

Trivial Units in Abelian Group Algebras

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Abstract: We find a necessary and sufficient condition when all units in arbitrary abelian group algebras are trivial by reducing it to the torsion case. This is a supplement to our recent results in (Extracta Math., 2008). In particular, we extend theorems due to Ritter-Sehgal (Math. Proc. Royal Irish Acad., 2005) and Herman-Li (Proc. Amer. Math. Soc., 2006).

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1. INTRODUCTION

Suppose that RG is the group algebra of an abelian group G over a commutative ring R with identity 1_R (sometimes, when there are no doubts, we shall simply write 1), and suppose $V(RG)$ is its subgroup of normalized units. A problem of interest in group rings theory is to describe only in terms associated with R and G when all invertible elements in RG (often called *units*) are trivial, i.e., they are of the type rg , where $r \in R^*$, the multiplicative group (also called *the unit group*) of R , and $g \in G$. Specifically, we can state in an explicit form the following equivalent reformulation.

PROBLEM. Find a criterion only in terms of R and G when the equality $V(RG) = G$ holds.

To avoid any successive verbosity, we shall briefly give the needed notations and terminology. As usual, G_0 denotes the maximal torsion part of G consisting of all elements in G of finite order with p -primary component G_p . Imitating [11], for the set of all primes \mathbb{P} we also define its subsets like these: $\text{supp}(G) = \{p : G_p \neq 1\}$, $\text{inv}(R) = \{p : p \cdot 1_R \in R^*\}$ and $\text{zd}(R) = \{p : \exists 0 \neq r \in R \text{ such that } pr = 0\}$; notice that if G is torsion-free, that is, $G_0 = 1$ we have $\text{supp}(G) = \emptyset$ and vice versa. Traditionally, ζ_n is reserved for the primitive n -th root of unity whenever $n \in \mathbb{N}$.

Likewise, we shall say that $\text{char}(R) = p \in \mathbb{P}$ if $pr = 0$ for every $r \in R$ (it is worth noting that this is tantamount to $p \cdot 1_R = 0$), that $\text{char}(R) = l \in \mathbb{N} \setminus \{1\}$ if for every $r \in R$ we have $lr = 0$ and $kr \neq 0$ for any $k < l$ (equivalently l is the minimal natural with the property $l \cdot 1_R = 0$), and that $\text{char}(R) = 0$ if $mr \neq nr$ for every $r \in R$ whenever $m, n \in \mathbb{N}$, i.e., does not exist a positive integer i with $ir = 0$ whenever $r \in R$; note that if $\text{char}(R) = 0$ we have $\text{zd}(R) = \emptyset$ and reversely. Finally, R is said to be *indecomposable* if $\text{id}(R) = \{e \in R : e^2 = e\} \equiv \{0, 1\}$ and *reduced* if $N(R) = \{r \in R : \exists n \in \mathbb{N} \text{ such that } r^n = 0\} \equiv \{0\}$. All other notions are standard and follow essentially those from [9].

And so, the aforementioned problem for trivial units was intensively studied by many authors (see, e.g., [7], [8], [10], [13] and [1], [2], [3], [4]). Especially, in [4] we gave a systematic exploration of this subject by establishing comprehensive characterization when RG contains only trivial units in the cases when $\text{supp}(G) \cap \text{inv}(R) \neq \emptyset$, or $\text{char}(R)$ divides the orders of the torsion elements in G , or $\text{char}(R) = l > 1$ (in particular, $\text{char}(R) = p$, a prime).

The objective of the present paper is to obtain a valuable criterion when $V(RG) = G$ is true without any additional limitations on R and G , respectively. However, it will not be in a final form, and will reduce the general case to the torsion one, although shall show that some determinative conditions must be fulfilled as well. For some other generalizations of trivial units the reader can see [5] and [6].

As direct consequences of our chief result, we shall prove some assertions pertaining to trivial units in group algebras KG over integral domains K of zero characteristic such that no prime divisor of the order of any element in G_0 inverts in K , that is, in other words, $\text{supp}(G) \cap \text{inv}(K) = \emptyset$.

