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Weighted Composition Operators Between Besov Spaces

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Abstract: In this paper, we study the boundedness, the compactness and the essential norm of weighted composition operators between Besov spaces. We also find estimates for the essential norm of weighted composition operators between S^p spaces.

Key words: Besov spaces, Carleson measure, essential norm, S^p spaces, weighted composition operators.

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1. Introduction

Let $H(\mathbf{D})$ denote the space of holomorphic functions on the unit disk $\mathbf{D}$. Suppose φ and ψ are holomorphic functions defined on $\mathbf{D}$ such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. The weighted composition operator $W_{\varphi,\psi}$ is defined as follows:

$$W_{\varphi,\psi}(f)(z) = \psi(z) f(\varphi(z))$$

for all f holomorphic on $\mathbf{D}$.

For the study of weighted composition operators one can refer to [3], [5], [7], [9], [15], [16], [17], [21] and references therein.

Fix any $a \in \mathbf{D}$ and let $\sigma_a(z)$ be the Möbius transformation, defined by

$$\sigma_a(z) = \frac{a - z}{1 - \overline{a} z}\,, \qquad z \in \mathbf{D}\,.$$

We denote the set of all Möbius transformations on $\mathbf{D}$ by G. Also, the inverse of σ_a under composition is again σ_a for $z \in \mathbf{D}$. Further, we have

$$\left|\sigma_a'(z)\right| = \frac{1 - |a|^2}{|1 - \overline{a} z|^2}$$

and

$$1 - |\sigma_a(z)|^2 = \frac{\left(1 - |a|^2\right)\left(1 - |z|^2\right)}{|1 - \overline{a} z|^2} = \left(1 - |z|^2\right)\left|\sigma_a'(z)\right| \tag{1.1}$$

for all $a, z \in \mathbf{D}$.

Fix $1 < p < \infty$ and $-1 < q < \infty$. Then f is in the Besov type space $\mathbf{B}_{p,q}$ if

$$\|f\|_{\mathbf{B}_{p,q}} = \left(\int_{\mathbf{D}} \left|f^{'}(z)\right|^{p} \left(1-|z|^2\right)^{q} \mathrm{d}A(z)\right)^{\frac{1}{p}} < \infty\,, \tag{1.2}$$

where $\mathrm{d}A(z)$ denotes the Lebesgue area measure on $\mathbf{D}$.

Also, if we take $1 < p < \infty$ and $q = p-2$ in (1.2), then we get the analytic Besov space $\mathbf{B}_p$. That is, an analytic function f is in the analytic Besov space $\mathbf{B}_p$ if

$$\|f\|_{\mathbf{B}_{p}} = \left(\int_{\mathbf{D}} \left|f^{'}(z)\right|^{p} \left(1-|z|^2\right)^{p-2} \mathrm{d}A(z)\right)^{\frac{1}{p}} < \infty\,. \tag{1.3}$$

Again, if $p = 2$ and $-1 < q < \infty$ in (1.2), then we get the weighted Dirichlet spaces $\mathbf{D}_q$, and for $1 \le p \le 2$ and $q = 0$, we get the Dirichlet type spaces $\mathbf{D}^p$. Also, for $1 \le p < \infty$, $\mathbf{B}_{p,p}$ is the Bergman space A^p.

We can see that $|f(0)| + \|f\|_{p,q}$ is a norm on $\mathbf{B}_{p,q}$, that makes it a Banach space. Moreover, we can observe that, for f to be in $\mathbf{B}_{p,q}$ or $\mathbf{B}_p$, it is necessary that the derivative of f belong to the weighted Bergman spaces A^p_q or A^p_{p-2}. Also, for $1 < p < q < \infty$, we have the relation $\mathbf{B}_p \subset \mathbf{B}_q$.

The Besov space $\mathbf{B}_p$ is invariant under Möbius transformations. That is, if $f \in \mathbf{B}_p$, then $f \circ \varphi \in \mathbf{B}_p$, for all $\varphi \in G$. Moreover from the definition of norm on $\mathbf{B}_p$, we have

$$\|f \circ \varphi\|_{\mathbf{B}_p} = \|f\|_{\mathbf{B}_p} \tag{1.4}$$

for all $f \in \mathbf{B}_p$, see [1].

Let μ be a positive measure on $\mathbf{D}$. Then the space $\mathbf{D}_p(\mu)$ is defined as the space of all holomorphic functions $f \in H(\mathbf{D})$ for which $f^{'} \in L^p(\mathbf{D}, \mu)$. Also, the norm on $\mathbf{D}_p(\mu)$ is defined as

$$\|f\|^p_{\mathbf{D}_p(\mu)} = \int_{\mathbf{D}} \left|f^{'}(z)\right|^p \mathrm{d}\mu(z)\,.$$

For $1 \le p \le \infty$, we denote by $H^p(\mathbf{D})$ the Hardy space of the unit disk $\mathbf{D}$, see [4]. We denote by S^p, the space of those holomorphic functions on $\mathbf{D}$ with first derivative in the Hardy space $H^p(\mathbf{D})$. For each such function f we define the S^p norm of f by

$$\|f\|_{S^p} = |f(0)| + \|f^{'}\|_{H^p}\,.$$

We note that S^p is a Banach space with this norm.

In this paper we study weighted composition operators between Besov spaces by using Carleson measures.

