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Type II-Λ-Weak Radon-Nikodym Property in a Banach Space Associated with a Compact Metrizable Abelian Group

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Abstract: Let G be a compact metrizable abelian group with normalized Haar measure λ, Γ the dual group of G and Λ a subset of Γ. Let X be a Banach space and $f : G \longrightarrow X$ be a Pettis integrable function with respect to λ. It has been shown that the set $\{\hat{f}(\gamma) : \gamma \in \Lambda\}$ of the Fourier coefficients of f is a relatively norm compact subset of X. We have shown by a counter-example that the converse of this result is not true, in general. We have introduced the idea of type II-Λ-Weak Radon-Nikodym property (type II-Λ-WRNP) of X and have shown that the converse is true for X having this property when Λ is a Riesz set. We have also obtained several necessary and sufficient conditions for X to possess this property when Λ is a Riesz set.

Key words: Compact metrizable abelian group, Pettis integrable functions, Riesz sets, type II-Λ-weak Radon-Nikodym property.

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1. Introduction

In [7], Edgar introduced the idea of Λ-Radon-Nikodym property in a Banach space associated with a compact metrizable abelian group. Dowling called it type I-Λ-Radon- Nikodym property and also introduced another such property called type II-Λ-Radon Nikodym property in [6]. He used these properties to give new characterizations of Riesz subsets and Rosenthal subsets of countable discrete abelian groups. We introduced in [17] the idea of type I-Λ-weak Radon-Nikodym property in a Banach space under the name Λ-weak Radon-Nikodym property.

The object of the present paper is to introduce the idea of type II-Λ-weak Radon-Nikodym property in a Banach space associated with a compact metrizable abelian group. It is observed that type II-Λ-Radon-Nikodym property implies type II-Λ-weak Radon-Nikodym property which, in turn, implies type I-Λ-weak Radon-Nikodym property.

It has been shown that the set of Fourier coefficients of a Pettis integrable function is relatively norm compact and it satisfies some other conditions. The converse of this result has been shown to be true for a Banach space possessing type II-Λ-weak Radon-Nikodym property where Λ is a Riesz set. Several sufficient conditions have been obtained for a Banach space to possess this property and they have been shown to be necessary also when and only when Λ is a Riesz set.

2. Notations and terminologies

Throughout this paper, G will denote a compact metrizable abelian group (under multiplication), $\beta(G)$ is the σ-algebra of Borel subsets of G, and λ is the normalized Haar measure on $\beta(G)$. Let $\Gamma = \hat{G}$ be the dual group of G, the set of continuous homomorphisms $\gamma : G \longrightarrow \mathbb{C}$ with $|\gamma(z)| = 1$, for all $z \in G$. Then Γ is a countable discrete abelian group [14].

Let X be a complex Banach space with dual X'. By a vector measure, we always mean a finitely additive set function from $\beta(G)$ to X.

The set of all X-valued countably additive vector measures of bounded variation defined on $\beta(G)$ is denoted by $V^1(G;X)$. The set of all X-valued vector measures with bounded average range (with respect to Haar measure λ) is denoted by $V^\infty(G;X)$. $V^1(G;X)$ is a Banach space under the variation norm whereas $V^\infty(G;X)$ is a Banach space under the norm

$$\|\mu\|_\infty = \sup\{\|\mu(E)\|/\lambda(E) : E \in \beta(G),\ \lambda(E) > 0\}.$$

Every $\mu \in V^\infty(G;X)$ is countably additive, λ-continuous and of bounded variation, and as such $V^\infty(G;X) \subset V^1(G;X)$.

A function $f : G \longrightarrow X$ is said to be scalarly integrable if $x'f \in L^1(G)$ for each $x' \in X'$. Let us recall that every scalarly integrable function is Dunford integrable [3, p. 52, Lemma II.3.1]. The value of the Dunford integral $\mathrm{D}-\int_E f d\lambda$, $E \in \beta(G)$, lies in X''. If $\mathrm{D}-\int_E f d\lambda$ belongs to X for each $E \in \beta(G)$, then f is called Pettis integrable and we denote it by $\mathrm{P}-\int_E f d\lambda$.

The Fourier coefficients of a Dunford integrable function $f : G \longrightarrow X$ are defined as

$$\hat{f}(\gamma) = \mathrm{D}-\int_G \bar{\gamma} f d\lambda, \quad \gamma \in \Gamma.$$

The Fourier coefficients of a Pettis integrable or a Bochner integrable function are similarly defined.

The Fourier coefficients of a bounded vector measure $\mu : \beta(G) \longrightarrow X$ are defined as

$$\hat{\mu}(\gamma) = \int_G \bar{\gamma} d\mu, \quad \gamma \in \Gamma.$$

The set of all Pettis integrable functions from G to X is denoted by $P(G;X)$. It becomes a normed linear space under the Pettis norm

$$\|f\|_P = \sup_{\|x'\|\leq 1} \int_G |x'f| d\lambda = \sup_{\|x'\|\leq 1} \|x'f\|_1 < \infty, \quad f \in P(G;X).$$

Every $f \in P(G;X)$ induces a countably additive, λ-continuous vector measure $\mu_f : \beta(G) \longrightarrow X$ of σ-finite variation, defined by

$$\mu_f(E) = \text{ P}-\int_E f d\lambda,$$

for all $E \in \beta(G)$.

