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Description of Derivations on Locally Measurable Operator Algebras of Type I*

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Abstract: Given a type I von Neumann algebra M let $LS(M)$ be the algebra of all locally measurable operators affiliated with M. We give a complete description of all derivations on the algebra $LS(M)$. In particular, we prove that if M is of type I_∞ then every derivation on $LS(M)$ is inner.

Key words: von Neumann algebras, non commutative integration, measurable operator, locally measurable operator, Hilbert – Kaplansky module, type I algebra, derivation, inner derivation.

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1. Introduction

The present paper is devoted to a complete description of derivations on the algebra of locally measurable operators $LS(M)$ affiliated with a type I von Neumann algebra M.

Given a (complex) algebra A, a linear operator $D : A \to A$ is called a *derivation* if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$. Each element $a \in A$ generates a derivation $D_a : A \to A$ defined as $D_a(x) = ax - xa$, $x \in A$. Such derivations are called *inner* derivations.

It is well known that all derivation on a von Neumann algebra are inner and therefore are norm continuous. But the properties of derivations on the unbounded operator algebra $LS(M)$ seem to be very far from being similar. Indeed, the results of [2] and [5] show that in the commutative case where $M = L^\infty(\Omega, \Sigma, \mu)$, with (Ω, Σ, μ) any non atomic measure space with a finite

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measure μ, the algebra $LS(M) \cong L^0(\Omega, \Sigma, \mu)$ of all complex measurable functions on (Ω, Σ, μ) admits non zero derivations. It is clear that these derivations are discontinuous in the measure topology (i.e., the topology of convergence in measure), and thus are non inner. It seems that the existence of such pathological examples deeply depends on the commutativity of the underlying algebra M. Indeed, the main result of our previous paper [1] states that if M is a type I von Neumann algebra, then any derivation D on $LS(M)$, which is identically zero on the center Z of the von Neumann algebra M (i.e., D is Z-linear), is inner, i.e., $D(x) = ax - xa$ for an appropriate element $a \in LS(M)$.

In the mentioned paper [1] we have also constructed an example of a non inner derivation on the algebra $LS(M)$, where M is a homogeneous type I_n algebra $L^\infty(\Omega)\bar{\otimes} M_n(\mathbb{C})$. In this case $LS(M)$ coincides with the algebra $M_n(L^0(\Omega))$ of all $n \times n$ matrices over the algebra $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$. Namely, given any non zero derivation $\delta : L^0(\Omega) \to L^0(\Omega)$ and a matrix $(\lambda_{ij})_{i,j=1}^n \in M_n(L^0(\Omega))$, $\lambda_{ij} \in L^0(\Omega), i, j = \overline{1, n}$, put

$$D_\delta((\lambda_{ij})_{i,j=1}^n) = (\delta(\lambda_{ij}))_{i,j=1}^n.$$

Then it is clear that D_δ defines a derivation on $M_n(L^0(\Omega))$, which coincides with δ on the center of $M_n(L^0(\Omega))$.

In the present paper we prove that for type I von Neumann algebras the above construction (1) gives the general form of the pathological derivations and these only exist in type I_{fin} cases, while for type I_∞ von Neumann algebras M all derivation on $LS(M)$ are inner. Moreover we prove that an arbitrary derivation D on $LS(M)$ for a type I von Neumann algebra M, can be uniquely decomposed into the sum $D_a + D_\delta$ where the derivation D_a is inner and the derivation D_δ is of the form above. In such a decomposition δ is defined uniquely, and the element a is unique up to a central summand.

2. Preliminaries

Let (Ω, Σ, μ) be a measurable space and suppose that the measure μ has the direct sum property, i. e. there is a family $\{\Omega_i\}_{i \in J} \subset \Sigma$, $0 < \mu(\Omega_i) < \infty$, $i \in J$, such that for any $A \in \Sigma$, $\mu(A) < \infty$, there exist a countable subset $J_0 \subset J$ and a set B with zero measure such that $A = \bigcup\limits_{i \in J_0} (A \cap \Omega_i) \cup B$.

We denote by $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ the algebra of all (equivalence classes of) complex measurable functions on (Ω, Σ, μ) equipped with the topology of convergence in measure. Then $L^0(\Omega)$ is a complete commutative regular algebra with the unit $\mathbf{1}$ given by $\mathbf{1}(\omega) = 1$, $\omega \in \Omega$.

Recall that a net $\{\lambda_\alpha\}$ in $L^0(\Omega)$ (o)-converges to $\lambda \in L^0(\Omega)$ if there exists a net $\{\xi_\alpha\}$ monotone decreasing to zero such that $|\lambda_\alpha - \lambda| \leq \xi_\alpha$ for all α.