2. Bounded and compact weighted composition operators

In this section, we characterize boundedness and compactness of $W_{\varphi,\psi}$ by using Carleson measures.

Definition 2.1. Take $0 < p < \infty$. A positive measure μ on $\mathbf{D}$ is called a p-Carleson measure in $\mathbf{D}$ if

$$\sup_{I \subset \partial \mathbf{D}} \frac{\mu(S(I))}{|I|^p} < \infty \,, \tag{2.1}$$

where $|I|$ denotes the arc length of I and $S(I)$ denotes the Carleson square based on I,

$$S(I) = \left\{ z \in \mathbf{D} \,:\, 1 - |I| \le |z| < 1 \,,\ \frac{z}{|z|} \in I \right\}.$$

Again, μ is called a vanishing p-Carleson measure if

$$\lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^p} = 0 \,. \tag{2.2}$$

Take $h \in (0,1)$ and $\theta \in [0, 2\pi)$. If we set

$$S(h,\theta) = \left\{ z \in \mathbf{D} \,:\, \left| z - \mathrm{e}^{\mathrm{i}\theta} \right| < h \right\} \,,$$

then we can see that (2.1) and (2.2) are equivalent to

$$\sup_{h \in (0,1),\, \theta \in [0,2\pi)} \frac{\mu(S(h,\theta))}{h^p} < \infty \tag{2.3}$$

and

$$\lim_{h \to 0} \sup_{\theta \in [0,2\pi)} \frac{\mu(S(h,\theta))}{h^p} = 0 \,, \tag{2.4}$$

respectively.

Suppose φ is a holomorphic mapping defined on $\mathbf{D}$. Let $\varphi(\mathbf{D}) \subseteq \mathbf{D}$ and $\psi \in \mathbf{B}_q$ be such that $\psi(z)\varphi'(z)(1 - |z|^2) \in L^q(\mathbf{D}, \lambda)$, where $\mathrm{d}\lambda(z)$ is the Möbius invariant measure defined by $\mathrm{d}\lambda(z) = (1 - |z|^2)^{-2} \mathrm{d}A(z)$. We define the measures μ_q and ν_q on $\mathbf{D}$ by

$$\mu_q(E) = \int_{\varphi^{-1}(E)} \left| \psi(z)\varphi'(z) \right|^q \left(1 - |z|^2\right)^{q-2} \mathrm{d}A(z) \tag{2.5}$$

and

$$\nu_q(E) = \int_{\varphi^{-1}(E)} \left|\psi'(z)\right|^q \left(1-|z|^2\right)^{q-2} \mathrm{d}A(z)\,, \tag{2.6}$$

where E is a measurable subset of the unit disk $\mathbf{D}$.

Take $\psi \in A^q_{q-2}$, then we can define the measure $\nu_{q,\psi}$ on $\mathbf{D}$ by

$$\nu_{q,\psi}(E) = \int_{\varphi^{-1}(E)} |\psi(z)|^q \left(1-|z|^2\right)^{q-2} \mathrm{d}A(z)\,. \tag{2.7}$$

Definition 2.2. Take $1 < p < \infty$. Let μ be a positive measure on $\mathbf{D}$. Then the measure μ is p-Carleson measure for $\mathbf{B}_p$ if there is a constant $K > 0$ such that

$$\int_{\mathbf{D}} \left|f'(\omega)\right|^p \mathrm{d}\mu(\omega) \le K\|f\|^p_{\mathbf{B}_p}$$

for all $f \in \mathbf{B}_p$. That is, the inclusion operator i from $\mathbf{B}_p$ into $\mathbf{D}_p(\mu)$ is bounded.

Further, the measure μ is a vanishing p-Carleson measure for $\mathbf{B}_p$ if the inclusion operator i from $\mathbf{B}_p$ into $\mathbf{D}_p(\mu)$ is compact.

The following characterization of p-Carleson measures is given in [1].

Theorem 2.3. *Take $1 < p < \infty$. Let μ be a positive measure on $\mathbf{D}$. Then the following statements are equivalent:*

(1) *The measure μ is a p-Carleson measure for $\mathbf{B}_p$.*

(2) *There exists a constant $K < \infty$ such that*

$$\mu(S(h,\theta)) \le Kh^p$$

for all $\theta \in [0, 2\pi)$ and $h \in (0,1)$.

(3) *There exists a constant $C < \infty$ such that*

$$\int_{\mathbf{D}} \left|\sigma'_a(z)\right|^p \mathrm{d}\mu(z) \le C$$

for all $a \in \mathbf{D}$.

Theorem 2.4. ([22, Proposition 3.4]) *Take $1 < p < \infty$. Let μ be a positive measure on $\mathbf{D}$. Then the following statements are equivalent:*

(1) *The measure* μ *is a vanishing* p*-Carleson measures for* $\mathbf{B}_p$.

(2) *For all* $a \in \mathbf{D}$, *we have*

$$\lim_{|a|\to 1} \int_{\mathbf{D}} \left(\frac{1-|a|^2}{|1-\overline{a}z|^2} \right)^p \mathrm{d}\mu(z) = 0\,.$$

Using [8, page 163] and [2, Lemma 2.1], we can easily prove the following lemma.

