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Subnormality and Moment Problems†

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Abstract: There exists a strong connection between the concept of (sub)normality and that of moment problem. They interact very often, sometimes in a subtle, unexpected way. It is possible to use a (sub)normality result, providing eventually a spectral measure, used to solve a moment problem. Conversely, there are situations when the solution to a moment problem leads to the existence of a normal extension for some operators. The present work endeavor to present several results sustaining the interplay mentioned above, as well as the necessary background to understand those phenomena, both is a bounded or a unbounded context.

Key words: Bounded and unbounded (sub)normal operators; spectral measure; moment problems; algebraic sets; algebras of fractions.

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1. Bounded subnormal operators

1.1. Introduction We start by presenting the concepts of operator-valued positive measure and spectral measure. More details can be found in [8], [12] or [25].

Let Ω be an arbitrary set and let $\mathcal{P}(\Omega)$ be the family of all subsets of Ω. We recall that a nonempty family $\Sigma \subset \mathcal{P}(\Omega)$ is said to be a *σ-algebra* if it has the following properties:

(i) $\emptyset \in \Sigma$;

(ii) if $S \in \Sigma$, then $\Omega \setminus S \in \Sigma$;

(iii) if $S_k \in \Sigma, k = 1, 2, 3, \ldots$, then $\bigcup_{k\geq 1} S_k \in \Sigma$.

From now on we assume that Ω is a (topological) Hausdorff space. We denote by $\mathrm{Bor}(\Omega)$ the smallest σ-algebra in $\mathcal{P}(\Omega)$ containing all open (equivalently closed) subsets of Ω. Each element of $S \in \mathrm{Bor}(\Omega)$ is called a *Borel subset* of Ω.

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Let also $\mathcal{H}$ be a fixed complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators acting in $\mathcal{H}$. The identity on $\mathcal{H}$ will be denoted by $I_{\mathcal{H}}$ or simply by I.

DEFINITION 1.1. An *operator-valued positive measure* (or simply a *positive measure*) on Ω is a map $F : \mathrm{Bor}(\Omega) \to \mathcal{B}(\mathcal{H})$ with the following properties:

(1) $F(S) \geq 0$ for all $S \in \mathrm{Bor}(\Omega)$;

(2) $F(\emptyset) = 0$, $F(\Omega) = I$;

(3) for every sequence $(S_k)_{k\geq 1}$ of mutually disjoint Borel sets and each $x \in \mathcal{H}$, the series $\sum_{k\geq 1} F(S_k)x$ is convergent in $\mathcal{H}$ and one has

$$F\left(\bigcup_{k\geq 1} S_k\right) x = \sum_{k\geq 1} F(S_k)\, x\,.$$

If, moreover,

(4) for every pair $S_1, S_2 \in \mathrm{Bor}(\Omega)$ we have

$$F\left(S_1 \bigcap S_2\right) = F(S_1)\, F(S_2)\,,$$

then F is said to be *a spectral measure on* Ω.

Remarks 1.1. (a) If $F : \mathrm{Bor}(\Omega) \to \mathcal{B}(\mathcal{H})$ is a positive measure, then for each finite family $(S_k)_{k=1}^m$ consisting of mutually disjoint Borel subsets, we have

$$F\left(\bigcup_{k\geq 1}^m S_k\right) = \sum_{k\geq 1}^m F(S_k)\,,$$

as a consequence of (2) and (3). In particular, if $S_1 \subset S_2$, if $F(S_2) - F(S_1) = F(S_2 \setminus S_1) \geq 0$.

(b) If $F : \mathrm{Bor}(\Omega) \to \mathcal{B}(\mathcal{H})$ is a spectral measure, then $F(S)$ is an orhogonal projection for all $S \in \mathrm{Bor}(\Omega)$. Indeed, $F(S)^* = F(S)$ by (1) and $F(S)^2 = F(S)$, via (4).

Given a positive measure $F : \mathrm{Bor}(\Omega) \to \mathcal{B}(\mathcal{H})$, for all pairs $(x, y) \in \mathcal{H}$ we set

$$F_{x,y}(S) = \langle F(S)x, y\rangle\ , \qquad S \in \mathrm{Bor}(\Omega)\,,$$

which are scalar measures. Note that $F_{x,x}$ is positive for all $x \in \mathcal{H}$.

An important tool for the study of (operator-valued) positive measures is the following classical theorem, due to Naimark (see [21]):

THEOREM 1.1. *Let* $F : \mathrm{Bor}(\Omega) \to \mathcal{B}(\mathcal{H})$ *be a positive measure. There exist a Hilbert space* $\mathcal{K} \supset \mathcal{H}$ *and a spectral measure* $E : \mathrm{Bor}(\Omega) \to \mathcal{B}(\mathcal{K})$ *such that* $F(S) = PE(S)|\mathcal{H}$ *for all* $S \in \mathrm{Bor}(\Omega)$, *where* P *is the orthogonal projection of* $\mathcal{K}$ *onto* $\mathcal{H}$.

We recall that a complex-valued map f on Ω is said to be a *Borel function* if $f^{-1}(A) \in \mathrm{Bor}(\Omega)$ for each $A \in \mathrm{Bor}(\mathbb{C})$. Let $B(\Omega)$ be the set (in fact, an algebra) of all Borel functions on Ω and $B^\infty(\Omega)$ the subset (in fact, a subalgebra) of all bounded functions from $B(\Omega)$.