Denote by ∇ the complete Boolean algebra of all idempotents from $L^0(\Omega)$, i. e. $\nabla = \{\tilde{\chi}_A : A \in \Sigma\}$, where $\tilde{\chi}_A$ is the element from $L^0(\Omega)$ which contains the characteristic function of the set A. A *partition of the unit* in ∇ is a family (π_α) of orthogonal idempotents from ∇ such that $\bigvee_\alpha \pi_\alpha = \mathbf{1}$.

A complex linear space E is said to be normed by $L^0(\Omega)$ if there is a map $\|\cdot\| : E \longrightarrow L^0(\Omega)$ such that for any $x, y \in E, \lambda \in \mathbb{C}$, the following conditions are fulfilled:

1) $\|x\| \geq 0; \|x\| = 0 \Longleftrightarrow x = 0$;
2) $\|\lambda x\| = |\lambda|\|x\|$;
3) $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(E, \|\cdot\|)$ is called a lattice-normed space over $L^0(\Omega)$. A lattice-normed space E is called d-decomposable, if for any $x \in E$ with $\|x\| = \lambda_1 + \lambda_2$, $\lambda_1, \lambda_2 \in L^0(\Omega)$, $\lambda_1\lambda_2 = 0, \lambda_1, \lambda_2 \geq 0$, there exist $x_1, x_2 \in E$ such that $x = x_1 + x_2$ and $\|x_i\| = \lambda_i$, $i = 1, 2$. A net (x_α) in E is said to be (bo)-convergent to $x \in E$, if the net $\{\|x_\alpha - x\|\}$ (o)-converges to zero in $L^0(\Omega)$. A lattice-normed space E which is d-decomposable and complete with respect to the (bo)-convergence is called a *Banach – Kantorovich space*.

It is known that every Banach–Kantorovich space E over $L^0(\Omega)$ is a module over $L^0(\Omega)$ and $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in L^0(\Omega)$, $x \in E$ (see [6, Chapter II]).

Let $\mathcal{K}$ be a module over $L^0(\Omega)$. A map $\langle\cdot,\cdot\rangle : \mathcal{K} \times \mathcal{K} \to L^0(\Omega)$ is called an $L^0(\Omega)$-valued inner product, if for all $x, y, z \in \mathcal{K}$, $\lambda \in L^0(\Omega)$, the following conditions are fulfilled:

1) $\langle x, x\rangle \geq 0$; $\langle x, x\rangle = 0 \Leftrightarrow x = 0$;
2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$;
3) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$;
4) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

If $\langle\cdot,\cdot\rangle : \mathcal{K} \times \mathcal{K} \to L^0(\Omega)$ is an $L^0(\Omega)$-valued inner product, then $\|x\| = \sqrt{\langle x, x\rangle}$ defines an $L^0(\Omega)$-valued norm on $\mathcal{K}$. The pair $(\mathcal{K}, \langle\cdot,\cdot\rangle)$ is called a *Hilbert–Kaplansky module* over $L^0(\Omega)$, if $(\mathcal{K}, \|\cdot\|)$ is a Banach – Kantorovich space over $L^0(\Omega)$ (see [6, Chapter III]).

Let X be a Banach space. A map $s : \Omega \to X$ is said to be simple, if $s(\omega) = \sum\limits_{k=1}^{n} \chi_{A_k}(\omega)c_k$, where $A_k \in \Sigma, A_i \cap A_j = \emptyset$, $i \neq j$, $c_k \in X$, $k = \overline{1, n}$, $n \in \mathbb{N}$. A map $u : \Omega \to X$ is said to be measurable, if for every $A \in \sum$ with $\mu(A) < \infty$ there is a sequence (s_n) of simple maps such that $\|s_n(\omega) - u(\omega)\|_X \to 0$ almost everywhere on A.

Let $\mathcal{L}(\Omega, X)$ be the set of all measurable maps from Ω into X, and let $L^0(\Omega, X)$ denote the space of all equivalence classes with respect to the equality almost everywhere. Denote by $\hat{u}$ the equivalence class from $L^0(\Omega, X)$ which contains the measurable map $u \in \mathcal{L}(\Omega, X)$. Further we shall identify the element $u \in \mathcal{L}(\Omega, X)$ and the class $\hat{u}$. Note that the function $\omega \to \|u(\omega)\|_X$ is measurable for any $u \in \mathcal{L}(\Omega, X)$. The equivalence class containing the function $\|u(\omega)\|$ is denoted by $\|\hat{u}\|$. For $\hat{u}, \hat{v} \in L^0(\Omega, X), \lambda \in L^0(\Omega)$ put $\hat{u} + \hat{v} = \widehat{u(\omega) + v(\omega)}, \lambda\hat{u} = \widehat{\lambda(\omega)u(\omega)}$.

It is known [6, Chapter III] that $(L^0(\Omega, X), \|\cdot\|)$ is a Banach – Kantorovich space over $L^0(\Omega)$.