2. MAIN RESULT

We are now able to proceed by proving the following general statement.

THEOREM 1. *Let G be a non-identity abelian group and R a commutative ring with identity. Then $V(RG) = G$ if and only if R is indecomposable and reduced, $V(RG_0) = G_0$ and one of the following holds:*

- (a) $G = G_0$;
- (b) $G \neq G_0$, $\text{supp}(G) \cap (\text{inv}(R) \cup \text{zd}(R)) = \emptyset$.

Proof. “ \Rightarrow ” If we assume for a moment, in a way of contradiction, that

there exist $\{0, 1\} \neq r^2 = r \in R$ and $0 \neq f \in R$ with $f^n = 0$ for some natural n , then the elements $1 - r + rg$ where $1 \neq g \in G$ and $1 + f - fg = (1 + f)(1 - (1 + f)^{-1}fg)$ where $1 \neq g \in G$, obviously lie in $V(RG) \setminus G$. Indeed, the first has inverse $1 - r + rg^{-1}$, while the second is a normalized unit since it is well known that the sum of a unit and a nilpotent is again a unit. That is why, R is both without non-trivial idempotent and nilpotent elements.

Next, it is self-evident that $V(RG_0) \subseteq V(RG) = G$ and so $V(RG_0) = V(RG_0) \cap G = G_0$, as claimed. (We pause to note that $V(RG)_0 = V(RG_0)$ because $V(RG) = G$ obviously secures that $V(RG)_0 = G_0$.)

Further, if (a) is satisfied we are done. Thus, assume that G is not torsion, that is, there is $a \in G$ such that $a^n \neq 1$ for each $n \in \mathbb{N}$. If we suppose that $\text{supp}(G) \cap \text{inv}(R) \neq \emptyset$, then it is well known that there exists $e \in RF \setminus \{0, 1\}$ with $e^2 = e$, where $F \leq G_0$ is finite (see, for example, [11]). Consequently, the element $1 - e + ea \in V(RG) \setminus G$ because $(1 - e + ea)(1 - e + ea^{-1}) = 1$ and $ea \neq e$ due to $a \notin G_0$. This substantiates our claim that $\text{supp}(G) \cap \text{inv}(R) = \emptyset$ is valid, in fact.

Now, we shall show that $\text{supp}(G) \cap \text{zd}(R) = \emptyset$. In order to do this, assume the contrary, i.e., that $\text{supp}(G) \cap \text{zd}(R) \neq \emptyset$. Therefore, there exist $0 \neq r \in R$ with the property $pr = 0$ and $1 \neq g_p \in G_p$ with the property $g_p^p = 1$. So, it is easily checked that $r(1 - g_p)^p = 0 = (r(1 - g_p))^p$, whence $r(1 - g_p) = r - rg_p$ is a nilpotent. In fact, if $p = 2$ we have

$$r(1 - g_2)^2 = r(2 - 2g_2) = 2r - 2rg_2 = 0,$$

whereas if $p \geq 3$ we have

$$r(1 - g_p)^p = r(1 - pg_p + \cdots - g_p^p) = r(-pg_p + \cdots) = 0.$$

So, the claim sustained. This leads to $1 + r - rg_p$ is a normalized unit which does not belong to G whenever $r \neq \pm 1$. Thereby $V(RG) = G$ is impossible in this case. If now $r = \pm 1$, hence $p \cdot 1 = 0$, then $\text{char}(R) = p$ and because $G \neq G_p$ (otherwise $G = G_p$ implies $G = G_0$ which is false), the element $1 + g - gg_p$ for $g \in G \setminus G_p$ is a nontrivial unit of $V(RG)$. This allows us to conclude that $V(RG) \neq G$ which is the desired contradiction.

“ \Leftarrow ” Since R is indecomposable and reduced with

$$\text{supp}(G) \cap (\text{inv}(R) \cup \text{zd}(R)) = \emptyset,$$

we wish apply [11] to deduce that $V(RG) = GV(RG_0)$. Furthermore, $V(RG_0) = G_0$ forces that $V(RG) = G$, as wanted. ■

As immediate consequences, we yield the following three statements (compare with [4, p. 59, Problem]).