Since the normalized Haar measure on a compact abelian group G is a finite Radon measure and hence a perfect measure [18, p. 9, Prop. 1-3-2], it follows from [4, p. 149] that the induced vector measure μ_f of an $f \in P(G;X)$ has a relatively norm compact range in X.

If for a vector measure $\mu : \beta(G) \longrightarrow X$, there exists an $f \in P(G;X)$ such that

$$\mu(E) = \text{ P}-\int_E f d\lambda,$$

for all $E \in \beta(G)$, then f is said to be the Pettis derivative of μ. Thus every $f \in P(G;X)$ is the Pettis derivative of its induced vector measure μ_f.

If $f \in P(G;X)$ is the Pettis derivative of a vector measure $\mu : \beta(G) \longrightarrow X$, then it is easy to verify that $\hat{f}(\gamma) = \hat{\mu}(\gamma)$, for all $\gamma \in \Gamma$.

The set of all $f \in P(G;X)$ whose induced vector measures are of bounded variation is denoted by $P^1(G;X)$ so that $P^1(G;X) \subset P(G;X)$.

A function $f : G \longrightarrow X$ is said to be scalarly essentially bounded if $x'f \in L^\infty(G)$ for each $x' \in X'$. The set of all scalarly essentially bounded Pettis integrable functions from G to X is denoted by $P^\infty(G;X)$. Thus $P^\infty(G;X) \subset P^1(G;X) \subset P(G;X)$. If $f \in P(G;X)$, then it can be shown that $f \in P^\infty(G;X)$ if and only if the induced vector measure $\mu_f \in V^\infty(G;X)$.

As usual, we shall denote by $L(L^1(G),X)$ (resp. $L(C(G),X)$) the space of all bounded linear operators from $L^1(G)$ (resp. $C(G)$) to X which is a Banach space under the operator norm. Fourier coefficients of a $T \in L(L^1(G), X)$ (resp. $L(C(G), X)$) are defined by

$$\hat{T}(\gamma) = T(\bar{\gamma}), \quad \gamma \in \Gamma.$$

It is easy to see that the Banach spaces $V^\infty(G; X)$ and $L(L^1(G), X)$ are isometrically isomorphic under the correspondence

$$\mu(E) = T(\chi_E),$$

for all $E \in \beta(G)$, or equivalently

$$T(\phi) = \int_G \phi d\mu,$$

for all $\phi \in L^1(G)$, where $\mu \in V^\infty(G; X)$ and $T \in L(L^1(G), X)$.

If $T \in L(L^1(G), X)$ corresponds to $\mu \in V^\infty(G; X)$, then

$$\hat{T}(\gamma) = T(\bar{\gamma}) = \int_G \bar{\gamma} d\mu = \hat{\mu}(\gamma),$$

for all $\gamma \in \Gamma$.

If $\Lambda \subset \Gamma$, then we define

$$V^1_\Lambda(G; X) = \{\mu \in V^1(G; X) : \hat{\mu}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}$$

and

$$P_\Lambda(G; X) = \{f \in P(G; X) : \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}.$$

Similar definitions are used for $V^\infty_\Lambda(G; X)$, $L^1_\Lambda(G; X)$, $P^\infty_\Lambda(G; X)$, $L^\infty_\Lambda(G; X)$, $P^1_\Lambda(G; X)$, $L_\Lambda(L^1(G), X)$ and $L_\Lambda(C(G), X)$. It is clear that

$$L^\infty(G; X) \subset P^\infty(G; X) \subset V^\infty(G; X) = L(L^1(G), X),$$

and hence

$$L^\infty_\Lambda(G; X) \subset P^\infty_\Lambda(G; X) \subset V^\infty_\Lambda(G; X) = L_\Lambda(L^1(G), X).$$

We define

$$V^1_{\Lambda,\,\mathrm{ac}}(G; X) = \{\mu \in V^1_\Lambda(G; X) : \mu \text{ is absolutely continuous with respect to } \lambda\}.$$

If $\Lambda \subseteq \Gamma$, then Λ is called a Riesz subset of Γ if $V^1_\Lambda(G) = L^1_\Lambda(G)$. It is easy to show that if Λ is a Riesz subset of Γ and X is a Banach space then

$$V^1_\Lambda(G; X) = V^1_{\Lambda,\,\mathrm{ac}}(G; X).$$

A sequence $\{i_n\}$ of measurable functions $i_n : G \longrightarrow \mathbb{R}$ is called a good approximate identity on G [7, p. 202] if it satisfies the following properties (without loss of generality, by [10, p. 298, Theorem 33.12]):

a) $i_n \geq 0$ for $n = 1, 2, \ldots$,

b) $\int_G i_n d\lambda = 1$ for $n = 1, 2, \ldots$,

c) $\operatorname{supp} \hat{i}_n$ is finite and $0 \leq \hat{i}_n \leq 1$ on Γ for $n = 1, 2, \ldots$,

d) $\lim_{n\to\infty} \int_U i_n d\lambda = 1$ for all neighborhoods U of 1 in G.

Then we note that modifying each i_n if necessary on a set of measure 0, one has

$$i_n(t) = \sum_{\gamma\in\Gamma} \hat{i}_n(\gamma)\gamma(t) = \sum_{\gamma\in\operatorname{supp}\hat{i}_n} \hat{i}_n(\gamma)\gamma(t),$$

for all $\gamma \in \Gamma$.

It should be noted that a compact metrizable abelian group always possesses a good approximate identity [10, p. 298, Theorem 33.12].