If H is a Hilbert space, then $L^0(\Omega, H)$ can be equipped with an $L^0(\Omega)$-valued inner product $\langle x, y\rangle = \widehat{\langle x(\omega), y(\omega)\rangle_H}$, where $\langle\cdot,\cdot\rangle_H$ is the inner product on H, such that $(L^0(\Omega, H), \langle\cdot,\cdot\rangle)$ becomes a Hilbert – Kaplansky module over $L^0(\Omega)$. Moreover the space

$$L^\infty(\Omega, H) = \{x \in L^0(\Omega, H) : \langle x, x\rangle \in L^\infty(\Omega)\}$$

is a Hilbert – Kaplansky module over $L^\infty(\Omega)$.

An operator $T : L^0(\Omega, H) \to L^0(\Omega, H)$ is said to be $L^0(\Omega)$-linear if $T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$ for all $\lambda_1, \lambda_2 \in L^0(\Omega)$, $x_1, x_2 \in L^0(\Omega, H)$. An $L^0(\Omega)$-linear operator $T : L^0(\Omega, H) \to L^0(\Omega, H)$ is $L^0(\Omega)$-bounded if there exists an element $c \in L^0(\Omega)$ such that $\|T(x)\| \leq c\|x\|$ for any $x \in L^0(\Omega, H)$. For an $L^0(\Omega)$-bounded $L^0(\Omega)$-linear operator T we put $\|T\| = \sup\{\|T(x)\| : \|x\| \leq \mathbf{1}\}$.

Denote by $B(L^0(\Omega, H))$ the algebra of all $L^0(\Omega)$-bounded $L^0(\Omega)$-linear operators on $L^0(\Omega, H)$ and denote by $B(L^\infty(\Omega, H))$ the algebra of all $L^\infty(\Omega)$-bounded $L^\infty(\Omega)$-linear operators on $L^\infty(\Omega, H)$.

Let $B(H)$ be the algebra of all bounded linear operators on a Hilbert space H and let M be a von Neumann algebra in $B(H)$ with the operator norm $\|\cdot\|_M$. Denote by $P(M)$ the lattice of projections in M.

A linear subspace $\mathcal{D}$ in H is said to be *affiliated* with M (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for every unitary u from the commutant

$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$

of the von Neumann algebra M.

A linear operator x on H with the domain $\mathcal{D}(x)$ is said to be *affiliated* with M (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$.

A linear subspace $\mathcal{D}$ in H is said to be *strongly dense* in H with respect to the von Neumann algebra M, if

1) $\mathcal{D}\eta M$;
2) there exists a sequence of projections $\{p_n\}_{n=1}^{\infty}$ in $P(M)$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathcal{D}$ and $p_n^{\perp} = \mathbf{1} - p_n$ is finite in M for all $n \in \mathbb{N}$, where $\mathbf{1}$ is the identity in M.

A closed linear operator x acting in the Hilbert space H is said to be *measurable* with respect to the von Neumann algebra M, if $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in H. Denote by $S(M)$ the set of all measurable operators with respect to M.

A closed linear operator x in H is said to be *locally measurable* with respect to the von Neumann algebra M, if $x\eta M$ and there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in M such that $z_n \uparrow \mathbf{1}$ and $z_n x \in S(M)$ for all $n \in \mathbb{N}$.

It is well-known [7] that the set $LS(M)$ of all locally measurable operators with respect to M is a unital *-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator.

Note that if M is a finite von Neumann algebra then $S(M) = LS(M)$.

Let M be a von Neumann algebra with a faithful normal finite trace τ. Consider the topology t_τ of convergence in measure or *measure topology* on $S(M)$, which is defined by the following neighborhoods of zero:

$$V(\varepsilon, \delta) = \{x \in S(M) : \exists e \in P(M), \tau(e^{\perp}) \leq \delta, xe \in M, \|xe\|_M \leq \varepsilon\},$$

where ε, δ are positive numbers.

It is well-known that $S(M)$ equipped with the measure topology is a complete metrizable topological *-algebra.

Recall that a von Neumann algebra M is of *type I* if it is isomorphic to a von Neumann algebra with an abelian commutant, or, equivalently M admits a faithful abelian projection.

If a von Neumann algebra M with the center $L^{\infty}(\Omega)$ is homogeneous of type I_α, where α is a cardinal number, then M is $*$-isomorphic to the algebra $B(L^{\infty}(\Omega, H))$, where $\dim H = \alpha$, while the algebra $LS(M)$ is $*$-isomorphic to $B(L^0(\Omega, H))$ (see for details [1]).

It should be noted that if $\dim H = n, n \in \mathbb{N}$, then $L^0(\Omega, H)$ is an n-homogeneous module over $L^0(\Omega)$ and thus it is isomorphic to $\prod_{i=1}^n L^0(\Omega)$, therefore the algebra $B(L^0(\Omega, H))$ of $L^0(\Omega)$-linear operators on $L^0(\Omega, H)$ is isomorphic to the algebra $M_n(L^0(\Omega))$ of all $n \times n$ matrices over $L^0(\Omega)$.