LEMMA 2.5. *Let* φ *be a holomorphic mapping defined on* $\mathbf{D}$ *such that* $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. *Take* $\psi \in \mathbf{B}_q$ *such that* $\psi(z)\varphi^{'}(z)(1-|z|^2) \in L^q(\mathbf{D},\lambda)$. *Then*

$$\int_{\mathbf{D}} g\, \mathrm{d}\mu_q = \int_{\mathbf{D}} \left|\psi(z)\varphi^{'}(z)\right|^q (g\circ\varphi)(z) \left(1-|z|^2\right)^{q-2} \mathrm{d}A(z)$$

and

$$\int_{\mathbf{D}} g\, \mathrm{d}\nu_q = \int_{\mathbf{D}} \left|\psi^{'}(z)\right|^q (g\circ\varphi)(z) \left(1-|z|^2\right)^{q-2} \mathrm{d}A(z)\,,$$

where g *is an arbitrary measurable positive function in* $\mathbf{D}$.

The next result is essential for the proof of Theorem 2.8. The proof follows by similar lines as in the case of composition operators on Besov spaces [22, Lemma 3.8].

LEMMA 2.6. *Given* $1 \leq p, q < \infty$, *let* φ *be a holomorphic mapping defined on* $\mathbf{D}$ *with* $\varphi(\mathbf{D}) \subseteq \mathbf{D}$ *and* $\psi \in \mathbf{B}_q$ *be such that* $W_{\varphi,\psi} : \mathbf{B}_p \to \mathbf{B}_q$ *is bounded. Then* $W_{\varphi,\psi} : \mathbf{B}_p \to \mathbf{B}_q$ *is compact (weakly compact) if and only if whenever* $\{f_n\}$ *is a bounded sequence in* $\mathbf{B}_p$ *converging to zero uniformly on compact subsets of* $\mathbf{D}$, *then* $\|W_{\varphi,\psi}(f_n)\|_{\mathbf{B}_q} \to 0$ *(respectively,* $\{W_{\varphi,\psi}(f_n)\}$ *is a weak null sequence in* $\mathbf{B}_q$*).*

THEOREM 2.7. *Take* $1 < p \leq q < \infty$. *Let* $\varphi \in \mathbf{B}_p$ *be such that* $\varphi(\mathbf{D}) \subseteq \mathbf{D}$ *and* $\psi \in A^q_{q-2}$. *Also, suppose that the measure* $\nu_{q,\psi}$ *is a vanishing* q*-Carleson measure for* $\mathbf{B}_q$. *Then* $W_{\varphi,\psi}$ *defines a bounded operator from* $\mathbf{B}_p$ *into* A^q_{q-2}. *Moreover,* $W_{\varphi,\psi} : \mathbf{B}_p \to A^q_{q-2}$ *is compact.*

Proof. We prove the compactness only. Let $\{f_n\}$ be a bounded sequence in $\mathbf{B}_p$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. Since the measure $\nu_{q,\psi}$ is a vanishing q-Carleson measure for $\mathbf{B}_q$, the inclusion map $i : \mathbf{B}_q \to$

$L^q(\mathbf{D}, \nu_{q,\psi})$ is compact. Since $\mathbf{B}_p \subset \mathbf{B}_q$, we have $\|f_n\|_{L^q(\mathbf{D},\nu_{q,\psi})} \to 0$ as $n \to \infty$. Therefore, by Lemma 2.5, we have

$$\|W_{\varphi,\psi}(f_n)\|^q_{A^q_{q-2}} = \int_{\mathbf{D}} |\psi|^q |f_n \circ \varphi|^q \left(1 - |z|^2\right)^{q-2} \mathrm{d}A(z)$$
$$= \int_{\mathbf{D}} |f_n|^q \, \mathrm{d}\nu_{q,\psi} \quad \longrightarrow \ 0 \ \text{ as } \ n \to \infty\,.$$

Thus, $W_{\varphi,\psi} : \mathbf{B}_p \to A^q_{q-2}$ is compact. ∎

THEOREM 2.8. *Take $1 < p \le q < \infty$. Let $\varphi, \psi \in \mathbf{B}_p$ be such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. Also, suppose that the measure ν_q is a vanishing q-Carleson measure for $\mathbf{B}_q$. Then $W_{\varphi,\psi}$ exists as a bounded operator from $\mathbf{B}_p$ into $\mathbf{B}_q$ if and only if $W_{\varphi,\psi\varphi'}$ exists as a bounded operator from A^p_{p-2} into A^q_{q-2}.*

Proof. First, suppose $W_{\varphi,\psi} : \mathbf{B}_p \to \mathbf{B}_q$ is bounded. Then there exists a constant $C > 0$ such that

$$\|W_{\varphi,\psi}(f)\|_{\mathbf{B}_q} \le C\|f\|_{\mathbf{B}_p} \qquad \text{for all } f \in \mathbf{B}_p\,.$$

Also, by Theorem 2.7, we can find a constant $M > 0$ such that

$$\|W_{\varphi,\psi'}(f)\|_{A^q_{q-2}} \le M\|f\|_{\mathbf{B}_p} \qquad \text{for all } f \in \mathbf{B}_p\,.$$