Given a positive measure $F : \mathrm{Bor}(\Omega) \to \mathcal{B}(\mathcal{H})$ and a function $f \in B^\infty(\Omega)$, we can construct, as in the scalar case, the "integral" of the function f with respect to the measure F, which is an element of $\mathcal{B}(\mathcal{H})$ denoted by $\int_\Omega f \,\mathrm{d}F$. Moreover, the map

$$B^\infty(\Omega) \ni f \longmapsto \int_\Omega f \,\mathrm{d}F \in \mathcal{B}(\mathcal{H}) \tag{1}$$

is linear, unital, contractive and involutive. If, in addition, the measure F is a spectral measure, the map (1) is also multiplicative. As both $B^\infty(\Omega)$ and $\mathcal{B}(\mathcal{H})$ are C^*-algebras, the map (1) is therefore a unital C^*-algebra homomorphism, provided that F is a spectral measure. In particular, setting $N_f = \int_\Omega f \,\mathrm{d}F$ for some $f \in B^\infty(\Omega)$, the adjoint N_f^* of N_f is precisely $N_{\bar{f}}$ and we have that $N_f^* N_f = N_f N_f^*$ $(= N_{|f|^2})$. In other words, the operator N_f is normal. The converse is also true:

THEOREM 1.2. *Let* $N \in \mathcal{B}(\mathcal{H})$ *be a normal operator and let* $\sigma(N)$ *be the spectrum of* N. *There exists a unique spectral measure* $E_N : \mathrm{Bor}(\sigma(N)) \to \mathcal{B}(\mathcal{H})$ *such that*

$$N = \int_{\sigma(N)} z \,\mathrm{d}E_N(z)\,.$$

Theorem 1.2 is a fundamental result in operator theory, known as the *spectral theorem for normal operators* (see, for instance, [25]).

Normal operators form a class in $\mathcal{B}(\mathcal{H})$ whose members are fairly well understood. For this reason, mathematicians have tried to study some larger classes, related to that of normal operators. One of them is the class of subnormal operators, which will be discussed in the next section.

1.2. Subnormal operators It was Halmos (see [16]) who first isolated the class of subnormal operators as restrictions of normal operators to invariant subspaces (which are not necessarily normal). More precisely, we have the following:

Definition 1.2. An operator $S \in \mathcal{B}(\mathcal{H})$ is said to be *subnormal* if there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a normal operator $N \in \mathcal{B}(\mathcal{K})$ such that $S = N|\mathcal{H}$.

In particular, $\mathcal{H}$ is invariant under N and $S^k = N^k|\mathcal{H}$ for all integers $k \geq 1$.

Example 1.1. One of the simplest yet interesting example of a subnormal operator is the following. Let $\mathbb{T}$ be the unit circle in the complex plane and let $L^2(\mathbb{T})$ be the Hilbert of all square integrable complex-valued functions with respect to the normalized Lebesgue measure on $\mathbb{T}$. The (multiplication) operator N on $L^2(\mathbb{T})$, given by $Nf(\zeta) = \zeta f(\zeta)$, is normal, as it is easily seen. The Hardy space $H^2(\mathbb{D})$, consisting of those functions from $L^2(\mathbb{T})$ having an analytic extension to unit disk in the complex plane $\mathbb{D}$, is a closed subspace of $L^2(\mathbb{T})$, invariant under N. Thus the operator $S = N|H^2(\mathbb{D})$ is a subnormal operator which is non trivial in the sense that S is not normal.

The next result is a (very elegant and useful) criterion of subnormality, due to Halmos and Bram (see [10]).

Theorem 1.3. *An operator $S \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if*

$$\sum_{j,k=0}^{n} \left\langle S^j x_k, S^k x_j \right\rangle \geq 0 \tag{HB}$$

for all finite collections of vectors $x_0, x_1, \ldots, x_n \in \mathcal{H}$.

The necessity of the (HB) condition is easily obtained. Indeed, it is clear that

$$\sum_{j,k=0}^{n} \left\langle S^j x_k, S^k x_j \right\rangle = \sum_{j=0}^{n} \left\| N^{j*} x_j \right\|^2 \geq 0\,,$$

where N is any normal extension of S. The sufficiency is much more elaborated (see [16] or [13]).

Another criterion of subnormality, due to M. Embry (see [13]), is expressed in terms of positive measures. Specifically, we have the following:

THEOREM 1.4. *An operator $S \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if there exists a positive measure $F : \mathrm{Bor}\big([0, \|S\|]\big) \to \mathcal{B}(\mathcal{H})$ such that*

$$S^{*n} S^n = \int_0^{\|S\|} t^n \, \mathrm{d}F(t)$$

for all integers $n \geq 0$.

With the terminology from the theory of operator moment problems (see [19], [22], [31], ...) the operator S is subnormal if and only if the sequence $(S^{*n} S^n)_{n\geq 0}$ is a Hausdorff operator moment sequence.

The result of Halmos-Bram (Theorem 1.3) has been extended to arbitrary families of operators. To exhibit such a result, due to Ito (see [17]), let Γ be a fixed (non-null) abelian semi-group. A *representation* of Γ in $\mathcal{B}(\mathcal{H})$ is a map $\Gamma \ni \gamma \mapsto S_\gamma \in \mathcal{B}(\mathcal{H})$ such that $S_{\gamma_1} S_{\gamma_2} = S_{\gamma_1+\gamma_2}$ for all $\gamma_1, \gamma_2 \in \Gamma$ and $S_0 = I$. Such a representation is said to satisfy the *Halmos-Bram-Ito-condition* (briefly (HBI)-condition) if

$$\sum_{j,k=1}^{n} \big\langle S_{\gamma_j} x_k, S_{\gamma_k} x_j \big\rangle \geq 0$$

for all finite collections $x_1, \ldots, x_n \in \mathcal{H}$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$.

A representation $\Gamma \ni \gamma \mapsto S_\gamma \in \mathcal{B}(\mathcal{H})$ is said to have a *normal extension* if there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a representation $\Gamma \ni \gamma \mapsto N_\gamma \in \mathcal{B}(\mathcal{K})$ such that N_γ is normal and $S_\gamma = N_\gamma | \mathcal{H}$ for all $\gamma \in \Gamma$.

A version of the Halmos-Bram theorem, due to Ito, sounds like that:

THEOREM 1.5. *A representation $\Gamma \ni \gamma \mapsto S_\gamma \in \mathcal{B}(\mathcal{H})$ has a normal extension if and only if it satisfies the* (HBI)*-condition.*

An interesting application of the Halmos-Bram criterion is the following result, due to A. Atzmon (see [7]):

THEOREM 1.6. *Let $(\alpha_{m,n})_{m,n\geq 0}$ be a double sequence of complex numbers. There exists a (scalar) positive measure on the closed unit disk $\overline{\mathbb{D}}$ such that*

$$\alpha_{m,n} = \int_{\overline{\mathbb{D}}} z^m \bar{z}^n, \qquad m, n \geq 0$$

if and only if

$$\sum_{m,n,j,k\geq 0} \alpha_{m+j,n+k} \, c_{n,j} \, \bar{c}_{m,k} \ \geq 0$$

for every double sequence $(c_{n,j})_{n,j\geq 0}$ *with finite support, and*

$$\sum_{m,n\geq 0} \bigl(\alpha_{m,n} - \alpha_{m+1,n+1}\bigr) w_m \bar{w}_n \geq 0$$

for every sequence $(w_n)_{n\geq 0}$ *with finite support.*

This is, in fact a solution of the moment problem in the closed unit disk $\overline{\mathbb{D}}$ (see the next section for formal definitions), whose proof, as given by A. Atzmon, uses Theorem 1.3.