Thus if M is a von Neumann algebra of type I_n, $n \in \mathbb{N}$, with the center $L^\infty(\Omega)$, then $LS(M) = S(M)$ is $*$-isomorphic to the algebra $M_n(L^0(\Omega))$. If moreover M admits a faithful normal finite trace then one can obtain the following direct proof of the mentioned isomorphism.

PROPOSITION 2.1. *Let M be a von Neumann algebra of type I_n, $n \in \mathbb{N}$, with a faithful normal finite trace τ and let $Z(S(M))$ denote the center of the algebra $S(M)$. Then $S(M) \cong M_n(Z(S(M)))$.*

Proof. Let $\{e_{ij} : i, j \in \overline{1, n}\}$ be matrix units in $M_n(Z)$. Consider the *-subalgebra in $S(M)$ generated by the set

$$Z(S(M)) \cup \{e_{ij} : i, j \in \overline{1, n}\}.$$

This *-subalgebra consists of elements of the form

$$\sum_{i,j=1}^{n} \lambda_{ij} e_{ij},\ \lambda_{i,j} \in Z(S(M)),\ i, j = \overline{1, n}$$

and it is *-isomorphic with $M_n(Z(S(M))) \subseteq S(M)$. Since τ is finite and M is t_τ-dense in $S(M)$, it is sufficient to show that the subalgebra $M_n(Z(S(M)))$ is closed in $S(M)$ with respect to the topology t_τ. The center $Z(S(M))$ is t_τ-closed in $S(M)$ and therefore the subalgebra

$$e_{11} Z(S(M)) e_{11} = Z(S(M)) e_{11},$$

is also t_τ-closed in $S(M)$.

Consider a sequence $x_m = \sum\limits_{i,j=1}^{n} \lambda_{ij}^{(m)} e_{ij}$ in $M_n(Z(S(M)))$ such that $x_m \to x \in S(M)$ in the topology t_τ. Fixing $k, l \in \overline{1, n}$ we have that $e_{1k} x_m e_{l1} \to e_{1k} x e_{l1}$. Since $e_{1k} x_m e_{l1} = \lambda_{kl}^{(m)} e_{11}$ one has $\lambda_{kl}^{(m)} e_{11} \to e_{1k} x e_{l1}$. The t_τ-closedness of $Z(S(M)) e_{11}$ in $S(M)$ implies that

$$\lambda_{kl}^{(m)} e_{11} \to \lambda_{kl} e_{11}$$

for an appropriate $\lambda_{kl} \in Z(S(M))$. Multiplying by e_{k1} from the left side and by e_{1l} from the right side we obtain that $\lambda_{kl}^{(m)} e_{kl} \to \lambda_{kl} e_{kl}$. Therefore $x_m \to \sum_{i,j=1}^{n} \lambda_{ij} e_{ij}$ and thus $x = \sum_{i,j=1}^{n} \lambda_{ij} e_{ij}$. This implies that $S(M) \cong M_n(Z(S(M)))$. The proof is complete. ∎

Note that the algebra $LS(M)$ has the following remarkable property: given any family $\{z_i\}_{i\in I}$ of mutually orthogonal central projections in M with $\bigvee_{i\in I} z_i = \mathbf{1}$ and a family of elements $\{x_i\}_{i\in I}$ in $LS(M)$ there exists a unique element $x \in LS(M)$ such that $z_i x = z_i x_i$ for all $i \in I$. This element is denoted by $x = \sum_{i\in I} z_i x_i$ (see [7]).

The last assertion enables us to obtain the following important property of the algebra $LS(M)$ in the case of type I von Neumann algebra M, which follows from Proposition 2.1 and the spectral resolution of operators.

PROPOSITION 2.2. *Let M be a type I von Neumann algebra. Then for any element $x \in LS(M)$ there exists a countable family of mutually orthogonal central projections $\{z_k\}_{k\in\mathbb{N}}$ in M such that $\bigvee_k z_k = \mathbf{1}$ and $z_k x \in M$ for all k.*

It is known [8] that given a type I von Neumann algebra M there exists a unique (cardinal-indexed) family of central orthogonal projections $(q_\alpha)_{\alpha\in J}$ in $P(M)$ with $\sum_{\alpha\in J} q_\alpha = \mathbf{1}$ such that $q_\alpha M$ is a homogeneous type I_α von Neumann algebra. In this case M is a *-isomorphic with the C^*-product of the algebras $B(L^\infty(\Omega_\alpha, H_\alpha)$, i.e.

$$M \cong \bigoplus_{\alpha\in J} B(L^\infty(\Omega_\alpha, H_\alpha)).$$

The direct product

$$\prod_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha)$$

equipped with the coordinate-wise algebraic operations and inner product forms a Hilbert−Kaplansky module over $L^0(\Omega) \cong \prod_{\alpha\in J} L^0(\Omega_\alpha)$.