Take $f \in A^p_{p-2}$ and let the function $g \in \mathbf{B}_p$ be such that $g' = f$ and $g(0) = 0$. Also, we have

$$\begin{aligned}
\left\|W_{\varphi,\psi\varphi'}(f)\right\|_{A^q_{q-2}} &= \left\|\psi\varphi' f \circ \varphi\right\|_{A^q_{q-2}} \\
&= \left\|\psi\varphi' f \circ \varphi + \psi' g \circ \varphi - \psi' g \circ \varphi\right\|_{A^q_{q-2}} \\
&\le \left\|(\psi g \circ \varphi)'\right\|_{A^q_{q-2}} + \left\|\psi' g \circ \varphi\right\|_{A^q_{q-2}} \\
&= \|W_{\varphi,\psi}(g)\|_{\mathbf{B}_q} + \left\|W_{\varphi,\psi'}(g)\right\|_{A^q_{q-2}} \\
&\le (C + M)\|g\|_{\mathbf{B}_p} = (C + M)\|f\|_{A^p_{p-2}}\,.
\end{aligned}$$

Thus, $W_{\varphi,\psi\varphi'} : A^p_{p-2} \to A^q_{q-2}$ is bounded.

Conversely, suppose $W_{\varphi,\psi\varphi'} : A^p_{p-2} \to A^q_{p-2}$ is bounded. Again, by Theorem 2.7, $W_{\varphi,\psi'} : \mathbf{B}_p \to A^q_{q-2}$ is bounded. Take $f \in \mathbf{B}_p$ such that $f(0) = 0$. Then, we have

$$\begin{aligned}\|W_{\varphi,\psi}(f)\|_{\mathbf{B}_q} &= \left\|(\psi f \circ \varphi)'\right\|_{A^q_{q-2}} \\ &= \left\|\psi\varphi' f' \circ \varphi + \psi' f \circ \varphi\right\|_{A^q_{q-2}} \\ &\leq \left\|W_{\varphi,\psi\varphi'}(f')\right\|_{A^q_{p-2}} + \left\|W_{\varphi,\psi'}(f)\right\|_{A^q_{q-2}} < \infty\,.\end{aligned}$$

∎

By using Theorem 2.8 and Theorem 1 of [6], we can prove the following theorem.

THEOREM 2.9. *Take $1 < p \leq q < \infty$. Let $\varphi, \psi \in \mathbf{B}_p$ be such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. Also, suppose that the measure ν_q is a vanishing q-Carleson measure for $\mathbf{B}_q$. Then $W_{\varphi,\psi}$ exists as a bounded operator from $\mathbf{B}_p$ into $\mathbf{B}_q$ if and only if*

$$\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \left(\frac{1-|a|^2}{|1-\overline{a}\omega|^2}\right)^q \mathrm{d}\mu_q(\omega) < \infty\,.$$

THEOREM 2.10. *Take $1 < p \leq q < \infty$. Let $\varphi, \psi \in \mathbf{B}_p$ be such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. Also, suppose that the measure ν_q is a vanishing q-Carleson measure for $\mathbf{B}_q$. Then $W_{\varphi,\psi} : \mathbf{B}_p \to \mathbf{B}_q$ is compact if and only if $W_{\varphi,\psi\varphi'} : A^p_{p-2} \to A^q_{q-2}$ is compact.*

Proof. First, suppose $W_{\varphi,\psi} : \mathbf{B}_p \to \mathbf{B}_q$ is compact. Let $\{f_n\}$ be a bounded sequence in A^p_{p-2} such that $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. For each n, let us consider the function $g_n \in \mathbf{B}_p$ such that $g_n' = f_n$ and $g_n(0) = 0$. The sequence $\{g_n\}$ also converges to zero uniformly on compact subsets of $\mathbf{D}$ as $n \to \infty$. Further, $W_{\varphi,\psi} : \mathbf{B}_p \to \mathbf{B}_q$ is compact, so $\|W_{\varphi,\psi}(g_n)\|_{\mathbf{B}_q} \to 0$ as $n \to \infty$. Again, by Theorem 2.7, $W_{\varphi,\psi'} : \mathbf{B}_p \to A^q_{q-2}$ is compact, so $\|W_{\varphi,\psi'}(g_n)\|_{A^q_{q-2}}$ also converges to zero as $n \to \infty$. Also, we have

$$\begin{aligned}\left\|W_{\varphi,\psi\varphi'}(f_n)\right\|_{A^q_{q-2}} &= \left\|\psi\varphi' f_n \circ \varphi\right\|_{A^q_{q-2}} \\ &\leq \left\|\psi\varphi' f_n \circ \varphi + \psi' g_n \circ \varphi\right\|_{A^q_{q-2}} + \left\|\psi' g_n \circ \varphi\right\|_{A^q_{q-2}}\end{aligned}$$

$$= \left\| (\psi g_n \circ \varphi)' \right\|_{A^q_{q-2}} + \left\| W_{\varphi,\psi'}(g_n) \right\|_{A^q_{q-2}}$$
$$= \left\| W_{\varphi,\psi}(g_n) \right\|_{\mathbf{B}_q} + \left\| W_{\varphi,\psi'}(g_n) \right\|_{A^q_{q-2}}$$
$$\leq (C+M) \left\| g_n \right\|_{\mathbf{B}_p} = (C+M) \left\| f_n \right\|_{A^p_{p-2}} .$$

Therefore, $\left\| W_{\varphi,\psi\varphi'}(f_n) \right\|_{A^q_{q-2}} \to 0$ as $n \to \infty$. Thus, $W_{\varphi,\psi\varphi'} : A^p_{p-2} \to A^q_{q-2}$ is compact.