Let Λ be a subset of Γ and $\{a_\gamma\}_{\gamma\in\Lambda}$ be a bounded subset of X. Let us define, for each positive integer n, the function $F_n : G \longrightarrow X$ by

$$F_n(z) = \sum_{\gamma\in\Lambda} \hat{i}_n(\gamma)a_\gamma\gamma(z), \quad z \in G.$$

It is obvious that $F_n \in L^\infty_\Lambda(G; X)$ for each n. The sequence $\{F_n\}$ is said to be associated with the bounded set $\{a_\gamma\}_{\gamma\in\Lambda}$.

3. Main results

Lemma 1. *If $\Lambda \subset \Gamma$ and $\{a_\gamma\}_{\gamma\in\Lambda}$ is a bounded subset of X with the associated sequence $\{F_n\}$, then*

$$\hat{F}_n(\gamma) = \begin{cases} \hat{i}_n(\gamma)a_\gamma & \text{for } \gamma \in \Lambda, \\ 0 & \text{for } \gamma \notin \Lambda, \end{cases}$$

and hence

$$\hat{F}_n(\gamma) \longrightarrow \begin{cases} a_\gamma & \text{for } \gamma \in \Lambda, \\ 0 & \text{for } \gamma \notin \Lambda, \end{cases}$$

in the norm topology of X.

Proof. The proof is straightforward and follows from the orthogonality relation as given in [14, p. 10] and from the fact $\lim \hat{i}_n(\gamma) = 1$ for $\gamma \in \Gamma$. ∎

Theorem 2. *Let* $\mu : \beta(G) \longrightarrow X$ *be a finitely additive bounded vector measure and* Λ *be any subset of* Γ. *Let* $S = \{\hat{\mu}(\gamma) : \gamma \in \Lambda\}$. *Then*

(a) *S is a bounded subset of X,*

(b) *if μ is countably additive, then S is relatively weakly compact,*

(c) *if μ is countably additive and has a relatively norm compact range, then S is relatively norm compact in X,*

(d) *if μ is countably additive and $\hat{\mu}(\gamma) = 0$ for all $\gamma \notin \Lambda$, then the sequence $\{F_n\}$ associated with the set S is bounded in $P(G; X)$,*

(e) *if μ is countably additive and $\hat{\mu}(\gamma) = 0$ for all $\gamma \notin \Lambda$, then $\mu \in V^1_\Lambda(G; X)$ if and only if the sequence $\{F_n\}$ associated with the set $\{\hat{\mu}(\gamma) : \gamma \in \Gamma\}$ is bounded in $L^1_\Lambda(G; X)$.*

Proof. We prove Part (d) only as the proofs of the other parts are easy. For example, (b) follows from [11, p. 264, Lemma 2].

Part (d): As in the proof of Theorem 1 in [1, p. 111], we have that for each $x' \in X'$, $\|x'F_n\|_1 \leq |x'\mu|(G)$, and hence

$$\sup_{\|x'\| \leq 1} \|x'F_n\|_1 \leq \sup_{\|x'\| \leq 1} |x'\mu|(G) = \|\mu\|(G),$$

where $\|\mu\|(\cdot)$ is the semivariation of μ. Thus $\|F_n\|_P \leq \|\mu\|(G)$ for all n. This shows that the associated sequence is bounded in $P(G; X)$. ∎

Corollary 3. (a) *If $f : G \longrightarrow X$ is a Pettis integrable function, then for any subset Λ of Γ, the set $\{\hat{f}(\gamma)\}_{\gamma \in \Lambda}$ is relatively norm compact in X.*

(b) *If $f \in P_\Lambda(G; X)$, then the sequence $\{F_n\}$ associated with $\{\hat{f}(\gamma)\}_{\gamma \in \Lambda}$ is bounded in $P(G; X)$.*

Remark. If $f : G \longrightarrow X$ is a Dunford integrable function, then for any subset Λ of Γ, the set $\{\hat{f}(\gamma)\}_{\gamma \in \Lambda}$ is bounded in X'', the proof being straightforward as

$$\sup_{\gamma \in \Lambda} \|\hat{f}(\gamma)\| \leq \sup_{\|x'\| \leq 1} \int_G |x'f| d\lambda < \infty.$$

Theorem 4. *If Λ is a subset of Γ and $\{a_\gamma\}_{\gamma \in \Lambda}$ is a subset of X, then the following statements are equivalent:*

(a) *The set $\{a_\gamma\}_{\gamma \in \Lambda}$ is relatively weakly compact in X and the corresponding associated sequence $\{F_n\}$ is bounded in $L^1_\Lambda(G; X)$.*

(b) *The set* $\{a_\gamma\}_{\gamma\in\Lambda}$ *is bounded in* X *and the corresponding associated sequence* $\{F_n\}$ *is bounded in* $L^1_\Lambda(G;X)$.

(c) *There exists a* $\mu \in V^1_\Lambda(G;X)$ *such that* $\hat{\mu}(\gamma) = a_\gamma$ *for all* $\gamma \in \Lambda$.

(d) *There exists an absolutely summing operator* $T : C(G) \longrightarrow X$ *such that* $\hat{T}(\gamma) = a_\gamma$ *for all* $\gamma \in \Lambda$, *and* $\hat{T}(\gamma) = 0$ *for* $\gamma \notin \Lambda$.

Proof. (a) ⇒ (b) It is trivial.

(b) ⇒ (c) It follows from Theorem 2 of [1].