In [1] we have proved that if the von Neumann algebra M is *-isomorphic with $\bigoplus_{\alpha\in J} B(L^\infty(\Omega_\alpha, H_\alpha))$ then the algebra $LS(M)$ is *-isomorphic with

$$B(\prod_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha)) \cong \prod_{\alpha\in J} B(L^0(\Omega_\alpha, H_\alpha)),$$

i.e.

$$LS(M) \cong \prod_{\alpha\in J} B(L^0(\Omega_\alpha, H_\alpha)).$$

Indeed, let Φ be a *-isomorphism between M and $B(\bigoplus\limits_{\alpha\in J} L^\infty(\Omega_\alpha, H_\alpha))$. Take $x \in B(\prod\limits_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha))$ and let $||x||$ be its $L^0(\Omega)$-valued norm. Consider a family of mutually orthogonal projections $\{z_n\}_{n\in\mathbb{N}}$ in $L^\infty(\Omega)$ with $\bigvee z_n = \mathbf{1}$ such that $z_n||x|| \in L^\infty(\Omega)$ for all $n \in \mathbb{N}$. Then $z_n x \in M$ for all $n \in \mathbb{N}$ and $\sum\limits_n z_n\Phi(z_n x) \in LS(M)$. Put

$$\Psi : x \to \sum_n z_n\Phi(z_n x).$$

It is clear that Ψ is a well-defined *-homomorphism from $B(\prod\limits_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha))$ into $LS(M)$. Since given any element $x \in LS(M)$ there exists a sequence of mutually orthogonal central projections $\{z_n\}$ in M such that $z_n x \in M$ for all $n \in \mathbb{N}$ and $x = \sum\limits_n z_n x$, this implies that Ψ is a *-isomorphism between $LS(M)$ and $B(\prod\limits_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha))$.

It is known [3] that $B(\prod\limits_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha))$ is a C^*-algebra over $L^0(\Omega)$ and therefore there exists a map $||\cdot|| : LS(M) \to L^0(\Omega)$ such that for all $x, y \in LS(M), \lambda \in L^0(\Omega)$ one has

$$\begin{gathered} ||x|| \geq 0, ||x|| = 0 \Leftrightarrow x = 0; \\ ||\lambda x|| = |\lambda|||x||; \\ ||x + y|| \leq ||x|| + ||y||; \\ ||xy|| \leq ||x||||y||; \\ ||xx^*|| = ||x||^2. \end{gathered}$$

This map $||\cdot|| : LS(M) \to L^0(\Omega)$ is called the *center-valued* norm on $LS(M)$.

The above isomorphism enables us to obtain the following necessary and sufficient condition for a derivation on the algebra $LS(M)$ to be inner.

Theorem 2.3. ([1]) *Let M be a type I von Neumann algebra with the center Z. A derivation D on the algebra $LS(M)$ is inner if and only if it is Z-linear, or equivalently, it is identically zero on Z.*

3. Main results

Let A be an algebra with the center Z and let $D : A \to A$ be a derivation. Given any $x \in A$ and a central element $z \in Z$ we have

$$D(zx) = D(z)x + zD(x)$$

and

$$D(xz) = D(x)z + xD(z).$$

Since $zx = xz$ and $zD(x) = D(x)z$, it follows that $D(z)x = xD(z)$ for any $x \in A$. This means that $D(z) \in Z$, i.e. $D(Z) \subseteq Z$. Therefore given any derivation D on the algebra A we can consider it restriction δ onto the center Z :

$$\delta : z \to D(z),\ z \in Z.$$

This simple but important property of derivations is crucial in our further considerations.

Let M be a homogeneous von Neumann algebra of type I_n, $n \in \mathbb{N}$, with the center $Z = L^\infty(\Omega)$. Then M is *-isomorphic with the algebra of $M_n(L^\infty(\Omega))$ of $n \times n$ matrices over the algebra $L^\infty(\Omega)$, while $LS(M) \cong B(L^0(\Omega, H))$ is isomorphic with the algebra $M_n(L^0(\Omega))$ of all $n \times n$ matrices over the algebra $L^0(\Omega)$, and the center of $M_n(L^0(\Omega))$ can be identified with $L^0(\Omega)$.

In the papers [2], [5] the existence of non zero derivations on $L^0(\Omega)$ has been proven in the case of a non atomic measure space (Ω, Σ, μ). Given a derivation $\delta : L^0(\Omega) \to L^0(\Omega)$ consider the elementwise derivation D_δ on $M_n(L^0(\Omega))$ defined as:

$$D_\delta((\lambda_{ij})_{i,j=1}^n) = (\delta(\lambda_{ij}))_{i,j=1}^n, \tag{1}$$

where $(\lambda_{ij})_{i,j=1}^n \in M_n(L^0(\Omega))$.

A straightforward calculation shows that D_δ is indeed a derivation on $M_n(L^0(\Omega))$ and its restriction onto the center of $M_n(L^0(\Omega))$ coincides with δ.