1.3. Moment problems in semi-algebraic sets Let $t = (t_1, \ldots, t_n)$ denote the variable in the real Euclidean space $\mathbb{R}^n$, and let $\mathcal{P}_n$ be the algebra of all polynomial functions in $t_1, \ldots, t_n$, with complex coefficients. Let also $\mathbb{Z}_+^n$ be the set of all n-tuples of nonnegative integers (i.e., multi-indices). If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ is an arbitrary multi-index, we put $t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$.

Let $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ be an n-sequence of real numbers. We set

$$L_\gamma(t^\alpha) = \gamma_\alpha\,, \qquad \alpha \in \mathbb{Z}_+^n\,, \tag{2}$$

and extend L_γ to $\mathcal{P}_n$ by linearity.

The n-sequence $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ is said to be *positive semi-definite* if L_γ is positive semi-definite (that is, $L_\gamma(p\bar{p}) \geq 0$ for all $p \in \mathcal{P}_n$).

Let $K \subset \mathbb{R}^n$ be a closed set. The n-sequence γ is said to be a K-*moment sequence* [9] if there exists a positive Borel measure μ on K such that $t^\alpha \in L^1(\mu)$ and $\gamma_\alpha = \int_K t^\alpha \,\mathrm{d}\mu(t)$ for all $\alpha \in \mathbb{Z}_+^n$. When such a measure μ exists, then it is called a *representing measure* of the sequence γ.

To solve the K-*moment problem* means to characterize those n-sequences of real numbers $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ which possess a representing measure on K (see [9]).

Let $\mathcal{P} = \{p_1, \ldots, p_m\}$ be a finite family in $\mathcal{P}_n$ consisting of polynomials with real coefficients, and let

$$K_{\mathcal{P}} = \bigl\{s \in \mathbb{R}^n \,:\, p_j(s) \geq 0,\ j = 1, \ldots, m\bigr\}. \tag{3}$$

A closed subset $K \subset \mathbb{R}^n$ will be called (in this text) *semi-algebraic* if there exists a family $\mathcal{P}$ such that $K = K_{\mathcal{P}}$.

Fix $\mathcal{P} = \{p_1, \ldots, p_m\}$ a finite family in $\mathcal{P}_n$, as above. Suppose that $K = K_{\mathcal{P}}$ is compact. In this case, with no loss of generality we may and shall assume that $0 \leq p(s) \leq 1$ for all $p \in \mathcal{P}_n$ and $s \in K$. We also assume that $\{0, 1\} \subset \mathcal{P}$.

We denote by $\Delta_{\mathcal{P}}$ the set of all products of the form

$$q_1 \cdots q_k (1 - r_1) \cdots (1 - r_l)$$

for polynomials $q_1, \ldots, q_k, r_1, \ldots, r_l \in \mathcal{P}$ and integers $k, l \geq 1$.

We clearly have $p|K \geq 0$ for all $p \in \Delta_{\mathcal{P}}$. Note also that the set $\Delta_{\mathcal{P}}$ can be explicitly constructed in terms of $\mathcal{P}$.

The following assertion has been proved by the author of this text (see [33] and [35]; see also [11], [23], [27], ... for related results).

Theorem 1.7. *Suppose that $K = K_{\mathcal{P}}$ is compact and that the family $\mathcal{P}$ generates the algebra $\mathcal{P}_n$.*

An n-sequence of real numbers $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ is a K-moment sequence if and only if the linear form L_γ is nonnegative on the set $\Delta_{\mathcal{P}}$.

In addition, the representing measure of a K-moment sequence is uniquely determined.

Corollary. *Let $\mathcal{P}$ and $K = K_{\mathcal{P}}$ be as in the previous statement. Let also $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ be a K-moment sequence, and let μ be the representing measure of γ. Assume that there exists an $r \in P(\mathbb{R}^n)$ such that $L_\gamma(rp) \geq 0$ for all $p \in \Delta_{\mathcal{P}}$. Then*

$$\operatorname{supp}(\mu) \subset \big\{ s \in K \,:\, r(s) \geq 0 \big\}.$$

If $L_\gamma(rp) = 0$ for some $r \in P(K)$ and for all $p \in \Delta_{\mathcal{P}}$, then

$$\operatorname{supp}(\mu) \subset \big\{ s \in K \,:\, r(s) = 0 \big\}.$$

Remarks 1.2. (1) There are versions of the previous results stated for a family $\mathcal{P}$ consisting of polynomials with complex coefficients (see [35]).

(2) Most of the classical results concerning the moment problems (e.g., the Hausdorff moment problem, the trigonometric moment problems etc., in one or several variables (see also [2], [9], ...) are particular cases of these results.

(3) Results concerning the decomposition of positive polynomials on semi-algebraic compact sets can be obtained as applications of the moment results from above (see [33] and [35]).

1.4. Moments and subnormality That there exists a strong connection between the moment problem and subnormality has been known for a longtime (see [31], [13], [1], [7], [6], to quote only a few).

In this section, we present some statements concerning the subnormality of certain (multi)operators, as well as some operator moment problems, obtained via the results from the previous section (see also [6]).

As in the first two sections, let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators acting on $\mathcal{H}$.

If $T = (T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{H})^n$ is a commuting multioperator (briefly, a c.m.), for every $p \in \mathcal{P}_{2n}$, $p(\overline{z}, z) = \sum_{\alpha,\beta} c_{\alpha,\beta}\, \overline{z}^\alpha z^\beta$, we set

$$p(T^*, T) = \sum_{\alpha,\beta} c_{\alpha,\beta}\, T^{*\alpha} T^\beta, \tag{4}$$

with $T^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$ for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$.