(c) ⇒ (d) Let there exist a $\mu \in V^1_\Lambda(G;X)$ such that $\hat{\mu}(\gamma) = a_\gamma$ for all $\gamma \in \Lambda$. Then there exists a bounded linear operator $T : C(G) \longrightarrow X$ whose representing measure is μ [3, p. 6, Theorem I.1.13 and p. 153, Definition VI.2.2]. Clearly $\hat{T}(\gamma) = \hat{\mu}(\gamma)$ for all $\gamma \in \Gamma$. Hence $\hat{T}(\gamma) = a_\gamma$ for all $\gamma \in \Lambda$, and $\hat{T}(\gamma) = 0$ for $\gamma \notin \Lambda$. Since μ is of bounded variation, T is absolutely summing [3, p. 162, Theorem VI.3.3].

(d) ⇒ (c) Let there exist an absolutely summing operator $T : C(G) \longrightarrow X$ such that $\hat{T}(\gamma) = a_\gamma$ for all $\gamma \in \Lambda$, and $\hat{T}(\gamma) = 0$ for $\gamma \notin \Lambda$. Then T is weakly compact [3, p. 164, Corollary VI.3.5]. So there exist a countably additive vector measure $\mu : \beta(G) \longrightarrow X$ such that $\hat{\mu}(\gamma) = \hat{T}(\gamma)$ for all $\gamma \in \Gamma$ [3, p. 152, Theorem VI.2.1 and p. 153, Theorem VI.2.5]. Hence $\hat{\mu}(\gamma) = a_\gamma$ for all $\gamma \in \Lambda$ and $\hat{\mu}(\gamma) = 0$ for all $\gamma \notin \Lambda$. Since T is absolutely summing, it follows from [3, p. 162, Theorem VI.3.3] that μ is of bounded variation. Thus $\mu \in V^1_\Lambda(G;X)$ with $\hat{\mu}(\gamma) = a_\gamma$ for all $\gamma \in \Lambda$.

(c) ⇒ (a) It follows from Theorem 2. ∎

LEMMA 5. *Let* $f : G \longrightarrow X$ *be a Dunford integrable function. If there exists a countably additive vector measure* $\mu : \beta(G) \longrightarrow X$ *such that* $\hat{\mu}(\gamma) = \hat{f}(\gamma)$ *for all* $\gamma \in \Gamma$, *then* f *is Pettis integrable with* μ *as its induced vector measure.*

Proof. The proof is straightforward. ∎

Combining Theorem 4 and Lemma 5, we have the following important result:

COROLLARY 6. *Let* $f : G \longrightarrow X$ *be a scalarly integrable function such that* $\hat{f}(\gamma) \in X$ *for all* $\gamma \in \Gamma$. *If* X *contains no copy of* c_0, *then* f *is Pettis integrable.*

Proof. By hypothesis and the density of D (trigonometric polynomials) inside $C(G)$, the operator $T : C(G) \longrightarrow X$ given by $\phi \longrightarrow$ D$-\int_G \bar{\phi} f d\lambda$ is well defined and bounded. By [3, p. 159, Theorem VI.2.15], T must be weakly compact and thus its representing measure μ is countably additive and takes its values in X [3, p. 153, Theorem VI.2.5]. It is easy to see that $T(\gamma) = \hat{f}(\gamma)$ for all $\gamma \in \Gamma$. Finally, since $\hat{\mu}(\gamma) = T(\gamma)$ (applying [3, p. 152, Theorem VI.2.1(iii)]), one gets $\hat{\mu}(\gamma) = \hat{f}(\gamma)$ for all $\gamma \in \Gamma$ and Lemma 5 completes the proof. ∎

Following Dinculeanu [5, p. 73, Definition 7], we define the convolution of a vector-valued function f with a scalar-valued function ϕ.

Definition 7. Let $f : G \longrightarrow X$ be a vector-valued function and $\phi : G \longrightarrow C$ be a scalar-valued function. Let G_0 be the set of all points $t \in G$ such that the mapping $s \longrightarrow f(s)\phi(ts^{-1})$ is scalarly integrable. We define the convolution $f \star \phi : G_0 \longrightarrow X''$ by

$$(f \star \phi)(t) = \mathrm{D}-\int_G f(s)\phi(ts^{-1})d\lambda(s)$$

for $t \in G_0$. Similarly $\phi \star f$ is defined and $f \star \phi = \phi \star f$.

According to Dinculeanu [5, p. 73, Definition 7], if the mapping $s \longrightarrow f(s)\phi(ts^{-1})$ is Pettis integrable for all $t \in G_0$, then the convolution $f \star \phi : G_0 \longrightarrow X \subset X''$ is defined by

$$(f \star \phi)(t) = \mathrm{P}-\int_G f(s)\phi(ts^{-1})d\lambda(s)$$

for $t \in G_0$.

Lemma 8. *If $f : G \longrightarrow X$ is scalarly integrable and $\phi \in L^\infty(G)$, then $f * \phi$ is defined everywhere on G with values in X'' and*

$$\|(f \star \phi)(t)\| \leq \|f\|_P \|\phi\|_\infty$$

for $t \in G$.