Conversely, suppose $W_{\varphi,\psi\varphi'} : A^p_{p-2} \to A^q_{q-2}$ is compact. Again, by Theorem 2.7, $W_{\varphi,\psi'} : \mathbf{B}_p \to A^q_{q-2}$ is compact. Let g_n be the same sequence as in the direct part. Then, we have

$$\begin{aligned} \left\| W_{\varphi,\psi}(g_n) \right\|_{\mathbf{B}_q} &= \left\| (\psi g_n \circ \varphi)' \right\|_{A^q_{q-2}} \\ &= \left\| \psi \varphi' g_n' \circ \varphi + \psi' g_n \circ \varphi \right\|_{A^q_{q-2}} \\ &\leq \left\| W_{\varphi,\psi\varphi'}(f_n) \right\|_{A^q_{q-2}} + \left\| W_{\varphi,\psi'}(g_n) \right\|_{A^q_{q-2}} \quad \longrightarrow \; 0 \;\text{ as }\; n \to \infty . \end{aligned}$$

Thus, $W_{\varphi,\psi} : \mathbf{B}_p \to \mathbf{B}_q$ is compact. ∎

By using Theorem 2.10 and Corollary 1 of [6], we can prove the following theorem.

THEOREM 2.11. *Take $1 < p \leq q < \infty$. Let $\varphi, \psi \in \mathbf{B}_p$ be such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. Also, suppose that the measure ν_q is a vanishing q-Carleson measure for $\mathbf{B}_q$. Let $W_{\varphi,\psi}$ exists as a bounded operator from $\mathbf{B}_p$ into $\mathbf{B}_q$. Then the weighted composition operator $W_{\varphi,\psi}$ is compact from $\mathbf{B}_p$ into $\mathbf{B}_q$ if and only if*

$$\limsup_{|a| \to 1} \int_{\mathbf{D}} \left(\frac{1-|a|^2}{|1-\overline{a}\omega|^2} \right)^q \mathrm{d}\mu_q(\omega) = 0 .$$

THEOREM 2.12. *Let $1 < p < \infty$. Let $\varphi, \psi \in \mathbf{B}_p$ be such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. Also, suppose that the measure ν_p is a vanishing p-Carleson measure for $\mathbf{B}_p$. Then $W_{\varphi,\psi} : \mathbf{B}_p \to \mathbf{B}_p$ is bounded (compact) if and only if the measure μ_p is a bounded (respectively vanishing) p-Carleson measure for $\mathbf{B}_p$.*

Proof. We only prove the boundedness. Suppose first that $W_{\varphi,\psi} : \mathbf{B}_p \to \mathbf{B}_p$ is bounded. Then by Theorem 2.8, $W_{\varphi,\psi\varphi'}$ is a bounded operator on A^p_{p-2}. Also, by Theorem 2.7, $W_{\varphi,\psi'} : \mathbf{B}_p \to A^p_{p-2}$ is bounded.

Let $f \in \mathbf{B}_p$ such that $f(0) = 0$. Then, by using Lemma 2.5, we have

$$\left\|W_{\varphi,\psi\varphi'}(f')\right\|^p_{A^p_{p-2}} = \int_{\mathbf{D}} \left|\psi\varphi'(z)\right|^p \left|f'(\varphi(z))\right|^p \left(1-|z|^2\right)^{p-2} \mathrm{d}A(z)$$
$$= \int_{\mathbf{D}} \left|f'(\omega)\right|^p \mathrm{d}\mu_p(\omega)\,.$$

Therefore, we can find a constant $C > 0$ such that

$$\int_{\mathbf{D}} \left|f'(\omega)\right|^p \mathrm{d}\mu_p(\omega) \le C\|f\|^p_{\mathbf{B}_p}.$$

That is, the inclusion operator $i : \mathbf{B}_p \to \mathbf{D}_p(\mu)$ is bounded. Thus the measure μ_p is a bounded p-Carleson measure for $\mathbf{B}_p$.

Conversely, suppose that μ_p is a bounded p-Carleson measure for $\mathbf{B}_p$. Our aim is to prove that $W_{\varphi,\psi} : \mathbf{B}_p \to \mathbf{B}_p$ is bounded. Also, we have $(\psi(f\circ\varphi))' = \psi\varphi'(f'\circ\varphi) + \psi'(f\circ\varphi)$. Take $f \in \mathbf{B}_p$. So by Lemma 2.5, we have

$$\int_{\mathbf{D}} \left|\psi\varphi'(z)\right|^p \left|f'(\varphi(z))\right|^p \left(1-|z|^2\right)^{p-2} \mathrm{d}A(z) = \int_{\mathbf{D}} \left|f'(\omega)\right|^p \mathrm{d}\mu_p(\omega) < \infty\,.$$

Also, by using Theorem 2.7, we get

$$\int_{\mathbf{D}} \left|\psi'(z)\right|^p \left|f(\varphi(z))\right|^p \left(1-|z|^2\right)^{p-2} \mathrm{d}A(z) = \int_{\mathbf{D}} |f(\omega)|^p \,\mathrm{d}\nu_p(\omega) < \infty\,.$$

Hence $W_{\varphi,\psi} : \mathbf{B}_p \to \mathbf{B}_p$ is bounded. ∎

We can also prove the compactness of $W_{\varphi,\psi}$ by using Theorem 2.7 and Theorem 2.10.