COROLLARY 2. *Suppose G is an abelian non-identity group and R is the ring of integers of an algebraic number field L . Then $V(RG) = G$ if and only if one of the following points holds:*

- (1) $G_0 = 1$;
- (2) $G_0 \neq 1$,
 - (2.1) $\exp(G_0) = 2$, and $L = \mathbb{Q}$ or $L = \mathbb{Q}(\sqrt{-d})$ where $d = 1$ or d is a square-free positive integer;
 - (2.2) $\exp(G_0) = 4$, and $R = \mathbb{Z}$ or $R = \mathbb{Z}(\sqrt{-1})$;
 - (2.3) $\exp(G_0) = 3$ or $\exp(G_0) = 6$, and $R = \mathbb{Z}$ or $R = \mathbb{Z}[\zeta_3]$ where

$$\zeta_3 = \frac{-1 + \sqrt{-3}}{2}.$$

Proof. First of all, observe that $\text{char}(R) = 0$, whence $\text{zd}(R) = \emptyset$, and $\text{supp}(G) \cap (\text{inv}(R) \cup \text{zd}(R)) = \emptyset$. Moreover, R is an integral domain. So, if G is torsion-free, $V(RG) = G$ is true appealing to [10] or [11].

Let now G be not torsion-free. Henceforth, we may employ Theorem 1 combined with [8, Theorem 1.1]. ■

COROLLARY 3. *Suppose G is an abelian non-identity group whose G_0 is finite and R is an integral domain of characteristic zero such that $\text{supp}(G) \cap \text{inv}(R) = \emptyset$. Then $V(RG) = G$ if and only if one of the following clauses holds:*

- (1) $G_0 = 1$;
- (2) $G_0 \neq 1$,
 - (2.1) $\exp(G_0) = 2$, and $R_2^* = \{a \in R^* : a \equiv 1 \pmod{2}\}$ is torsion;
 - (2.2) $\exp(G_0) = 3$, and R_3^* is torsion, where
$$R_3^* = \{a + b\zeta_3 \in R[\zeta_3] : a + b\zeta_3 \equiv 1 \pmod{(\zeta_3 - 1)}, a^2 + b^2 - ab \in R^*\};$$
 - (2.3) $\exp(G_0) = 4$, and R_4^* is torsion, where
$$R_4^* = \{a + b\zeta_4 \in R[\zeta_4] : a + b\zeta_4 \equiv 1 \pmod{(\zeta_4 - 1)}, a^2 + b^2 \in R^*\};$$
 - (2.4) $\exp(G_0) = 6$, and R_2^* and R_3^* are torsion.

Proof. Follows in the same manner as above according to Theorem 1 and [13, Theorem 1]. ■

COROLLARY 4. *Suppose G is an abelian group for which $|G_0| = 2$ and R is a field with $|R| = 2$. Then $V(RG) = G$ if and only if $G = G_0$.*

Proof. Observe that $\text{char}(R) = 2$, so $2 \in \text{zd}(R)$. Moreover, it is easily checked that $\text{supp}(G) \cap \text{inv}(R) = \emptyset$, while $\text{supp}(G) \cap \text{zd}(R) \neq \emptyset$. Hereafter, utilizing Theorem 1, we infer that $G \neq G_0$ is impossible. So, G has to be torsion of cardinality 2. ■

It is also worthwhile noticing that another verification may be like this:

Since G_0 is cyclic of order 2, one may write $G = G_0 \times M$ for some $M \leq G$. Besides, because $\text{char}(R) = 2$, we write also that

$$V(RG) = V(RM) \times V_0(RG).$$

But, since R is a field, it is well known that $V(RM) = M$. Therefore, $V(RG) = G$ is obviously equivalent to $V_0(RG) = G_0$. However, if we assume that $|G| \geq 3$, then the element $1 + g - gg_0 \in V_0(RG) \setminus G$ whenever $g \in G \setminus G_0$ and $g_0 \in G_0 \setminus \{1\}$, a contradiction. Notice also that $V(RG_0) = G_0$.