We recall that a c.m. $T \in \mathcal{B}(\mathcal{H})^n$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a c.m. $N \in \mathcal{L}(\mathcal{K})^n$ consisting of normal operators (which is called a *normal extension* of T) such that $T_j = N_j \mid \mathcal{H}$, $j = 1, \ldots, n$. Among all normal extensions of a subnormal c.m. T, there exists a minimal one, which is unique up to unitary equivalence. In that case one also have $\|T_j\| = \|N_j\|$, $j = 1, \ldots, n$ (see [17] for details).

Let $K = K_{\mathcal{P}}$ be a semi-algebraic compact subset of $\mathbb{R}^n$ (see the previous section). Let also τ be the mapping

$$\mathbb{C}^n \ni z = (z_1, \ldots, z_n) \;\longmapsto\; \big(|z_1|^2, \ldots, |z_n|^2\big) \in \mathbb{R}^n. \tag{5}$$

Note that the set $\tau^{-1}(K) \subset \mathbb{C}^n$ is also compact. With this notation, we have the following (see [33, Theorem 3.1]):

Theorem 1.8. *The commuting multioperator $T \in \mathcal{B}(\mathcal{H})^n$ has a normal extension $N \in \mathcal{B}(\mathcal{K})^n$ ($\mathcal{K} \supset \mathcal{H}$), whose joint spectrum lies in $\tau^{-1}(K)$, if and only if $(p \circ \tau)(T^*, T) \geq 0$ for all $p \in \Delta_{\mathcal{P}}$.*

The next result is an enlarged version of [33, Theorem 3.4]. The method of proof is similar to that for Theorem 1.8.

Theorem 1.9. *Let $\Gamma = (\Gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ be a sequence of bounded self-adjoint operators on $\mathcal{H}$, with $\Gamma_0 = I$. Let also $L_\Gamma : \mathcal{P}_n \to \mathcal{B}(\mathcal{H})$ be the mapping*

$$L_\Gamma(p) = \sum_\alpha c_\alpha \Gamma_\alpha \qquad \text{if} \quad p(t) = \sum_\alpha c_\alpha t^\alpha\,.$$

Then there exists a uniquely determined operator-valued positive measure F_Γ on K such that $L_\Gamma(p) = \int_K p\,\mathrm{d}F_\Gamma$ for all $p \in P(K)$ if and only if $L_\Gamma(p) \geq 0$ for all $p \in \Delta_{\mathcal{P}}$.

In the affirmative case, assume, moreover, that there exists an $r \in \mathcal{P}_n$ with real coefficients such that $L_\Gamma(rp) \geq 0$ for all $p \in \Delta_{\mathcal{P}}$. Then

$$\operatorname{supp}(F_\Gamma) \subset \big\{s \in K \,:\, r(s) \geq 0\big\}.$$

If $L_\Gamma(rp) = 0$ for some $r \in \mathcal{P}_n$ and for all $p \in \Delta_{\mathcal{P}}$, then

$$\operatorname{supp}(F_\Gamma) \subset \big\{s \in K \,:\, r(s) = 0\big\}.$$

Remark 1.1. Theorem 1.9 contains, as a particular case, the following classical result of [31]:

A sequence $(\Gamma_k)_{k\in\mathbb{Z}_+}$ in $\mathcal{B}(\mathcal{H})$ can be represented as

$$\Gamma_k = \int_0^1 t^k\,\mathrm{d}F(t),\ k \geq 0\,,$$

for a certain operator-valued positive measure F on $[0,1]$, if and only if

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} \Gamma_{j+k} \geq 0$$

for all integers $m, k \geq 0$. See also [19] for further connections.

A consequence of Theorem is the following (see also [33, Theorem 3.4]):

Corollary. *Let $T \in \mathcal{B}(\mathcal{H})^n$ be a subnormal c.m., and assume that the support of the representing measure F_T of T is contained in the semi-algebraic compact set $K = K_{\mathcal{P}}$.*

If there exists an $r \in \mathcal{P}_n$ with real coefficients such that $((pr)\circ\tau)(T^*, T) \geq 0$ for all $p \in \Delta_{\mathcal{P}}$, then

$$\operatorname{supp}(F_T) \subset \big\{s \in K \,:\, r(s) \geq 0\big\}.$$

If there exists an $r \in \mathcal{P}_n$ such that $(r \circ \tau)(T^*, T) = 0$, then one also has $(r \circ \tau)(N^*, N) = 0$, where N is the minimal normal extension of T.

2. Unbounded subnormal operators

2.1. Unbounded normal operators The results and concepts from this section are classical. More details can be found in [12] or [25] (see also [26]).

Let $\mathcal{H}$ be a fixed complex Hilbert space. We consider linear transformations of the form $T : D(T) \subset \mathcal{H} \to \mathcal{H}$, where $D(T)$ is the *domain* (of *definition*) of T. Let also $R(T) = T(D(T))$ and $G(T) = \{(x, Tx) \in \mathcal{H} \times \mathcal{H} : x \in D(T)\}$ be the *range* and the *graph* of T, respectively. If $\overline{D(T)} = \mathcal{H}$, the operator T is said to be *densely defined*. If $\overline{G(T)} = G(T)$, the operator T is said to be *closed*. If $\overline{G(T)}$ is the graph of an operator $\bar{T}$, then T is said to be *closable* and $\bar{T}$ (which extends T) is the *closure* of T.

Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be densely defined. One defines the *adjoint* T^* of T in the following way. The domain of T^* is given by

$$D(T^*) = \big\{y \in \mathcal{H} : \exists\, M_y \geq 0 \text{ such that } |\langle Tx, y\rangle| \leq M_y\|x\|,\ x \in D(T)\big\}.$$

For each $y \in D(T^*)$, the linear functional $f_y(x) = \langle Tx, y\rangle$, $x \in D(T)$, has a bounded extension to $\mathcal{H}$, thus by a classical theorem by Riesz there exists a (unique) vector $y_* \in \mathcal{H}$ such that $\langle Tx, y\rangle = \langle x, y_*\rangle$ for all $x \in D(T)$. We set $T^*(y) = y_*$, which is linear on $D(T^*)$. The operator $T^* : D(T^*) \subset \mathcal{H} \to \mathcal{H}$ is called the *adjoint* of T.

The next three results are classical (see [12] and/or [25]).

