III.2, III.3] for details of some such results.

For generators of cosine functions, we have the following corollary of Theorem 3.12 and Theorem 4.2.

Corollary. *If A generates a cosine function with Kisyński space V and $B : V \to X$ is bounded, then $A + B$ generates a cosine function.*

4.2. Differentiable semigroups A C_0-semgroup T with generator B is said to be (*immediately*) *differentiable* if T is differentiable on $(0, \infty)$ in operator-norm, or equivalently in the strong operator topology. This is equivalent to saying that $T(t)$ maps X into $D(B)$, and then $T'(t) = BT(t)$, a bounded operator, for each $t > 0$. Holomorphic semigroups are differentiable, and some degenerate differential operators generate differentiable semigroups which are not holomorphic (see [24], for example). Moreover, C_0-semigroups which are eventually differentiable (i.e., differentiable on (t_0, ∞) for some $t_0 > 0$) arise naturally in the study of delay equations [9], [10] and Volterra equations [8].

The definition of a differentiable semigroup can be expressed in terms of functional calculus as follows. For $t > 0$, let $g_t(z) = ze^{tz}$. Then g_t is holomorphic on each left half-plane, and moreover g_t can be regularised by means of a function of the form $(\omega + z)^{-3}$. Thus $g_t(B)$ is defined as a closed operator, and indeed $g_t(B)$ is a closed extension of $T(t)B$. Now T is differentiable if and only if $g_t(B) \in \mathcal{B}(X)$ for each $t > 0$.

A necessary condition for T to be differentiable is that g_t is bounded on $\sigma(B)$. Writing $\beta = 1/t$, this requires $\sigma(B)$ to be disjoint from a region of the form $E_{\beta,c}$ defined below (sometimes known as an *exponential region*, and sometimes as a *logarithmic region*). This explains part of the characterisation of generators of C_0-semigroups given in Theorem 4.3, originally due to Pazy [48].

For $\beta > 0$ and $c \in \mathbb{R}$, let

$$E_{\beta,c} = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq c - \beta \log(|\operatorname{Im} \lambda| + 1)\}.$$

Theorem 4.3. [49, Theorem 2.4.7]. *Let T be a C_0-semigroup with generator B. Then T is differentiable if and only if, for each $\beta > 0$, there exist c_β, c'_β such that $E_{\beta,c_\beta} \subseteq \rho(B)$ and*

$$\|R(\lambda, B)\| \leq c'_\beta(|\lambda| + 1) \quad \text{for all } \lambda \in E_{\beta,c_\beta}. \tag{4.2}$$

In this case, T is given by a contour integral of the form (2.2) for $\exp(tB)$ using E_{β,c_β} instead of a sector:

$$T(t) = \frac{1}{2\pi i} \int_{\partial E_{\beta,c_\beta}} e^{\lambda t} R(\lambda, B)\, d\lambda,$$

where this integral, and the corresponding integral for T', are absolutely convergent if $t > 3/\beta$.

The nature of the resolvent estimate (4.2) is somewhat surprising. The linear growth of the resolvent can be replaced by polynomial growth of any order. However, differentiability of T does not imply boundedness of the resolvent of B in any $E_{\beta,c}$ [48, p.1136].

Let B be the generator of a differentiable semigroup. The question was raised whether the semigroup generated by $B + Q$ is differentiable for every $Q \in B(X)$. Pazy [48] observed that this is so if $\|R(\lambda, B)\| \to 0$ as $|\lambda| \to \infty$ through E_{β,c_β}. After a long interval Renardy [53] constructed a differentiable semigroup with a bounded perturbation which is not differentiable. Iley [31] recently proved that the converse of Pazy's observation is true.