3. Essential norm

In this section, we find estimates for the essential norm of $W_{\varphi,\psi}$. The following two lemmas are proved in [5].

Lemma 3.1. *Take* $0 < r < 1$ *and denote* $\mathbf{D}_r = \{z \in \mathbf{D} : |z| < r\}$. *Let* μ *be a positive Borel measure on* $\mathbf{D}$. *Take*

$$\|\mu\|_r = \sup_{|I|\le 1-r} \frac{\mu(S(I))}{|I|^p} \qquad \text{and} \qquad \|\mu\| = \sup_{I\subset\partial\mathbf{D}} \frac{\mu(S(I))}{|I|^p}\,,$$

where I *run through arcs on the unit circle. Let* μ_r *denotes the restriction of measure* μ *to the set* $\mathbf{D}\setminus\mathbf{D}_r$. *Further, if* μ *is a Carleson measure for some Besov space, so is* μ_r *and* $\|\mu_r\| \le M\|\mu\|_r$, *where* $M > 0$ *is a constant.*

Lemma 3.2. *For $0 < r < 1$ and $1 < p < \infty$, let*

$$\|\mu\|_r^* = \sup_{|a|\geq r} \int_{\mathbf{D}} |\sigma_a'(z)|^p \mathrm{d}\mu(z)\,.$$

Moreover, if μ is a Carleson measure for some Besov space, then $\|\mu_r\| \leq K\|\mu\|_r^*$, where K is an absolute constant.

Take $f(z) = \sum_{s=0}^{\infty} a_s z^s$ holomorphic on $\mathbf{D}$. For a positive integer n, define the operators $R_n f(z) = \sum_{s=n+1}^{\infty} a_s z^s$ and $K_n = I - R_n$, where I is the identity map.

Recall that the essential norm of an operator T is defined as:

$$\|T\|_e = \inf\left\{\|T-K\| \,:\, \text{where } K \text{ is compact operator}\right\}.$$

Lemma 3.3. *If T is a bounded linear operator on $\mathbf{B_p}$, then*

$$\lambda \limsup_{n\to\infty} \|TR_n\| \leq \|T\|_e \leq \liminf_{n\to\infty} \|TR_n\|$$

for some positive constant λ independent of T.

Proof. Since $(R_n + K_n)f = f$ for every n, where K_n is a compact operator, we have

$$\|T\|_e \leq \|TR_n + TK_n\|_e \leq \|TR_n\|_e \leq \|TR_n\|$$

so that

$$\|T\|_e \leq \lim_{n\to\infty} \inf \|TR_n\|\,. \tag{3.1}$$

Also, since K_n is a compact operator on $\mathbf{B_p}$, therefore we have

$$\begin{aligned} \|T-K_n\| &\geq \|(T-K_n)R_n\| \\ &= \|(TR_n - K_nR_n\| \geq \|TR_n\| - \|K_nR_n\|\,. \end{aligned} \tag{3.2}$$

Since $R_n \to 0$ pointwise if $p \geq 1$ [25, Corollary 6] and is therefore uniformly bounded. Hence $R_n \to 0$ uniformly on each relatively compact subset of $\mathbf{B_p}$.

Let

$$\langle f, g\rangle = \int_{\mathbf{D}} f'(z)\overline{g'(z)}\,\mathrm{d}A(z)$$

be the integral pairing that identifies $\mathbf{B_p^*}$ and $\mathbf{B_q}$. Then we see that

$$\langle R_n f, g\rangle = \langle f, R_n g\rangle$$

for all n, $f \in \mathbf{B_p}$ and $g \in \mathbf{B_q}$.

Now,

$$\|K_n R_n\| = \|(K_n R_n)^*\| = \|R_n^* K_n^*\| . \tag{3.3}$$

In particular, since K_n^* is compact and the norm of a functional in $\mathbf{B_p^*}$ is less than or equal to the norm of the generating function, it follows that R_n^* also converges to zero on each relatively compact subset of $\mathbf{B_p^*}$. Thus $\|R_n^* K_n^*\| \to 0$. So from (3.2) and (3.3), we get

$$\|T\|_e \geq \|T - K_n\| \geq \lambda \limsup_{n\to\infty} \|TR_n\|$$

for some positive constant λ independent of T. ∎

In the following theorem we give upper and lower estimates for the essential norm of a weighted composition operator.

Theorem 3.4. *Let $\varphi, \psi \in \mathbf{B}_p$ be such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. Also, suppose that the measure ν_p is a vanishing p-Carleson measure for $\mathbf{B}_p$. Suppose $W_{\varphi,\psi}$ is bounded on $\mathbf{B}_p$ Then there are absolute constants $C_1, C_2 \geq 1$ such that*

$$\limsup_{|a|\to 1} \|(W_{\varphi,\psi})\sigma_a\|_{\mathbf{B}_p}^p \;\leq\; \|W_{\varphi,\psi}\|_e^p \;\leq\; C_1 \limsup_{|a|\to 1} \Phi(a) \,+\, C_2 \limsup_{|a|\to 1} \Psi(a)\,,$$

where

$$\Phi(a) = \int_{\mathbf{D}} \left(\frac{1-|a|^2}{|1-\overline{a}\omega|^2}\right)^p \mathrm{d}\mu_p(\omega)$$

and

$$\Psi(a) = \int_{\mathbf{D}} \left(\frac{1-|a|^2}{|1-\overline{a}\omega|^2}\right)^p \mathrm{d}\nu_p(\omega)\,.$$

Proof. First we prove the upper estimate.