Proof. Easy. ∎

Lemma 9. *If $f : G \longrightarrow X$ is Pettis integrable and $\phi \in L^\infty(G)$, then $f \star \phi$ is defined everywhere on G with values in X and is Pettis integrable and weakly equivalent to a Bochner integrable function.*

Proof. The first part follows from [5, p. 73, Proposition 9]. For the second and third parts, we have $x'f \in L^1(G)$ for $x' \in X'$. Also $\phi \in L^\infty(G)$. Hence $x'f \star \phi = x'(f \star \phi)$ is a uniformly continuous, in particular, continuous scalar-valued function on G [14, p. 4]. So $f \star \phi$ is scalarly measurable and weakly continuous on G with values in X. Since G is compact, $f \star \phi$ has a weakly compact range in X. Hence the result follows from [2, p. 259, Corollary 19].

However a direct proof of the second part (i.e., that $f \star \phi$ is Pettis integrable) can be made as follows:

Since $x'f \star \phi(t) = \int_G \phi(ts^{-1}) < f(s), x' > ds$ we can see that this is a measurable map; moreover, $\|f \star \phi(t)\| \leq \|\phi\|_\infty \|f\|_P$ for all $t \in G$. Thus $f \star \phi$ is Dunford-integrable. Let $T : L^\infty(G) \longrightarrow X''$ be the operator defined by

$$T(g) = \mathrm{D}\text{–}\int_G g(f \star \phi) d\lambda.$$

We need to see that $T(L^\infty(G)) \subset X$. For each $x' \in X'$ we have, by Fubini's theorem and Pettis integrability of f:

$$\begin{aligned}\langle T(g), x'\rangle &= \int_G g(t)\langle f \star \phi(t), x'\rangle\, dt = \int_G g(t) \left(\int_G \phi(ts^{-1})\langle f(s), x'\rangle ds \right) dt \\ &= \int_G \left(\int_G g(t)\phi(ts^{-1}) dt \right) \langle f(s), x'\rangle\, ds = \int_G \langle h(s)f(s), x'\rangle\, ds \\ &= \Big\langle \mathrm{P}\text{–}\int_G hf d\lambda, x' \Big\rangle\end{aligned}$$

for $h(s) = \int_G g(t)\phi(ts^{-1})dt$. Since $\mathrm{P}\text{–}\int_G hf d\lambda \in X$, it follows that $Tg \in X$. ∎

THEOREM 10. *If $f : G \longrightarrow X$ is a scalarly integrable function, then for any good approximate identity $\{i_n\}$ on G, $i_n \star f$ is defined everywhere on G with values in X'' for each n. Let $\{F_n\}$ be the sequence associated with the bounded set $\{\hat{f}(\gamma)\}_{\gamma \in \Gamma}$ of X''. Then $F_n = i_n \star f$ and*

$$\widehat{(i_n \star f)}(\gamma) = \hat{i_n}(\gamma)\hat{f}(\gamma)$$

for each n, and $\hat{F}_n(\gamma) \longrightarrow \hat{f}(\gamma)$ for each $\gamma \in \Gamma$ in the norm topology of X''. Also

$$\|x'F_n - x'f\|_1 \longrightarrow 0$$

for each $x' \in X'$.

If f is Pettis integrable, then $i_n \star f$ takes its values in X for each n, and $\hat{F_n}(\gamma) \longrightarrow \hat{f}(\gamma)$ for each $\gamma \in \Gamma$ in the norm topology of X. Also $F_n \longrightarrow f$ in Pettis norm.

Proof. An easy calculation shows that $i_n \star f$ is defined everywhere on G with values in X''. By suitable modification of the arguments as given in the proof of (c) ⇒ (a) of the Theorem in [7, p. 203], we get $F_n = i_n \star f$.

The next part also follows easily. We only prove that $F_n \longrightarrow f$ in Pettis norm. Let us define $T_n : L^1(G) \longrightarrow L^1(G)$ by

$$T_n(\phi) = i_n \star \phi$$

for all $\phi \in L^1(G)$. Then $T_n \in L(L^1(G), L^1(G))$ for all n. Let $T(\phi) = \phi$ for all $\phi \in L^1(G)$. Then $T \in L(L^1(G), L^1(G))$. Now $i_n \star \phi \longrightarrow \phi$ implies that $T_n(\phi) \longrightarrow T(\phi) = \phi$ for each $\phi \in L^1(G)$. Therefore $T_n \longrightarrow T$ uniformly on every compact set of $L^1(G)$ [9, p. 43]. Since f is Pettis integrable, its induced vector measure has a relatively norm compact range. Hence the set $\{x'f : \|x'\| \leq 1\}$ is relatively norm compact in $L^1(G)$ [4, p. 149] and so $T_n \longrightarrow T$ uniformly on this set. Thus $T_n(x'f) \longrightarrow T(x'f)$ uniformly on $\{x' \in X' : \|x'\| \leq 1\}$. So

$$\sup_{\|x'\| \leq 1} \|T_n(x'f) - T(x'f)\|_1 \longrightarrow 0,$$

which implies that $\|F_n - f\|_P \longrightarrow 0$. ∎

THEOREM 11. *Let $\mu : \beta(G) \longrightarrow X$ be a countably additive and λ-continuous vector measure. If the sequence $\{F_n\}$ associated with the bounded set $\{\hat{\mu}(\gamma)\}_{\gamma \in \Gamma}$ has a weak λ-almost everywhere limit $f : G \longrightarrow X$, then f is Pettis integrable with μ as its induced vector measure.*

Proof. The scalar measure $x'\mu$ has a Radon-Nikodym derivative $\phi_{x'} \in L^1(G)$ with respect to λ such that $x'F_n \longrightarrow \phi_{x'}$ in $L^1(G)$ for each $x' \in X'$. Then, by hypothesis, it follows that $x'f = \phi_{x'}$ λ-almost everywhere. So f is scalarly integrable. An easy calculation shows that $\hat{f}(\gamma) = \hat{\mu}(\gamma)$ for all $\gamma \in \Gamma$. So by Lemma 5, f is Pettis integrable. ∎

Remark. If the measure $\mu : \beta(G) \longrightarrow X$ that appears in Theorem 11 is not λ-continuous, the function f is Dunford-integrable since

$$x'F_n(t) = \int_G i_n(ts^{-1})d(x'\mu)(s)$$

and
$$\int_G |x'f|d\lambda \leq \lim_n \int_G |x'F_n(t)|dt \leq |x'\mu|(G) \leq ||\mu||(G)$$
by Fatou's lemma.