Theorem 2.1. *If the operator $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ is densely defined and closed, then its adjoint $T^* : D(T^*) \subset \mathcal{H} \to \mathcal{H}$ is also densely defined and closed. Moreover, $T^{**} = T$.*

Definition 2.1. A densely defined closed operator N is said to be *normal* if $NN^* = N^*N$.

Although this definition is formally the same as the corresponding one for bounded operators, it explicitly means that

$$\begin{aligned} D(NN^*) &= \big\{x \in D(N^*) : N^*x \in D(N)\big\} \\ &= \big\{x \in D(N) : Nx \in D(N^*)\big\} = D(N^*N) \end{aligned}$$

and $NN^*x = N^*Nx$ for all $x \in D(NN^*) = D(N^*N)$.

Theorem 2.2. *A densely defined closed operator N is normal if and only if $D(N^*) = D(N)$ and $\|N^*x\| = \|Nx\|$ for all $x \in D(T)$.*

The most important property of a normal (not necessarily bounded) operator is the existence of an associated spectral measure (already mentioned in the bounded case; see Theorem 1.2).

THEOREM 2.3. *Let $N : D(N) \subset \mathcal{H} \to \mathcal{H}$ be a normal operator. There exists a uniquely determined spectral measure $E : \mathrm{Bor}(\mathbb{C}) \to \mathcal{B}(\mathcal{H})$ such that*

$$\langle Nx, y\rangle = \int_{\mathbb{C}} z \,\mathrm{d}E_{x,y}(z)\,, \qquad x \in D(N)\,,\ y \in \mathcal{H}\,.$$

In fact, defining $\sigma(N)$ as the set of those $z \in \mathbb{C}$ for which $zI - N$ has not a bounded inverse, which is the *spectrum* of N, one can prove that the support of the spectral measure E in Theorem 2.3 is precisely $\sigma(N)$.

2.2. INSUFFICIENCY OF THE HALMOS-BRAM CONDITION As in the bounded case, a linear transformation $S : D(S) \subset \mathcal{H} \to \mathcal{H}$ is said to be *subnormal* when there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a normal operator $N : D(N) \subset \mathcal{K} \to \mathcal{K}$ such that $D(S) \subset D(N)$ and $Sx = Nx$ for all $x \in D(S)$. A natural question is the following:

PROBLEM. Characterize the unbounded normal operators.

As we have seen, the Halmos-Bram condition is necessary and sufficient for the subnormality of a given bounded operator. This condition can be stated for unbounded transformations in the following way:

$$\sum_{j,k=0}^{n} \left\langle S^j x_k, S^k x_j \right\rangle \geq 0 \tag{HBU}$$

for all finite collections of vectors $x_0, x_1, \ldots, x_n \in D^\infty(S)$, where

$$D^\infty(S) = \bigcap_{k\geq 0} D\big(S^k\big)\,.$$

Although certainly necessary, the condition (HBU) is far from being sufficient for the subnormality of a given linear (unbounded) transformation. The following example, showing the insufficiency of the (HBU)-condition for subnormality, has been given by J. Stochel and F.H. Szafraniec (see [29]; see also [28] and [30] for other details concerning unbounded subnormal operators).

Example 2.1. Let $\gamma = (\gamma_{m,n})_{m,n\geq 0}$ be a positive semi-definite (see Section 1.3 for this and other details) double sequence of complex numbers. In the space of polynomials in two variables $\mathcal{P}_2$, we define the semi-inner product $\langle p, q\rangle_0 = L_\gamma(p\bar{q})$ (a classical idea which goes back to Gelfand and Naimark; see [15]). Put $\mathcal{I} = \{q \in \mathcal{P}_2 : \langle q, q\rangle_0 = 0\}$, which is an ideal in $\mathcal{P}_2$. Therefore, $\mathcal{H}_0$ is a $\mathcal{P}_2$-module. Moreover, setting $\langle \tilde{p}, \tilde{q}\rangle = \langle p, q\rangle_0$ for all $p, q \in \mathcal{P}_2$, where $\tilde{p} = p + \mathcal{I}$, $\mathcal{H}_0$ becomes an inner product space, whose completion will be denoted by $\mathcal{H}$.

We consider in $\mathcal{H}$ the operator $S : \mathcal{H}_0 \to \mathcal{H}$, given by $S\tilde{p} = zp + \mathcal{I}$, where $z = s + it$ is the complex variable and s, t the corresponding real variables. In fact, $\mathcal{H}_0$ is invariant under S, and so we have $D^\infty(S) = \mathcal{H}_0$, implying $D^\infty(S)$ dense in $\mathcal{H}$. In addition, $\mathcal{H}_0$ is algebraically generated by the set $\{S^k(1 + \mathcal{I}) : k \geq 0\}$, and we may say that $\tilde{1} = 1 + \mathcal{I}$ is a "cyclic vector" for S.

Assume now that S would have a normal extension N in a Hilbert space $\mathcal{K} \supset \mathcal{H}$. If E is the spectral measure on N, we would have $S^k(\tilde{1}) = N^k(\tilde{1})$ for all $k \geq 0$. Hence

$$\langle z^m, z^n\rangle_0 = \langle S^m\tilde{1}, S^n\tilde{1}\rangle = \int_{\mathbb{C}} z^m\bar{z}^n \,\mathrm{d}E_{\tilde{1},\tilde{1}}(z)$$

for all integers $m, n \geq 0$. This clearly implies that

$$\gamma_{m,n} = \int_{\mathbb{C}} s^m t^n \,\mathrm{d}\mu$$

for all integers $m, n \geq 0$, where $\mu = E_{\tilde{1},\tilde{1}}$, showing that γ is a moment sequence. But choosing as γ a positive semi-definite sequence which is not a moment sequence (see, for instance, [9] for the existence of such a sequence), we are led to a contradiction. On the other hand, with γ only a positive semi-definite sequence, and choosing $\tilde{q}_0, \ldots, \tilde{q}_n \in \mathcal{H}_0$ arbitrary, a simple calculation shows that

$$\sum_{j,k=0}^{n} \left\langle S^j\tilde{q}_k, S^k\tilde{q}_j\right\rangle = \left\|\sum_{k\geq 0} \bar{z}^k q_k\right\|_0^2 \geq 0\,,$$

which is precisely the (HBU)-condition for S.