THEOREM 4.4. *Let T be a C_0-semigroup with generator B. Suppose that for each $Q \in \mathcal{B}(X)$, the C_0-semigroup generated by $B + Q$ is differentiable. Then, for each $\beta > 0$, there exists c_β such that $E_{\beta,c_\beta} \subseteq \rho(B)$ and $\|R(\lambda, B)\| \to 0$ as $|\lambda| \to \infty$ through E_{β,c_β}.*

The proof of this result involves several steps which we sketch here.

First step. Any locally Lipschitz C_0-semigroup is differentiable, because the condition that $\|T(t)-T(s)\| \leq c|t-s|$ on some interval implies that $\|BT(t)x\| \leq c\|x\|$ for $x \in D(B)$, and hence $BT(t) \in \mathcal{B}(X)$, for t in the same interval. The converse is also true, because $\|T'(t+h)\| = \|T(h)BT(t)\|$ is uniformly bounded for $0 \leq h \leq 1$.

Second step. Assume that $B + Q$ generates a differentiable semigroup S_Q for each $Q \in \mathcal{K}(X)$. Let $t_0 \in (0, 1)$ be arbitrary. Define

$$F_n = \{Q \in \mathcal{K}(X) : \|S_Q(t) - S_Q(s)\| \leq n|t - s| \ (t_0 \leq s, t \leq 1)\} .$$

Then F_n is a closed subset of $\mathcal{K}(X)$, and $\mathcal{K}(X) = \bigcup_{n=1}^{\infty} F_n$ (by assumption and the first step). By the Baire Category Theorem, there exists m such that F_m contains a ball in $\mathcal{K}(X)$, with centre Q_0 and radius $\varepsilon > 0$. Thus the derivatives of the semigroups S_Q are bounded uniformly for $t \in [t_0, 1]$ and $\|Q - Q_0\| < \varepsilon$.

Third step. By examining the proof of Theorem 4.3, or by applying the statement of Theorem 4.3 to a direct sum, it follows from the second step that, for a fixed $\beta > 0$, the constants c_β, c'_β in Theorem 4.3 can be chosen uniformly for all the semigroups S_Q with $\|Q - Q_0\| < \varepsilon$. In particular, there exists c_β

such that $E_{\beta,c_\beta} \subseteq \rho(B+Q_0+K)$ whenever $K \in \mathcal{K}(X)$ and $\|K\| < \varepsilon$. It follows by an elementary argument, involving only rank-one operators K, that $\|R(\lambda, B+Q_0)\| \le \varepsilon^{-1}$ for all $\lambda \in E_{\beta,c_\beta}$.

Fourth step. It is elementary that $\|R(\lambda, B+Q_0)x\| \to 0$ as $|\lambda| \to \infty$ through E_{β,c_β}, for each $x \in D(B)$. Since $\|R(\lambda, B+Q_0)\|$ is bounded in this region (by the third step) and Q_0 is compact, it follows that $\|R(\lambda, B+Q_0)Q_0\| \to 0$. As in Subsection 4.1 this implies that $R(\lambda, B)$ exists and is uniformly bounded in a subregion $E_{\beta,\tilde{c}_\beta}$.

Final step. Now the resolvent of B is bounded and holomorphic in a region $E_{\beta,\tilde{c}_\beta}$ and it is bounded by $M/(a-\omega)$ for $\operatorname{Re}\lambda = a > \omega$, for some M and ω. By choosing suitable a depending on $\operatorname{Im}\lambda$, one can apply an approximate form of the classical three-lines theorem to obtain an estimate on the resolvent showing that, for each $\beta' < \beta$, there exists $c_{\beta'}$ such that $\|R(\lambda, B)\| \to 0$ as $|\lambda| \to \infty$ through $E_{\beta',c_{\beta'}}$.