Definition 12. Let G be a compact metrizable abelian group with dual group Γ. Let Λ be a subset of Γ. A Banach space X is said to have type II-Λ-weak Radon-Nikodym property (type II-Λ-WRNP) if every $\mu \in V^1_{\Lambda,\,\mathrm{ac}}(G;X)$ has a Pettis derivative in $P^1_\Lambda(G;X)$.

A Banach space X is said to have type I-Λ-weak Radon-Nikodym property (type I-Λ-WRNP) if every $\mu \in V^\infty_\Lambda(G;X)$ has a Pettis derivative in $P^\infty_\Lambda(G;X)$. It was introduced by us in [17] under the name Λ-weak Radon-Nikodym property.

Since $V^\infty_\Lambda(G;X) \subset V^1_{\Lambda,\,\mathrm{ac}}(G;X)$ and since Pettis derivative of a vector measure in $V^\infty_\Lambda(G;X)$ belongs to $P^\infty_\Lambda(G;X)$, it follows that type II-Λ-WRNP implies type I-Λ-WRNP, for any subset Λ of Γ.

If $G = \Pi$, the circle group, then $\Gamma = \mathbb{Z}$ (the set of all integers) and type II-$\mathbb{Z}$-WRNP is equivalent to usual WRNP and type II-$\mathbb{N}$-WRNP is equivalent to usual AWRNP [15].

If, in the above definition, Pettis derivative is replaced by Bochner derivative, then one gets type II-Λ-RNP [6].

It is obvious that type II-Λ-RNP implies type II-Λ-WRNP, but the converse is not true as AWRNP does not imply ARNP [12], which are equivalent to type II-$\mathbb{N}$-WRNP and type II-$\mathbb{N}$-RNP respectively. However in a separable Banach space, the converse is true.

If Λ is a finite subset of Γ then if $\mu \in V^1_{\Lambda,\,\mathrm{ac}}(G;X)$ and $f(t) = \Sigma_{\gamma\in\Lambda}\hat{\mu}(\gamma)\gamma(t)$ then it is clear that $f \in L^1_\Lambda(G;X)$ and $\hat{\mu}_f = \hat{\mu}$. Therefore $\mu_f = \mu$ which implies that f is the Bochner derivative of μ. Thus every Banach space has type II-Λ-RNP and hence type II-Λ-WRNP.

In [17] we presented some characterizations of type I-Λ-WRNP. Using Theorem 10, one can easily prove the following corollary.

Corollary 13. *Let $\Lambda \subset \Gamma$ and X be a complex Banach space. Then the following conditions are equivalent:*

(a) *X has type* I-Λ-WRNP.

(b) *If $\{a_\gamma\}_{\gamma\in\Lambda}$ is a bounded set in X such that the associated sequence $\{F_n\}$ is bounded in $L^\infty_\Lambda(G;X)$, then there exists an $f \in P^\infty_\Lambda(G;X)$ such that $F_n \longrightarrow f$ in Pettis norm.*

Now we give an example to show that the converse of Corollary 3(a) is not true, in general.

Example 14. Let us consider the complex Banach space X introduced by Ghoussoub, Maurey and Schachermayer [8, V.4] and let $G = \Pi$, the circle group for which the dual group is $\Gamma = \mathbb{Z}$. Then X does not have the AWRNP [12, p. 7, Remarks 2] and X'' has the WRNP [12, p. 3, Theorem 2.1]. So there exists a countably additive vector measure $\mu : \beta(G) \longrightarrow X$ of bounded variation whose negative Fourier coefficients vanish but μ has no Pettis derivative. Thus the set $S = \{\hat{\mu}(\gamma) : \gamma \in \Gamma\}$ cannot coincide with the set of Fourier coefficients of a Pettis integrable function. It follows by Theorem 2(d) that the sequence associated with S is bounded in $P(G, X)$. Again as X'' has the WRNP, it has the compact range property (CRP) [13]. So X has also CRP. Hence μ has a relatively norm compact range in X which implies that the set S is relatively norm compact in X. Thus we obtain a relatively norm compact set S in X, the associated sequence of which is bounded in $P(G, X)$, which does not coincide with the set of Fourier coefficients of a Pettis integrable function. This shows that the converse of Corollary 3(a) is not true, in general.

The following theorem shows that type II-Λ-WRNP guarantees the converse assertion when Λ is a Riesz set.

Theorem 15. *Let G be a compact metrizable abelian group with dual group Γ. Let Λ be a subset of Γ and let X be a Banach space. Let us consider the following statements:*

(a) *X has type* II-Λ-WRNP.