2.3. Moments in unbounded sets via algebras of fractions We first present some facts from [36] (see also [3] and [37]).

Let Ω be a compact Hausdorff space and let $C(\Omega)$ be the algebra of all complex-valued continuous functions on Ω, endowed with the natural norm $\|f\|_\infty = \sup_{\omega\in\Omega}|f(\omega)|$, $f \in C(\Omega)$. It is well known that every positive linear functional on $C(\Omega)$ has an integral representation. Specifically, if $\psi : C(\Omega) \to \mathbb{C}$ is linear and positive, then there exists a uniquely determined positive measure μ on Ω such that $\psi(f) = \int_\Omega f \,\mathrm{d}\mu$, $f \in C(\Omega)$. As a matter of fact, if $\psi : C(\Omega) \to \mathbb{C}$ is linear, then ψ is positive if and only if ψ is continuous and $\|\psi\| = \psi(1)$.

These features of positive linear functionals can be partially or totally recaptured in more general spaces, derived from the basic model $C(\Omega)$.

Let $\mathcal{Q}$ be a family of non-null positive elements of $C(\Omega)$. We say that $\mathcal{Q}$ is a *set of denominators* if: (i) $1 \in \mathcal{Q}$, (ii) $q', q'' \in \mathcal{Q}$ implies $q'q'' \in \mathcal{Q}$, and (iii) if $qh = 0$ for some $q \in \mathcal{Q}$ and $h \in C(\Omega)$, then $h = 0$.

Let $C(\Omega)/\mathcal{Q}$ denote the algebra of fractions with numerators in $C(\Omega)$, and with denominators in the family $\mathcal{Q}$, which is a unital $\mathbb{C}$-algebra (see, for instance, [32] for details). This algebra has a natural involution $f \to \bar{f}$, induced by the natural involution of $C(\Omega)$.

To define a natural topological structure on $C(\Omega)/\mathcal{Q}$, we note that for every $f \in C(\Omega)/\mathcal{Q}$ we can find a $q \in \mathcal{Q}$ such that $qf \in C(\Omega)$. If

$$C(\Omega)/q = \{f \in C(\Omega)/\mathcal{Q} : qf \in C(\Omega)\},$$

then we have $C(\Omega)/\mathcal{Q} = \cup_{q\in\mathcal{Q}} C(\Omega)/q$. Setting $\|f\|_{\infty,q} = \|qf\|_\infty$ for each $f \in C(\Omega)/q$, the pair $(C(\Omega)/q, \| * \|_{\infty,q})$ becomes a Banach space. For this reason, $C(\Omega)/\mathcal{Q}$ can be naturally regarded as an inductive limit of Banach spaces (see [24, Section V.2]).

Giving two elements $q, q' \in \mathcal{Q}$, we say that q' *divides* q (in $\mathcal{Q}$) if there exists $q'' \in \mathcal{Q}$ such that $q = q'q''$.

In each space $C(\Omega)/q$ we have a positive cone $(C(\Omega)/q)^+$ consisting of those elements $f \in C(\Omega)/q$ such that $qf \geq 0$ as a continuous function. We say that $f \in C(\Omega)/q$ is *positive* if $f \in (C(\Omega)/q)^+$.

The positive elements of the algebra $C(\Omega)/\mathcal{Q}$ are, by definition, the members of the cone $(C(\Omega)/\mathcal{Q})^+$, consisting of all finite sums of positive elements from the cones $(C(\Omega)/q)^+$, with $q \in \mathcal{Q}$ arbitrary. The positivity of a linear map on $C(\Omega)/\mathcal{Q}$ will be defined with respect to the positive cone $(C(\Omega)/\mathcal{Q})^+$.

In fact, a linear map $\psi : C(\Omega)/\mathcal{Q} \to \mathbb{C}$ is positive if and only if $\psi|(C(\Omega)/q)^+$ is positive for all $q \in \mathcal{Q}$.

Example 2.2. Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and let $\Omega = (\mathbb{C}_\infty)^n$, for a fixed integer $n \geq 1$, which is a compact Hausdorff space.

Let $\mathcal{P}_{2n}$ be the algebra of all polynomials in $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$, where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ is the current variable. (In fact, $\mathcal{P}_{2n}$ can be identified with the algebra of polynomials in $2n$ real variables and complex coefficients, and so this notation is compatible with that from the previous chapter; see also Section 1.4. For every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we define

$$r_\alpha(z) = \left(1 + |z_1|^2\right)^{\alpha_1} \cdots \left(1 + |z_n|^2\right)^{\alpha_n} \in \mathcal{P}_{2n}$$

and $q_\alpha(z) = 1/r_\alpha(z)$, $z \in \mathbb{C}^n$. We note that each function q_α has a unique continuous extension to Ω, by setting $q_\alpha(z) = 0$ for all $z \in \Omega \setminus \mathbb{C}^n$. Moreover, if $h \in C(\Omega)$ and $h(z)q_\alpha(z) = 0$ for all $z \in \mathbb{C}^n$ and some α, then $h = 0$, since $\mathbb{C}^n$ is dense in Ω. For this reason, it is clear that the set $\mathcal{Q}_{2n} = \left\{q_\alpha : \alpha \in \mathbb{Z}_+^n\right\}$ is a set of denominators, and we can form the algebra of fractions $C(\Omega)/\mathcal{Q}_{2n}$.

If $\xi, \eta \in \mathbb{Z}_+^n$ are given, let $\alpha \in \mathbb{Z}_+^n$ be such that $2\alpha_j > \xi_j + \eta_j$ when $\xi_j \neq \eta_j$, and $\alpha_j \geq \xi_j$ when $\xi_j = \eta_j$. In this case, the rational function

$$\frac{z^\xi \bar{z}^\eta}{r_\alpha(z)} = \prod_{j=1}^n \frac{z_j^{\xi_j} \bar{z}_j^{\eta_j}}{(1 + |z_j|^2)^{\alpha_j}}$$

has a (unique) continuous extension to Ω.