More precisely, the final step of the argument involves an application of the two-constant theorem for subharmonic functions and harmonic measure [52, Theorem 4.3.7]. In fact, explicit estimates are obtained for the rate of decay. For example, it was shown in [31] that

$$\|R(is, B)\| = o\left((\log|s|)^{-\gamma}\right) \quad \text{as } |s| \to \infty$$

for any $\gamma < 1$, when B is as in Theorem 4.4. Chill and Tomilov [12, Lemma 4.13] improved this to

$$\|R(is, B)\| = O\left(\frac{\log\log|s|}{\log|s|}\right) \quad \text{as } |s| \to \infty. \tag{4.3}$$

Król [38] has constructed an example showing that (4.3) is sharp.

There are variants of Theorem 4.3 and Theorem 4.4 in which immediately differentiable semigroups are replaced by eventually differentiable semigroups and the resolvent conditions hold for some $\beta > 0$ [31].

4.3. H^∞-CALCULUS ON SECTORS Suppose that A has bounded H^∞-calculus on a sector. We consider the question:

When does $\omega I + A + Q$ also have bounded H^∞-calculus, for some ω?

The answer is easily seen to be positive when $Q \in \mathcal{B}(X)$ or even when Q is relatively bounded with respect to A^α for some $\alpha < 1$ [40, Proposition 13.1]. The question is more interesting mathematically for relatively bounded perturbations $Q : D(A) \to X$. Then replacing A by $\omega I + A$, we may assume, without loss of generality, that A is invertible and sectorial and we may write $Q = SA$ where $S \in \mathcal{B}(X)$. In addition, we will assume that $A+SA$ is sectorial, or we will impose conditions that S is compact or of small norm, implying that $\omega I + A + SA$ is sectorial (Subsection 4.1).

A positive answer was obtained in [3] when S is rank-1, so $Qx = \langle Ax, x^* \rangle y$ for some $x^* \in X^*$ and $y \in X$. The proof in [3] relied directly on the definition of bounded H^∞-calculus, but the argument is not straightforward. Independently, Kalton [32] obtained the same result as part of a wider programme which we discuss below. There is a rather simple proof of the rank-1 case from the square-function estimates (Theorem 2.2), using the Morrison–Sherman–Woodbury formula for the resolvent of $A + Q$ in terms of the resolvent of A, x^* and y [55, Corollary 3.37].

It follows from the rank-1 case that our question has a positive answer when S is of finite rank, and even when S is a nuclear operator. Moreover Kalton obtained further results using the following notion of “triangular” operators (unrelated to triangular matrices).

An operator $S \in \mathcal{B}(X)$ is said to be *triangular* if there exists c such that

$$\left| \sum_{j=1}^{n} \sum_{k=1}^{j} \langle Sx_j, x_k^* \rangle \right| \le c \sup_{|\alpha_j|=1} \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| \sup_{|\beta_k|=1} \left\| \sum_{k=1}^{n} \beta_k x_k^* \right\|$$

whenever $n \in \mathbb{N}$, $x_j \in X$, $x_k^* \in X^*$. The least such c is the *triangular norm* $\Theta(S)$ of S. Taking $n = 1$ shows that $\Theta(S) \ge \|S\|$. The triangular operators form an ideal in $\mathcal{B}(X)$ and $\Theta(USV) \le \|U\| \Theta(S) \|V\|$.

Theorem 4.5. (Kalton [32]). *Let A be a sectorial operator with bounded H^∞-calculus on a sector. Then $A + SA$ has bounded H^∞-calculus on a sector whenever S is triangular and $\Theta(S)$ is sufficiently small.*

The proof of this theorem uses the square-function estimates of Theorem 2.2, as follows. For a C^1-function $F : I \to \mathcal{B}(X)$ where I is an interval, let

$$\|F\|_{bv} = \sup \left\{ \int_I \left| \langle F'(t)x, x^* \rangle \right| \, dt : \|x\| = 1, \|x^*\| = 1 \right\}.$$

Let A be sectorial with bounded H^∞-calculus and $|\theta| \in (\theta_{H^\infty}(A), \pi)$. Define $F_\theta : (0, \infty) \to \mathcal{B}(X)$ by

$$F_\theta(t) = AR(te^{i\theta}, A).$$

Then F_θ is bounded since A is sectorial, and $\|F_\theta\|_{bv} < \infty$ by Theorem 2.2.