(b) *Every $\mu \in V^1_\Lambda(G; X)$ has a Pettis derivative $f \in P^1_\Lambda(G; X)$.*

(c) *For every $\mu \in V^1_\Lambda(G; X)$, there exists an $f \in P^1_\Lambda(G; X)$ such that $\hat{\mu}(\gamma) = \hat{f}(\gamma)$ for all $\gamma \in \Gamma$.*

(d) *For each absolutely summing operator $T \in L_\Lambda(C(G), X)$, there exists an $f \in P^1_\Lambda(G; X)$ such that $\hat{T}(\gamma) = \hat{f}(\gamma)$ for all $\gamma \in \Gamma$.*

(e) *Each absolutely summing operator $T \in L_\Lambda(C(G), X)$ is representable by an $f \in P^1_\Lambda(G; X)$.*

(f) *If $\{a_\gamma\}_{\gamma\in\Lambda}$ is a bounded set in X such that the associated sequence $\{F_n\}$ is bounded in $L^1_\Lambda(G; X)$, then there exists an $f \in P^1_\Lambda(G; X)$ such that $\hat{f}(\gamma) = a_\gamma$ for $\gamma \in \Lambda$.*

(g) *If* $\{a_\gamma\}_{\gamma\in\Lambda}$ *is the same as in* (f)*, then there exists an* $f \in P^1_\Lambda(G;X)$ *such that* $F_n \longrightarrow f$ *in Pettis norm.*

(h) *If* $\{a_\gamma\}_{\gamma\in\Lambda}$ *is the same as in* (f)*, then there exists an* $f \in P^1_\Lambda(G;X)$ *such that for each* $x' \in X'$, $\|x'F_n - x'f\|_1 \longrightarrow 0$.

(i) *If* $\{a_\gamma\}_{\gamma\in\Lambda}$ *is the same as in* (f)*, then there exists an* $f \in P^1_\Lambda(G;X)$ *such that for each* $x' \in X'$, $x'F_n \longrightarrow x'f$ *weakly in* $L^1(G)$.

(j) *If* $\{a_\gamma\}_{\gamma\in\Lambda}$ *is the same as in* (f)*, then there exists an* $f \in P^1_\Lambda(G;X)$ *such that for each* $\gamma \in \Gamma$, $\hat{F}_n(\gamma) \longrightarrow \hat{f}(\gamma)$ *weakly in* X.

(k) *If* $\{a_\gamma\}_{\gamma\in\Lambda}$ *is the same as in* (f)*, then there exists an* $f \in P^1_\Lambda(G;X)$ *such that for each* $\gamma \in \Gamma$, $\hat{F}_n(\gamma) \longrightarrow \hat{f}(\gamma)$ *in the norm topology of* X.

Then (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f) $\Leftrightarrow$ (g) $\Leftrightarrow$ (h) $\Leftrightarrow$ (i) $\Leftrightarrow$ (j) $\Leftrightarrow$ (k) $\Rightarrow$ (a)*. If* Λ *is a Riesz set, then all the conditions are equivalent. Conversely, if all the conditions are equivalent, then* Λ *is a Riesz set.*

Proof. (b) $\Rightarrow$ (c) Let $\mu \in V^1_\Lambda(G;X)$. Then by (b), μ has a Pettis derivative $f \in P^1_\Lambda(G;X)$. Hence $\hat{\mu}(\gamma) = \hat{f}(\gamma)$ for all $\gamma \in \Gamma$ and (c) follows.

(c) $\Rightarrow$ (d) It follows from Theorem 4.

(d) $\Rightarrow$ (e) Let $T \in L_\Lambda(C(G),X)$ be an absolutely summing operator. Then by (d), there exists an $f \in P^1_\Lambda(G;X)$ such that

$$T(\gamma) = \hat{T}(\bar{\gamma}) = \hat{f}(\bar{\gamma}) = \mathrm{P}-\int_G f(z)\gamma(z)d\lambda(z)$$

for all $\gamma \in \Gamma$. Hence for any trigonometric polynomial

$$\phi(z) = \sum_{i=1}^{n} c_i\gamma_i(z), \quad \gamma_i \in \Gamma,\ c_i \in C,$$

we have

$$T(\phi) = \mathrm{P}-\int_G \phi(z)f(z)d\lambda(z).$$

Since trigonometric polynomials are dense in $C(G)$, it follows from [16, p. 246, Theorem 2.4 (b)] that

$$T(\phi) = \mathrm{P}-\int_G \phi(z)f(z)d\lambda(z)$$

for all $\phi \in C(G)$. This shows that T is representable by $f \in P^1_\Lambda(G;X)$ and (e) follows.

(e) $\Rightarrow$(f) Let $\{a_\gamma\}_{\gamma\in\Lambda}$ be a bounded set in X such that the associated sequence $\{F_n\}$ is bounded in $L^1_\Lambda(G;X)$. Then by Theorem 4, there exists an absolutely summing operator $T \in L_\Lambda(C(G),X)$ such that $\hat{T}(\gamma) = a_\gamma$ for all $\gamma \in \Lambda$. By (e), T is representable by an $f \in P^1_\Lambda(G;X)$. Therefore

$$a_\gamma = \hat{T}(\gamma) = \mathrm{P}-\int_G f(z)\bar{\gamma}(z)d\lambda(z) = \hat{f}(\gamma)$$

for all $\gamma \in \Lambda$ and (f) follows.

(f) $\Rightarrow$(g) It follows from Theorem 10.

The implications (g) $\Rightarrow$ (h) $\Rightarrow$ (i) are trivial and (i) $\Rightarrow$ (j) follows from the fact that $\Gamma \subset L^\infty(G)$.