Lemma 3.1. *Every derivation D on the algebra $M_n(L^0(\Omega))$ admits a unique decomposition*

$$D = D_a + D_\delta,$$

where D_a is an inner derivation and D_δ is a derivation of the form (1).

Proof. Given a derivation D on $M_n(L^0(\Omega))$, consider its restriction δ onto its center $L^0(\Omega)$ and extend it to the whole $M_n(L^0(\Omega))$ by the form (1) as D_δ. Put $D_1 = D - D_\delta$. Then given any $z \in Z = L^\infty(\Omega)$ we have

$$D_1(z) = D(z) - D_\delta(z) = D(z) - D(z) = 0,$$

i.e. D_1 is identically zero on Z and therefore it is Z-linear. Theorem 2.3 implies that D_1 is inner, i.e. $D_1 = D_a$ for an appropriate $a \in M_n(L^0(\Omega))$. Therefore $D = D_a + D_\delta$.

Now suppose that

$$D = D_{a_1} + D_{\delta_1} = D_{a_2} + D_{\delta_2}.$$

Then $D_{a_1} - D_{a_2} = D_{\delta_2} - D_{\delta_1}$. Since $D_{a_1} - D_{a_2}$ is identically zero on the center of $M_n(L^0(\Omega))$, then $D_{\delta_2} - D_{\delta_1}$ also is identically zero on the center of $M_n(L^0(\Omega))$. Thus $\delta_1 = \delta_2$ and hence $D_{a_1} = D_{a_2}$. The proof is complete. ∎

In order to consider the case of type I_∞ von Neumann algebra we need some auxiliary results.

Lemma 3.2. *Any derivation δ on the algebra $L^0(\Omega)$ commutes with the mixing operation on $L^0(\Omega)$, i.e.*

$$\delta(\sum_\alpha \pi_\alpha \lambda_\alpha) = \sum_\alpha \pi_\alpha \delta(\lambda_\alpha)$$

for an arbitrary family $(\lambda_\alpha) \subset L^0(\Omega)$ and every partition $\{\pi_\alpha\}$ of the unit in ∇.

Proof. Consider a family $\{\lambda_\alpha\}$ in $L^0(\Omega)$ and a partition of the unit $\{\pi_\alpha\}$ in $\nabla \subset L^0(\Omega)$. Since $\delta(\pi) = 0$ for any idempotent $\pi \in \nabla$, we have $\delta(\pi_\alpha) = 0$ for all α and thus $\delta(\pi_\alpha\lambda) = \pi_\alpha\delta(\lambda)$ for any $\lambda \in L^0(\Omega)$. Therefore for each π_{α_0} from the given partition of the unit we have

$$\pi_{\alpha_0}\delta(\sum_\alpha \pi_\alpha\lambda_\alpha) = \delta(\pi_{\alpha_0}\sum_\alpha \pi_\alpha\lambda_\alpha) = \delta(\pi_{\alpha_0}\lambda_{\alpha_0}) = \pi_{\alpha_0}\delta(\lambda_{\alpha_0}).$$

By taking the sum over all α_0 we obtain

$$\delta(\sum_\alpha \pi_\alpha\lambda_\alpha) = \sum_\alpha \pi_\alpha\delta(\lambda_\alpha).$$

The proof is complete. ∎

Recall [6] that a subset $K \subset L^0(\Omega)$ is called *cyclic*, if $\sum\limits_{\alpha\in J} \pi_\alpha u_\alpha \in K$ for each family $(u_\alpha)_{\alpha\in J} \subset K$ and for any partition of the unit $(\pi_\alpha)_{\alpha\in J}$ in ∇.

LEMMA 3.3. *Given any non trivial derivation $\delta : L^0(\Omega) \to L^0(\Omega)$ there exist a sequence $\{\lambda_n\}_{n=1}^\infty$ in $L^\infty(\Omega)$ with $|\lambda_n| \le \mathbf{1}$, $n \in \mathbb{N}$, and an idempotent $\pi \in \nabla$, $\pi \neq 0$ such that*

$$|\delta(\lambda_n)| \ge n\pi$$

for all $n \in \mathbb{N}$.