Upper estimate: By Lemma 3.3, we have

$$\|W_{\varphi,\psi}\|_e^p \;\leq\; \liminf_{n\to\infty} \|W_{\varphi,\psi} R_n\|_{\mathbf{B}_p}^p \;\leq\; \liminf_{n\to\infty} \sup_{\|f\|_{\mathbf{B}_p}\leq 1} \|(W_{\varphi,\psi} R_n) f\|_{B_p}^p.$$

Thus

$$\|(W_{\varphi,\psi} R_n) f\|_{\mathbf{B}_p}^p = |\psi(0)(R_n f(\varphi(0))| + \left\|(\psi R_n f \circ \varphi)'\right\|_{A_{p-2}}^p .$$

Now the term $|\psi(0)(R_n f(\varphi(0))|$ is bounded as $n \to \infty$. So, by using Lemma 2.5, we have

$$\begin{aligned}
\left\|(W_{\varphi,\psi}R_n)f\right\|_{\mathbf{B}_p}^p &= \int_{\mathbf{D}} \left|(\psi(z)(R_n f(\varphi(z))))'\right|^p \left(1-|z|^2\right)^{p-2} \mathrm{d}A(z) \\
&\le \int_{\mathbf{D}} \left|\psi(z)\varphi'(z)\right|^p \left|(R_n f)'(\varphi(z))\right|^p \left(1-|z|^2\right)^{p-2} \mathrm{d}A(z) \\
&\quad + \int_{\mathbf{D}} \left|\psi'(z)\right|^p |(R_n f)(\varphi(z))|^p \left(1-|z|^2\right)^{p-2} \mathrm{d}A(z) \\
&= \int_{\mathbf{D}} \left|(R_n f)'(\omega)\right|^p \mathrm{d}\mu_p(\omega) + \int_{\mathbf{D}} |(R_n f)(\omega)|^p \mathrm{d}\nu_p(\omega) \\
&= I_1 + I_2 \,. \qquad (3.4)
\end{aligned}$$

The last condition follows by using Theorem 2.7 and Theorem 2.8.

However, for any $0 < r < 1$, we first take the integral I_2,

$$\int_{\mathbf{D}} \left|(R_n f)(\omega)\right|^p \mathrm{d}\nu_p(\omega) = \int_{\mathbf{D}\setminus\mathbf{D}_r} \left|(R_n f)(\omega)\right|^p \mathrm{d}\nu_p(\omega) + \int_{\mathbf{D}_r} \left|(R_n f)(\omega)\right|^p \mathrm{d}\nu_p(\omega) \,.$$

Since the measure ν_p is a bounded p-Carleson measure. Using [5, Proposition 3], we have

$$\left|R_n f(\omega)\right| \le \left\|f\right\|_{\mathbf{B}_p} \left\|R_n K_\omega\right\|_{\mathbf{B}_q} .$$

Also we have

$$\left|(R_n f)'(\omega)\right| \le \left\|f'\right\|_{A^p_{p-2}} \left\|(R_n K_\omega)'\right\|_{A^q_{q-2}} .$$

Take $0 < r < 1$ and $|\omega| \le r$, $z \in \mathbf{D}$. Also, take the Taylor expansion of

$$K_\omega = \sum_{k=1}^{\infty} (k+1) w^{-k} z^k .$$

Using this Taylor expansion, we get that

$$\left|R_n K_\omega(z)\right| \le \sum_{k=n+1}^{\infty} (k+1) r^k \,.$$

Also, we have

$$\left|(R_n K_\omega)'(z)\right| \le \sum_{k=n+1}^{\infty} k(k+1) r^{k-1} .$$

Thus for any $\epsilon > 0$, we can find n large enough such that

$$\int_{\mathbf{D}} \left|(R_n K_\omega)'(z)\right|^q \left(1-|z|^2\right)^{q-2} \mathrm{d}A(z) < \epsilon^q . \tag{3.5}$$

Therefore, for a fixed r, we have

$$\sup_{\|f\|_{\mathbf{B_p}} \le 1} \int_{\mathbf{D}_r} \left|(R_n f)(\omega)\right|^p \mathrm{d}\nu_p(\omega) \quad \longrightarrow \; 0 \;\text{ as }\; n \to \infty .$$

On the other hand, let $\nu_{p,r}$ denote the restriction of measure ν_p to the set $\mathbf{D} \setminus \mathbf{D}_r$. So by using Lemma 3.2 and Theorem 2.3, we have

$$\begin{aligned}\int_{\mathbf{D}\setminus\mathbf{D}_r} \left|(R_n f)(\omega)\right|^p \mathrm{d}\nu_{p,r}(\omega) &\le K \left\|\nu_{p,r}\right\| \left\|(R_n f)\right\|_{\mathbf{B}_p}^p \\ &\le K\, M \left\|\nu_p\right\|_r^* \left\|f\right\|_{\mathbf{B}_p}^p \le K\, M \left\|\nu_p\right\|_r^* ,\end{aligned}$$

where K and M are absolute constants and $\left\|\nu_p\right\|_r^*$ is defined as in Lemma 3.2.