(j) $\Rightarrow$ (k) Let$\{a_\gamma\}_{\gamma\in\Lambda}$ be a bounded set in X such that the associated sequence $\{F_n\}$ is bounded in $L^1_\Lambda(G;X)$. Then by (j), there exists an $f \in P^1_\Lambda(G;X)$ such that for each $\gamma \in \Gamma$, $\hat{F}_n(\gamma) \longrightarrow \hat{f}(\gamma)$ weakly in X. By Lemma 1, for each $\gamma \in \Lambda$, $\hat{F}_n(\gamma) \longrightarrow a_\gamma$ in the norm topology of X and hence weakly in X. Consequently, for each $\gamma \in \Lambda$, $a_\gamma = \hat{f}(\gamma)$ and $\hat{F}_n(\gamma) \longrightarrow \hat{f}(\gamma)$ in the norm topology of X. Since $\hat{f}(\gamma) = 0$ for all $\gamma \notin \Lambda$, by Lemma 1, $\hat{F}_n(\gamma) \longrightarrow \hat{f}(\gamma)$ for all $\gamma \in \Gamma$ in the norm topology of X and (k) follows.

(k) $\Rightarrow$ (b) Let $\mu \in V^1_\Lambda(G;X)$ and $\hat{\mu}(\gamma) = a_\gamma$ for $\gamma \in \Lambda$. Then by Theorem 4, the set $\{a_\gamma\}_{\gamma\in\Lambda}$ is bounded in X and the corresponding associated sequence $\{F_n\}$ is bounded in $L^1_\Lambda(G;X)$. Hence by (k), there exists an $f \in P^1_\Lambda(G;X)$ such that for each $\gamma \in \Gamma$, $\hat{F}_n(\gamma) \longrightarrow \hat{f}(\gamma)$ in the norm topology of X and hence weakly in X. Therefore for each $x' \in X'$, $x'(\hat{F}_n(\gamma)) \longrightarrow x'(\hat{f}(\gamma))$.

Now it is easy to verify that for each $x' \in X'$ and for each n, $i_n \star (x'\mu) = x'F_n$ and hence $(\widehat{i_n \star x'\mu})(\gamma) = (\widehat{x'F_n})(\gamma)$, i.e., $\hat{i}_n(\gamma)(\widehat{x'\mu})(\gamma) = x'(\hat{F}_n(\gamma))$, for all $\gamma \in \Gamma$. Hence $x'(\hat{F}_n(\gamma)) \longrightarrow (\widehat{x'\mu})(\gamma)$ as $\hat{i}_n(\gamma) \longrightarrow 1$ for all $\gamma \in \Gamma$. Consequently

$$(\widehat{x'\mu})(\gamma) = x'\big(\hat{f}(\gamma)\big) = x'\big(\hat{\mu}_f(\gamma)\big) = (\widehat{x'\mu_f})(\gamma)$$

for all $\gamma \in \Gamma$ where μ_f is the induced vector measure of f. Hence by uniqueness Theorem [14, p. 17], $x'\mu = x'\mu_f$ for all $x' \in X'$ and so $\mu = \mu_f$. This implies that f is the Pettis derivative of μ and thus (b) follows.

(b) $\Rightarrow$ (a) It follows from Definition 13.

Now let Λ be a Riesz set. Then $V^1_{\Lambda,\,\mathrm{ac}}(G;X) = V^1_\Lambda(G;X)$ and hence (a) $\Rightarrow$ (b) and thus all the conditions are equivalent.

Conversely, let all the conditions be equivalent. Let X be a Banach space having type II-Λ-WRNP. Then (b) holds. Let $m \in V^1_\Lambda(G)$ and $x \in X$ with

$x \neq 0$. We define $\mu : \beta(G) \longrightarrow X$ by $\mu(E) = xm(E)$ for all $E \in \beta(G)$. Clearly $\mu \in V^1(G;X)$ and $\hat{\mu}(\gamma) = x\hat{m}(\gamma)$ for all $\gamma \in \Gamma$. Hence $\hat{\mu}(\gamma) = 0$ for $\gamma \notin \Lambda$ which implies that $\mu \in V^1_\Lambda(G;X)$. Hence by (b), μ has a Pettis derivative $f \in P^1_\Lambda(G;X)$. Therefore $\mu << \lambda$ and so $m << \lambda$ and hence has a Radon-Nikodym derivative with respect to λ. This shows that Λ is a Riesz set. ∎

THEOREM 16. *Let Λ be a subset of Γ. If for any bounded subset $\{a_\gamma\}_{\gamma\in\Lambda}$ of X, the associated sequence $\{F_n\}$ has a weak λ-almost everywhere limit $f : G \longrightarrow X$, then X has type* II-Λ-WRNP.

Proof. Let $\mu \in V^1_{\Lambda,\,\mathrm{ac}}(G;X)$. Then $\{\hat{\mu}(\gamma)\}_{\gamma\in\Lambda}$ is a bounded set in X. Hence by hypothesis, the associated sequence $\{F_n\}$ has a weak λ-almost everywhere limit $f : G \longrightarrow X$. By Theorem 11, f is Pettis integrable with μ as its induced vector measure. Since $\mu \in V^1_{\Lambda,\,\mathrm{ac}}(G;X)$, $f \in P^1_\Lambda(G;X)$. Also f is the Pettis derivative of μ. Hence X has type II-Λ-WRNP. ∎

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