Proof. Suppose that the set $\{\delta(\lambda) : \lambda \in L^0(\Omega), |\lambda| \le \mathbf{1}\}$ is order bounded in $L^0(\Omega)$. Then δ maps any uniformly convergent sequence in $L^\infty(\Omega)$ to an (o)-convergent sequence in $L^0(\Omega)$. The algebra $L^\infty(\Omega)$ coincides with the uniform closure of the linear span of idempotents from ∇. Since δ is identically zero on ∇ it follows that $\delta \equiv 0$ on $L^\infty(\Omega)$. Since δ commutes with the mixing operation and since every element $\lambda \in L^0(\Omega)$ can be represented as $\lambda = \sum\limits_\alpha \pi_\alpha\lambda_\alpha$, where $\{\lambda_\alpha\} \subset L^\infty(\Omega)$, and $\{\pi_\alpha\}$ is a partition of unit in ∇, we have $\delta(\lambda) = \delta(\sum\limits_\alpha \pi_\alpha\lambda_\alpha) = \sum\limits_\alpha \pi_\alpha\delta(\lambda_\alpha) = 0$, i.e. $\delta \equiv 0$ on $L^0(\Omega)$. This contradiction shows that the set $\{\delta(\lambda) : \lambda \in L^0(\Omega), |\lambda| \le \mathbf{1}\}$ is not order bounded in $L^0(\Omega)$. Further, since δ commutes with the mixing operations and the set $\{\lambda : \lambda \in L^0, |\lambda| \le \mathbf{1}\}$ is cyclic, the set $\{\delta(\lambda) : \lambda \in L^0(\Omega), |\lambda| \le \mathbf{1}\}$ is also cyclic. By [4, Proposition 3] there exist a sequence $\{\lambda_n\}_{n=1}^\infty$ in $L^\infty(\Omega)$ with $|\lambda_n| \le \mathbf{1}$ and an idempotent $\pi \in \nabla$, $\pi \neq 0$, such that $|\delta(\lambda_n)| \ge n\pi$, $n \in \mathbb{N}$. The proof is complete. ∎

Now we are in position to consider derivations on the algebra of locally measurable operators for type I_∞ von Neumann algebras.

THEOREM 3.4. *If M is a homogeneous von Neumann algebra of type I_α, $\alpha \ge \aleph_0$,then any derivation on the algebra $LS(M)$ is inner.*

Proof. Since M is homogeneous of type I_α, $\alpha \ge \aleph_0$, there exists a sequence of mutually orthogonal and mutually equivalent abelian projections $\{p_n\}_{n=1}^\infty$ in M with the central cover $\mathbf{1}$ (i.e. faithful projections).

For any bounded sequence $\Lambda = \{\lambda_k\}$ in Z define an operator x_Λ by

$$x_\Lambda = \sum_{k=1}^\infty \lambda_k p_k.$$

Then

$$x_\Lambda p_n = p_n x_\Lambda = \lambda_n p_n. \qquad (2)$$

Let D be a derivation on $LS(M)$, and let δ be its restriction onto the center of $LS(M)$, identified with $L^0(\Omega)$.

Take any $\lambda \in L^0(\Omega)$ and $n \in \mathbb{N}$. From the identity

$$D(\lambda p_n) = D(\lambda)p_n + \lambda D(p_n)$$

multiplying it by p_n from both sides we obtain

$$p_n D(\lambda p_n)p_n = p_n D(\lambda)p_n + \lambda p_n D(p_n)p_n.$$

Since p_n is a projection, one has that $p_n D(p_n)p_n = 0$, and since $D(\lambda) = \delta(\lambda) \in L^0(\Omega)$, we have

$$p_n D(\lambda p_n)p_n = \delta(\lambda)p_n. \qquad (3)$$

Now from the identity

$$D(x_\Lambda p_n) = D(x_\Lambda)p_n + x_\Lambda D(p_n),$$

in view of (3) one has similarly

$$p_n D(\lambda_n p_n)p_n = p_n D(x_\Lambda)p_n + \lambda_n p_n D(p_n)p_n,$$

i.e.

$$p_n D(\lambda_n p_n)p_n = p_n D(x_\Lambda)p_n. \qquad (4)$$

(3) and (4) imply

$$p_n D(x_\Lambda)p_n = \delta(\lambda_n)p_n.$$

Further for the center-valued norm $\|\cdot\|$ on $LS(M)$ (see Section 2) we have :

$$\|p_n D(x_\Lambda)p_n\| \leq \|p_n\|\|D(x_\Lambda)\|\|p_n\| = \|D(x_\Lambda)\|$$

and

$$\|\delta(\lambda_n)p_n\| = |\delta(\lambda_n)|.$$

Therefore

$$\|D(x_\Lambda)\| \geq |\delta(\lambda_n)|$$

for any bounded sequence $\Lambda = \{\lambda_n\}$ in Z.

If we suppose that $\delta \neq 0$ then by Lemma 3.3 there exist a bounded sequence $\Lambda = \{\lambda_n\}$ in Z and an idempotent $\pi \in \nabla$, $\pi \neq 0$, such that

$$|\delta(\lambda_n)| \geq n\pi$$

for any $n \in \mathbb{N}$. Thus, $\|D(x_\Lambda)\| \geq n\pi$ for all $n \in \mathbb{N}$, i.e. $\pi = 0-$that is a contradiction. Therefore $\delta \equiv 0$, i.e. D is identically zero on the center of $LS(M)$, and therefore it is Z-linear. By Theorem 2.3 D is inner. The proof is complete. ∎

Now let us consider the general case of type I von Neumann algebras.