Let S be a triangular operator on X with $\|S\| < \|F_\theta\|_\infty^{-1}$, and let

$$G_\theta(t) = (A+SA)R(te^{i\theta}, A+SA) = (I+S)AR(te^{i\theta}, A)(I - SAR(te^{i\theta}, A))^{-1}$$
$$= (I+S)F_\theta(t)(I - SF_\theta(t))^{-1} = (I+S)\sum_{n=0}^{\infty} F_\theta(t)(SF_\theta(t))^n.$$

We shall show in Proposition 4.3 that $\|F(I-SF)^{-1}\|_{bv} < \infty$ if $\|F\|_{bv} < \infty$ and $\Theta(S)$ is sufficiently small. Then

$$\|G_\theta\|_{bv} \le \|I+S\|\,\|F_\theta(I - SF_\theta)^{-1}\|_{bv} < \infty,$$

if $\Theta(S)$ is sufficiently small. So Theorem 4.5 follows from Theorem 2.2.

PROPOSITION 4.1. *Let $f : [a,b] \to X$ and $g : [a,b] \to X^*$ be continuous functions, and let S be a triangular operator on X. Then*

$$\int_a^b \left| \int_a^t \langle Sf(s), g(t)\rangle\, ds \right| dt$$
$$\le \Theta(S) \sup_{\|x^*\|=1} \left\{ \int_a^b |\langle f(s), x^*\rangle|\, ds \right\} \sup_{\|x\|=1} \left\{ \int_a^b |\langle x, g(t)\rangle|\, dt \right\}.$$

Proof. Let $I_1, \dots, I_n$ be a partition of $[a,b]$ into disjoint subintervals, and let

$$x_j = \int_{I_j} f(s)\, ds, \quad x_j^* = \int_{I_j} g(t)\, dt \qquad (j = 1, \dots, n).$$

Take $\varepsilon_j \in \mathbb{C}$ so that $|\varepsilon_j| = 1$ and $\varepsilon_j \sum_{k=1}^{j} \langle Sx_j, x_k^*\rangle \ge 0$. Then

$$\sum_{j=1}^n \left| \sum_{k=1}^j \langle Sx_j, x_k^*\rangle \right| = \sum_{j=1}^n \sum_{k=1}^j \langle S(\varepsilon_j x_j), x_k^*\rangle$$
$$\le \Theta(S) \sup_{|\alpha_j|=1} \left\| \sum_{j=1}^n \alpha_j x_j \right\| \sup_{|\beta_k|=1} \left\| \sum_{k=1}^n \beta_k x_k^* \right\|$$
$$= \Theta(S) \sup_{\substack{|\alpha_j|=1 \\ \|x^*\|=1}} \left| \sum_{j=1}^n \alpha_j \int_{I_j} \langle f(s), x^*\rangle\, ds \right| \sup_{\substack{|\beta_k|=1 \\ \|x\|=1}} \left| \sum_{k=1}^n \beta_k \int_{I_k} \langle x, g(t)\rangle\, dt \right|$$
$$\le \Theta(S) \sup_{\|x^*\|=1} \int_a^b |\langle f(s), x^*\rangle|\, ds \sup_{\|x\|=1} \int_a^b |\langle x, g(t)\rangle|\, dt.$$

Now the result follows by taking the limit as the mesh of the partition tends to 0. ∎

PROPOSITION 4.2. *Let $F, G : [a,b] \to \mathcal{B}(X)$ be C^1-functions, and S be a triangular operator on X. Let $(FSG)(t) = F(t)SG(t)$. Then*

$$\|FSG\|_{bv} \le \Theta(S)\left(2\|F\|_{bv}\|G\|_{bv} + \|F\|_{bv}\|G(a)\| + \|F(a)\|\,\|G\|_{bv}\right).$$