For a fixed $\alpha \in \mathbb{Z}_+^n$, let $\mathcal{P}_{2n,\alpha}$ be the linear space generated by the monomials $z^\xi \bar{z}^\eta$, with $\xi, \eta \in \mathbb{Z}_+^n$ satisfying the relations from above. For each $p \in \mathcal{P}_{2n,\alpha}$, the function p/r_α has a unique continuous extension to Ω, say h, allowing us to write $p(z) = h(z)/q_\alpha(z)$, $z \in \mathbb{C}^n$. In this way, the space $\mathcal{P}_{2n,\alpha}$ can be identified with a subspace of $C(\Omega)/q_\alpha$, and so $\mathcal{P}_{2n}$ is a subalgebra of the algebra of fractions $C(\Omega)/\mathcal{Q}_{2n}$.

The *Hamburger moment problem* in $\mathbb{C}^n$ means to characterize those multisequences $\gamma = (\gamma_{\xi,\eta})_{\xi,\eta \in \mathbb{Z}_+^n}$ for which there exists a positive measure μ on $\mathbb{C}^n$ with the property

$$\gamma_{\xi,\eta} = \int_{\mathbb{C}^n} z^\xi \bar{z}^\eta \, \mathrm{d}\mu(z)\,, \qquad \xi, \eta \in \mathbb{Z}_+^n.$$

This is equivalent to characterizing those linear forms $L : \mathcal{P}_{2n} \to \mathbb{C}$ for which there exists a positive measure μ on $\mathbb{C}^n$ such that $L(p) = \int_{\mathbb{C}^n} p \, \mathrm{d}\mu$ for all $p \in \mathcal{P}_{2n}$.

The positivity of linear forms on the algebra of fractions can be described in the following way (see [36, Theorem 2.4]):

THEOREM 2.4. *A linear map $\psi : C(\Omega)/\mathcal{Q}_{2n} \to \mathbb{C}$ is positive if and only if*

$$\|\psi_q\| = \psi(q^{-1}), \qquad q \in \mathcal{Q}_{2n},$$

where $\psi_q = \psi|C(\Omega)/q$.

If $\psi : C(\Omega)/\mathcal{Q}_{2n} \to \mathbb{C}$ is positive, there exists a positive measure μ on $\mathbb{C}^n$ such that $1/q$ is μ-integrable for all $q \in \mathcal{Q}_{2n}$ and $\psi(f) = \int_{\mathbb{C}^n} f \, \mathrm{d}\mu$ for all $f \in C(\Omega)/\mathcal{Q}_{2n}$.

Giving a solution to the Hamburger moment problem on $\mathbb{C}^n$ amounts to extending the linear functionals on $\mathcal{P}_{2n}$ to positive linear functionals on $C(\Omega)/\mathcal{Q}_{2n}$. The next result is an assertion in this sense (see [36, Theorem 3.7]).

THEOREM 2.5. *Let $\phi : \mathcal{P}_{2n} \to \mathbb{C}$ be linear. There exists a positive extension $\psi : C(\Omega)/\mathcal{Q}_{2n} \to \mathbb{C}$ of ϕ such that $\|\psi_\alpha\| = \|\phi_\alpha\|$ for all $\alpha \in \mathbb{Z}_+^n$, with $\phi_\alpha = \phi|\mathcal{P}_{2n,\alpha}$ and $\psi_\alpha = \psi|C(\Omega)/q_\alpha$, if and only if $\|\phi_\alpha\| = \phi(q_\alpha^{-1})$ for all $\alpha \in \mathbb{Z}_+^n$.*

That the condition from the statement is necessary can be easily obtained, as follows.

If $\phi(p) = \int_{\mathbb{C}^n} p \, \mathrm{d}\mu$ for all $p \in \mathcal{P}_{2n,\alpha}$ we have

$$|\phi(p)| \le \int_{\mathbb{C}^n} \left| \frac{p}{r_\alpha} \right| \mathrm{d}\mu \le \|p\|_\alpha \phi(r_\alpha),$$

whence $\|\phi_\alpha\| \le \phi(r_\alpha)$. But $r_\alpha \in \mathcal{P}_{2n,\alpha}$ and $\|r_\alpha\|_{\infty,q_\alpha} = 1$, and so $\|\phi_\alpha\| = \phi(r_\alpha)$.

2.4. POSITIVE MAPS ON ALGEBRAS OF FRACTIONS As before, let Ω be a compact Hausdorff space and let $\mathcal{H}$ be a complex Hilbert space. The next two results are essentially due to Arveson (see [4]).

THEOREM 2.6. *Let $\Psi : C(\Omega) \to \mathcal{B}(\mathcal{H})$ be linear, positive and unital. Then Ψ is completely positive and completely contractive.*

Let us explain the meaning of this statement. We know that Ψ is linear, $\Psi(f) \ge 0$ for all positive $f \in C(\Omega)$ and $\Psi(1) = I$. If $M_n(C(\Omega))$ is the algebra of all $n \times n$-matrices with entries in $C(\Omega)$, and if $\Psi_n : M_n(C(\Omega)) \to \mathcal{B}(\mathcal{H}^n)$ (where $\mathcal{H}^n$ is a direct sum of n copies of $\mathcal{H}$) is given by

$$\Psi_n((f_{jk})) = (\Psi(f_{jk})), \qquad (f_{jk}) \in M_n(C(\Omega)),$$

the hypothesis of the theorem from above implies that $\Psi_n \geq 0$ and $\|\Psi_n\| \leq 1$ for all integers $n \geq 1$. In other words, with the terminology from [4], the map Φ is completely positive and completely contractive.

THEOREM 2.7. *Let $\mathcal{M} \subset C(\Omega)$ be a linear subspace with $1 \in \mathcal{M}$. If $\Phi : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ is a unital, complete contraction, there exists a (completely) positive map $\Psi : C(\Omega) \to \mathcal{B}(\mathcal{H})$ which extends Φ.*

The hypothesis means that $\Phi(1) = I$ and $\|(\Phi(f_{jk}))\|_n \leq 1$ for all $(f_{jk}) \in M_n(C(\mathcal{M}))$ and $n \geq 1$, where the norm $\|(\Phi(f_{jk}))\|_n$ is computed in $\mathcal{B}(\mathcal{H}^n)$. This hypothesis implies the existence of a $\Psi : C(\Omega) \to \mathcal{B}(\mathcal{H})$ which extends Φ, having the properties from Theorem 2.6.