Let M be a type I von Neumann algebra and let $(q_\alpha)_{\alpha\in J} \subset P(M)$ be the orthogonal family of central projections with $\sum_{\alpha\in J} q_\alpha = \mathbf{1}$ such that $q_\alpha M$ is a homogeneous type I_α von Neumann algebra, i.e.

$$M \cong \bigoplus_{\alpha\in J} B(L^\infty(\Omega_\alpha, H_\alpha)).$$

As it was mentioned in Section 2 we have the *-isomorphism

$$LS(M) \cong \prod_{\alpha\in J} B(L^0(\Omega_\alpha, H_\alpha)).$$

Now let D be a derivation on the algebra $LS(M)$ and let δ be its restriction onto its center $S(Z)$. Since each q_α is a central projection we have that $D(q_\alpha) = 0$, for all $\alpha \in J$. Therefore $D(q_\alpha x) = q_\alpha D(x)$ for all $x \in LS(M)$ and $\alpha \in J$. This implies D maps each $q_\alpha LS(M)$ into itself and thus $D_\alpha = D|_{q_\alpha LS(M)}$ is a derivation on $q_\alpha LS(M)$. Since $q_\alpha S(Z) \cong L^0(\Omega_\alpha)$ for each α, and since δ maps each $q_\alpha S(Z)$ into $q_\alpha S(Z)$, it follows that δ induces a derivation δ_α on each $L^0(\Omega_\alpha)$.

Put $F = \{\alpha \in J : \alpha \in \mathbb{N}\}$. Let D_{δ_α} $(\alpha \in F)$ be the derivation on the matrix algebra $q_\alpha LS(M) \cong M_\alpha(L^0(\Omega_\alpha))$ constructed by the formula (1). From Lemma 3.1 we obtain that

$$D_\alpha = D_{a_\alpha} + D_{\delta_\alpha}, \tag{5}$$

where $a_\alpha \in q_\alpha LS(M)$. By Theorem 3.4 for each $\alpha \in J \setminus F$ we have

$$D_\alpha = D_{a_\alpha}, \tag{6}$$

where $a_\alpha \in q_\alpha LS(M)$, i.e. $\delta_\alpha \equiv 0$. For such α we put $D_{\delta_\alpha} \equiv 0$. Denote

$$a = \{a_\alpha\}_{\alpha\in J}. \tag{7}$$

Now define a derivation D_δ on $LS(M)$ by

$$D_\delta(x) = (D_{\delta_\alpha}(x_\alpha)), \quad x = (x_\alpha) \in LS(M). \tag{8}$$

Since $D = \{D_\alpha\}_{\alpha \in J}$, from (5) and (6) we obtain that

$$D = D_a + D_\delta,$$

where a and D_δ are defined in (7) and (8) respectively.

Therefore we obtain the following main result of the paper.

THEOREM 3.5. *Let M be a type I von Neumann algebra. Every derivation D on the algebra $LS(M)$ can be decomposed in a unique way as*

$$D = D_a + D_\delta,$$

where D_a is an inner derivation and D_δ is a derivation of the form (8).

In particular if M is an arbitrary type I_∞ von Neumann algebra then in the above notation we have that $F = \emptyset$ (i.e. $\alpha \geq \aleph_0$ for all $\alpha \in J$) and (6) implies

COROLLARY 3.6. *If M is a type I_∞ von Neumann algebra then any derivation on $LS(M)$ is inner.*

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REFERENCES

[1] S. ALBEVERIO, SH.A. AYUPOV, K.K. KUDAYBERGENOV, Derivations on the algebra of measurable operators affiliated with a type I von Neumann algebra, *Siberian Adv. Math.,* **18** (2008), 86 – 94.

[2] A.F. BER, V.I. CHILIN, F.A. SUKOCHEV, Non-trivial derivation on commutative regular algebras, *Extracta mathematicae,* **21** (2006), 107 – 147.

[3] V.I. CHILIN, I.G. GANIEV, K.K. KUDAYBERGENOV, The Gelfand – Naimark – Segal theorem for C^*-algebras over a ring measurable functions, *Vladikav. Math. J.,* **9** (2007), 16 – 22.

[4] I.G. GANIEV, K.K. KUDAYBERGENOV, The Banach – Steinhaus uniform boundedness principle for operators in Banach – Kantorovich spaces over L^0, *Siberian Adv. Math.,* **16** (2006), 42 – 53.

[5] A.G. KUSRAEV, Automorphisms and derivations on a universally complete complex f-algebra, *Sib. Math. J.,* **47** (2006), 77 – 85.

[6] A.G. Kusraev, "Dominated Operators", Kluwer Academic Publishers, Dordrecht, 2000.

[7] M.A. Muratov, V.I. Chilin, $*$-Algebras of unbounded operators affiliated with a von Neumann algebra, *J. Math. Sciences,* **140** (2007), 445 – 451.

[8] M. Takesaki, "Theory of Operator Algebras I", Springer-Verlag, New York-Heildelberg-Berlin, 1991.