Proof. We use the formula

$$\begin{aligned}(FSG)'(t) = F'(t)S(G(t) - G(a)) + (F(t) - F(a))SG(t)\\ + F'(t)SG(a) + F(a)SG'(t).\end{aligned} \tag{4.4}$$

Let $x \in X$ and $x^* \in X^*$ be unit vectors. By Proposition 4.1,

$$\begin{aligned}&\int_a^b \left|\langle F'(t)S(G(t) - G(a))x, x^*\rangle\right| \, dt = \int_a^b \left|\int_a^t \langle SG'(s)x, F'(t)^*x^*\rangle \, ds\right| dt\\ &\quad\le \Theta(S) \sup_{\|y^*\|=1}\left\{\int_a^b |\langle G'(s)x, y^*\rangle| \, ds\right\} \sup_{\|y\|=1}\left\{\int_a^b |\langle y, F'(t)^*x^*\rangle| \, dt\right\}\\ &\quad\le \Theta(S)\|F\|_{bv}\|G\|_{bv}.\end{aligned}$$

The second term on the right-hand side of (4.4) can be handled in a similar way. For the third term, we have

$$\begin{aligned}\int_a^b \left|\langle F'(t)SG(a)x, x^*\rangle\right| \, dt &\le \|F\|_{bv}\|SG(a)x\| \le \|F\|_{bv}\|S\|\|G(a)\|\\ &\le \|F\|_{bv}\Theta(S)\|G(a)\|,\end{aligned}$$

and similarly for the last term. ∎

PROPOSITION 4.3. *Let $F : (0,\infty) \to \mathcal{B}(X)$ be a bounded C^1-function such that $\|F\|_{bv}$ is finite. Let $r = \|F\|_\infty + 2\|F\|_{bv}$, and let S be a triangular operator with $\Theta(S) < 1/r$. Then*

$$\|F(I - SF)^{-1}\|_{bv} \le (1 - r\Theta(S))^{-2}\|F\|_{bv}.$$

Proof. Let $F_n = F(SF)^n$. Then $F(I - SF)^{-1} = \sum_{n=0}^\infty F_n$. A simple induction shows that $\|F_n'(t)\| \le (n+1)\|T\|^n\|F(t)\|^n\|F'(t)\|$, so $\sum_{n=0}^\infty F_n'$ converges to the derivative of $F(I - SF)^{-1}$, locally uniformly on $(0,\infty)$.

It follows from Proposition 4.2 that

$$\|F_n\|_{bv} \leq r\Theta(S)\|F_{n-1}\|_{bv} + \|S\|^n \|F\|_\infty^n \|F\|_{bv}.$$

Hence,

$$\|F_n\|_{bv} \leq (n+1)r^n\Theta(S)^n\|F\|_{bv}.$$

It follows that

$$\|F(I-SF)^{-1}\|_{bv} \leq \sum_{n=0}^{\infty}\|F_n\|_{bv} \leq (1-r\Theta(S))^{-2}\|F\|_{bv}.$$

∎

In order to apply Theorem 4.5 one needs conditions which ensure triangularity and which can be verified. Kalton established the following in [32].

Example 4.1.

1. Any 1-absolutely summing operator S is triangular and $\Theta(S) \leq \pi_1(S)$. See [56, Section III.F] for background material on the 1-absolutely summing norm π_1.

2. Let $a_n(S) := \inf\,(\|S-R\| : \operatorname{rank} R \leq n)$ be the approximation numbers of S, so S is compact if $\lim_{n\to\infty} a_n(S) = 0$ (and the converse holds if X has the approximation property). Then

$$\Theta(S) \leq c\sum_{n=1}^{\infty}(1+\log n)\frac{a_n(S)}{n}$$

   for some constant c. Using Theorem 4.5 and the positive result for finite-rank operators, it follows that $\omega I + A + SA$ has bounded H^∞-calculus if $\sum a_n(S)(\log n)/n$ converges.