3. CONCLUDING DISCUSSION

We foremost remark that it was proved in ([11], [12]) that if R is an integral domain of $\text{char}(R) = 0$ and G is an abelian group such that $\text{supp}(G) \cap \text{inv}(R) = \emptyset$, then $V(RG) = G \times F$ where $F \subseteq V(RG_0)$ with $F_0 = 1$.

We would like to conclude with two more remarks.

Remark 1. In the proof of our result [4, pp. 56–57, Theorem] we certainly absolutely *formally* assumed that the subring $mR \leq R$ with $\text{char}(mR) = p$, a prime, contains the same identity as that of R due to $\text{char}(R) = n > 1$ and $n = mp \neq p$. In fact, the proof is correct since if $V(RG) = G$, then RG has only trivial units, and hence its subring $(mR)G$ has also only trivial units although mR is without the identity of R (and even it may not contain a proper identity element); indeed, if $mr = 1_R$ for some $r \in R$, then $0 = nr = pmr = p \cdot 1_R$, i.e., $\text{char}(R) = \text{char}(mR)$, which is wrong since $m > 1$. However, $mR \subseteq R'$ such that R' contains a proper identity, $\text{char}(R') = p$ and $R'G$ is also without nontrivial units. Therefore, $V(R'G) = G$, as stated.

Remark 2. In [7], both Proposition 2 and Proposition 4 seem to be incorrect. Indeed, they contradict Proposition 8 from [7] when the coefficient ring is of finite characteristic greater than 1. Besides, the condition in Proposition 2 on RC_2 (C_2 is cyclic of order 2) to have only trivial units when $\text{char}(R) = 3$ and $\pm 1 \neq u \in R^*$, is imaginer (see also the final discussion after Proposition 8 in [7]). In fact,

$$2(u - 1) - 1 = -u \in R^* \quad \text{with } u - 1 \neq \{0, 1\},$$

so Proposition 1(iii) of [7] works to obtain a non-trivial unit for RC_2 .

By using this idea, we shall give a new confirmation of clause (2.1) (respectively, (b)) from our Main Theorem (respectively, Corollary) in [4], namely we shall show that if $\text{supp}(G) \cap \text{inv}(R) \neq \emptyset$ (respectively, $\text{char}(R) = p \neq 2$, a prime) and $|G| = 2$ then $V(RG) = G$ if and only if $|R^*| = 2$.

In fact, $2 \in \text{inv}(R)$ and hence $R^* = \{\pm 1_R, 2_R, \dots\}$, so that $|R^*| \geq 3$. We differ two basic cases:

Case 1: $\text{char}(R) = 3$. Note that $2 = -1$. If $V(RG) = G$, we claim that $|R^*| = 2$ since otherwise if $\pm 1 \neq r \in R^*$, then

$$2(r - 1) - 1 = 2r = -r \in R^* \quad \text{with } r - 1 \notin \{0, 1\}.$$

This enables us to consult with Proposition 1(iii) from [7].

Conversely, if $|R^*| = 2$ it follows that $V(RG) = G$ because $2r - 1 \in R^* = \{-1, 1\}$ ensures that $2r - 1 = -1$, i.e., $2r = 0$, i.e., $-r = 0$ and $r = 0$, or that $2r - 1 = 1$, i.e., $2r = 2$, i.e., $-r = -1$ and $r = 1$. That is why does not exist $r \notin \{0, 1\}$ with $2r - 1 \in R^*$, whence again Proposition 1(iii) from [7] works.

Case 2: $\text{char}(R) > 3$. Observing that $2 \cdot 2 - 1 = 3 \in R^*$ since $(3, p) = 1$ when $\text{char}(R) = p$ with $2 \neq \{0, 1\}$ and even more generally $2 \cdot \frac{3}{2} - 1 = 2 \in R^*$ when $\text{supp}(G) \cap \text{inv}(R) \neq \emptyset$ with $\frac{3}{2} \neq \{0, 1\}$, we once again appeal to Proposition 1(iii) of [7] to conclude the proof.

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