We fix now a set of denominators $\mathcal{Q} \subset C(\Omega)$ and a dense linear subspace $\mathcal{D} \subset H$. Let $\mathrm{SF}(\mathcal{D})$ denote the space of all sesquilinear forms on $\mathcal{D}$. Let $\psi : C(\Omega)/\mathcal{Q} \to \mathrm{SF}(\mathcal{D})$ having the following properties:

$$\psi(1)(x,y) = \langle x, y\rangle\ , \qquad x, y \in \mathcal{D}$$

and

$$\psi(f)(x,x) \geq 0\,, \qquad f \in C(\Omega)/\mathcal{Q}\,,\ \ f \geq 0\,,\ \ x \in \mathcal{D}\,.$$

We shall briefly say that such a map ψ is *unital* and *positive*.

We also consider a subspace $\mathcal{F} = \sum_{q\in\mathcal{Q}} \mathcal{F}_q \subset C(\Omega)/\mathcal{Q}$ with $\mathcal{F}_q \subset C(\Omega)/q$ for all $q \in \mathcal{Q}$, and a linear map $\phi : \mathcal{F} \to \mathrm{SF}(\mathcal{D})$, which is *unital*, that is, $\phi(1)(x,y) = \langle x,y\rangle$, $x, y \in \mathcal{D}$ (as above). We want to extend such a map ϕ to a map $\psi : C(\Omega)/\mathcal{Q} \to \mathrm{SF}(\mathcal{D})$, which should be positive. The main reason for having such an extension is that positivity of ψ implies the existence of a positive measure $F : \mathrm{Bor}(\Omega) \to \mathcal{B}(\mathcal{H})$ such that

$$\psi(f)(x,y) = \int_\Omega f \,\mathrm{d}F_{x,y}\,, \qquad f \in C(\Omega)/\mathcal{Q}\,,\ x, y \in \mathcal{D}$$

(see [3, Theorem 2.2]).

If $\phi : \mathcal{F} \to \mathrm{SF}(\mathcal{D})$ and $\psi : C(\Omega)/\mathcal{Q} \to \mathrm{SF}(\mathcal{D})$ are as above, we put $\phi_q = \phi|\mathcal{F}_q$, $\psi_q = \psi|C(\Omega)/q$, $\phi_{q,x} = \phi_q(*)(x,x)$ and $\psi_{q,x} = \psi_q(*)(x,x)$ for all $q \in \mathcal{Q}$ and $x \in \mathcal{D}$.

We have the following extension result (see [3, Theorem 2.5] for more general conditions):

THEOREM 2.8. *Let $\mathcal{F} = \sum_{q\in\mathcal{Q}} \mathcal{F}_q$ with $\mathcal{F}_q \subset C(\Omega)/q$ for all $q \in \mathcal{Q}$ and $1/q' \in \mathcal{F}_q$, for every $q' \in \mathcal{Q}$ which divides q in $\mathcal{Q}$. Let also $\phi : \mathcal{F} \to \mathrm{SF}(\mathcal{D})$ be linear and unital. The following statements are equivalent:*

(a) *The map* ϕ *extends to a unital positive map* $\psi : C(\Omega)/\mathcal{Q} \to \mathrm{SF}(\mathcal{D})$ *such that* $\|\psi_{q,x}\| = \|\phi_{q,x}\|$ *for all* $q \in \mathcal{Q}$ *and* $x \in \mathcal{D}$.

(b) $\phi\bigl(q^{-1}\bigr)(x,x) > 0$ *for* $q \in \mathcal{Q}$ *and* $x \in \mathcal{D} \setminus \{0\}$ *and, for all* $q \in \mathcal{Q}$, $x_1, \dots, x_n$, $y_1, \dots, x_n \in \mathcal{D}$ *with*

$$\sum_{j=1}^{n} \phi\bigl(q_j^{-1}\bigr)(x_j, x_j) \le 1\,,$$

$$\sum_{j=1}^{n} \phi\bigl(q_j^{-1}\bigr)(y_j, y_j) \le 1,$$

and all $(f_{jk}) \in M_n(\mathcal{F}_q)$ *with* $\|(qf_{jk})\|_{n,\infty} \le 1$, *we have*

$$\left|\sum_{j,k=1}^{n} \phi\bigl(f_{jk}\bigr)(x_k, y_j)\right| \le 1.$$

Let $T = (T_1, \dots, T_n)$ be a tuple of linear operators defined on a dense linear subspace $\mathcal{D} \subset H$. We also assume that $T_j\mathcal{D} \subset \mathcal{D}$ and $T_jT_kx = T_kT_jx$ for all $j,k = 1, \dots, n$ and $x \in \mathcal{D}$. We define a map $\phi_T : \mathcal{P}_{2n} \to \mathrm{SF}(\mathcal{D})$ by the equation

$$\phi_T\bigl(z^{\xi}\bar{z}^{\eta}\bigr)(x,y) = \left\langle T^{\xi}x, T^{\eta}y\right\rangle\,, \qquad \xi, \eta \in \mathbb{Z}_+^n\,, \quad x, y \in \mathcal{D}\,,$$

extended to $\mathcal{P}_{2n}$ by linearity.

As a consequence of Theorem 2.8, we give the following criterion of subnormality for a tuple T as above (see also [3, Theorem 3.4]):

Theorem 2.9. *The tuple* $T = (T_1, \dots, T_n)$ *admits a normal extension if and only if for all* $\alpha \in \mathbb{Z}_+^n$, $m \ge 1$, $x_1, \dots, x_n \in \mathcal{D}$ *and* $y_1, \dots, y_n \in \mathcal{D}$ *with*

$$\sum_{j=1}^{n} \phi_T(r_\alpha)(x_j, x_j) \le 1\,,$$

$$\sum_{j=1}^{n} \phi_T(r_\alpha)(y_j, y_j) \le 1\,,$$

and for all $p = (p_{jk}) \in M_m(\mathcal{P}_{2n,\alpha})$ *with* $\sup_{t \in \mathbb{C}^n} \|q_\alpha(t)p(t)\|_m \le 1$ *we have*

$$\left|\sum_{j,k=1}^{n} \phi_T\bigl(p_{jk}\bigr)(x_k, y_j)\right| \le 1\,.$$

Other criteria of subnormality for unbounded operators can be found in [30] (the case $n = 1$) and in [34], where the subject is treated via completely different methods.

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