3. Let X be a Hilbert space. Then S is triangular if and only if $\sum a_n(S)/n$ converges. Moreover there exist positive constants c, c' such that

$$c'\sum_{n=1}^{\infty}\frac{a_n(S)}{n} \leq \Theta(S) \leq c\sum_{n=1}^{\infty}\frac{a_n(S)}{n}$$

   for all triangular operators S. Again it follows that $\omega I + A + SA$ has bounded H^∞-calculus if $\sum a_n(S)/n$ converges.

4. Let $X = \ell^2$. There exist an operator A with bounded H^∞-calculus on a sector, and a compact operator S, such that, for each $\omega \in \mathbb{R}$, $\omega I+A+SA$ does not have bounded H^∞-calculus.

Kalton's construction in Example 4.1(4) showed that Theorem 4.5 cannot be improved by replacing the triangular operators by any larger operator ideal. On the other hand, for a given operator A with bounded H^∞-calculus on a sector, there are likely to be many bounded non-triangular operators S such that $A + SA$ has bounded H^∞-calculus on a sector. For example, this occurs if S is the identity operator (which is not triangular if X has an unconditional basis), or if S is any bounded operator commuting with $R(\lambda, A)$ and $A+SA$ is sectorial. Furthermore, Kalton and Weis [33] showed that $A+B$ has bounded H^∞-calculus if A and B have commuting resolvents, each has bounded H^∞-calculus on a sector, the calculus of one operator is R-bounded in an appropriate sense, and the sum of the two angles is less than π. Prüss and Simonett [51] have extended this to the case when the resolvents of A and B do not commute but they satisfy a suitable commutator condition. These results have many applications to questions of maximal regularity.

4.4. H^∞-CALCULUS ON STRIPS We begin this subsection with some heuristic discussion. Consider an operator of the form $\log A$ where A is an injective sectorial operator. In Subsection 4.3 we considered perturbations of the form $A+SA$ where S is triangular and small in some sense, so that $A+SA$ is sectorial. Then $\log A$ and $\log(A + SA)$ are of strong horizontal strip type and $\log(I + S)$ is bounded on X if $\|S\| < 1$. We can imagine that

$$\log(A + SA) = \log A + \log(I + S)$$

(although this will be false if A and S do not commute). This suggests that we should consider bounded perturbations of operators of strong horizontal strip type, and we might expect to obtain analogues of Kalton's perturbation results.

It is easy to see that if B is an operator of strong horizontal strip type and $S \in \mathcal{B}(X)$, then $B + S$ is of strong horizontal strip type. If iB generates a C_0-group, then $i(B + S)$ also generates a C_0-group, by Theorem 4.2. So it follows from Theorem 3.4 that if X is a Hilbert space and B has bounded H^∞-calculus on a strip, then $B + S$ has bounded H^∞-calculus on a larger strip. With more sophisticated arguments, this can be extended to UMD-spaces.

Theorem 4.6. [27], [55]. *If B is a strong horizontal strip type operator on a UMD-space X with bounded H^∞-calculus, and $S \in \mathcal{B}(X)$, then $B + S$ has bounded H^∞-calculus on a strip.*

This already shows that it is not possible to obtain an analogue of 4.1(4) on Hilbert space. Our heuristic argument above relied too much on ignoring non-commutativity of the operators. Nevertheless the following analogue of Theorem 4.5 is true, with a similar proof.

Theorem 4.7. [55]. *Let B be an operator of strong horizontal strip type with bounded H^∞-calculus, on any Banach space. Then $B + S$ has bounded H^∞-calculus on a strip whenever S is triangular and $\Theta(S)$ is sufficiently small.*

In the light of Theorem 4.6, it seems likely that Theorem 4.7 is true for a much larger class of perturbations than triangular operators, but such a result is not yet known.

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