Now, we take the integral I_1,

$$\begin{aligned}&\int_{\mathbf{D}} \left|(R_n f)'(\omega)\right|^p \mathrm{d}\mu_p(\omega) = \\ &\qquad \int_{\mathbf{D}\setminus\mathbf{D}_r} \left|(R_n f)'(\omega)\right|^p \mathrm{d}\mu_p(\omega) + \int_{\mathbf{D}_r} \left|(R_n f)'(\omega)\right|^p \mathrm{d}\mu_p(\omega) .\end{aligned}$$

Also, the measure μ_p is a bounded p-Carleson measure, because the operator $W_{\varphi,\psi}$ is bounded on $\mathbf{B}_p$.

Therefore, for a fixed r, we have

$$\sup_{\|f\|_{\mathbf{B_p}} \le 1} \int_{\mathbf{D}_r} \left|(R_n f)'(\omega)\right|^p \mathrm{d}\mu_p(\omega) \quad \longrightarrow \; 0 \;\text{ as }\; n \to \infty .$$

Again, let $\mu_{p,r}$ denote the restriction of measure μ_p to the set $\mathbf{D} \setminus \mathbf{D}_r$. So by using Lemma 3.2 and Theorem 2.3, we get

$$\begin{aligned}\int_{\mathbf{D}\setminus\mathbf{D}_r} \left|(R_n f)'(\omega)\right|^p \mathrm{d}\mu_{p,r}(\omega) &\le K \left\|\mu_{p,r}\right\| \left\|(R_n f)'\right\|_{A_{p-2}^p}^p \\ &\le K_1 M_1 \left\|\mu_p\right\|_r^* \left\|f\right\|_{\mathbf{B}_p}^p \le K_1 M_1 \left\|\mu_p\right\|_r^* ,\end{aligned}$$

where K_1 and M_1 are absolute constants and $\|\mu_p\|_r^*$ is defined as in Lemma 3.2. Therefore,

$$\lim_{n\to\infty} \inf \sup_{\|f\|_{\mathbf{B}_p}\le 1} \left\|(W_{\varphi,\psi}R_n)f\right\|_{\mathbf{B}_p}^p \le \lim_{n\to\infty} \inf KM\left\|\mu_p\right\|_r^* + \lim_{n\to\infty} \inf K_1M_1\left\|\nu_p\right\|_r^* .$$

Thus,

$$\left\|W_{\varphi,\psi}\right\|_e^p \le KM\left\|\mu_p\right\|_r^* + K_1M_1\left\|\nu_p\right\|_r^* .$$

Taking $r \to 1$, we have

$$\begin{aligned}
\left\|W_{\varphi,\psi}\right\|_e^p &\le KM \lim_{r\to 1} \left\|\mu_p\right\|_r^* + K_1M_1 \lim_{r\to 1} \left\|\nu_p\right\|_r^* \\
&= KM \limsup_{|a|\to 1} \int_{\mathbf{D}} \left|\sigma_a^{'}(\omega)\right|^p \mathrm{d}\mu_p(\omega) \\
&\quad + K_1M_1 \limsup_{|a|\to 1} \int_{\mathbf{D}} \left|\sigma_a^{'}(\omega)\right|^p \mathrm{d}\nu_p(\omega) \\
&= KM \limsup_{|a|\to 1} \int_{\mathbf{D}} \left(\frac{1-|a|^2}{1-\overline{a}\omega|^2}\right)^p \mathrm{d}\mu_p(\omega) \\
&\quad + K_1M_1 \limsup_{|a|\to 1} \int_{\mathbf{D}} \left(\frac{1-|a|^2\,{}^2}{|1-\overline{a}\omega|}\right)^p \mathrm{d}\nu_p(\omega) \\
&= KM \limsup_{|a|\to 1} \Phi(a) + K_1M_1 \limsup_{|a|\to 1} \Psi(a) ,
\end{aligned}$$

which is the desired upper bound.

Lower bound: The set $\{\sigma_a : a \in \mathbf{D}\}$ is bounded in $\mathbf{B}_p$. Also, $\sigma_a - a \to 0$ as $|a| \to 1$ uniformly on compact sets in $\mathbf{D}$, since

$$|\sigma_a(z) - a| = |z| \frac{1-|a|^2}{|1-\overline{a}z|} .$$

Also, fix a compact operator K on $\mathbf{B}_p$. Then $\|K(\sigma_a - a)\|_{\mathbf{B}_p} \to 0$ as $|a| \to 1$. Thus $\|K(\sigma_a)\|_{\mathbf{B}_p} \to 0$ as $|a| \to 1$. Therefore

$$\left\|W_{\varphi,\psi}\right\|_e^p \ge \left\|W_{\varphi,\psi} - K\right\|_{\mathbf{B}_p}^p \ge \limsup_{|a|\to 1} \left\|(W_{\varphi,\psi})\sigma_a\right\|_{\mathbf{B}_p}^p .$$

∎

By using Theorem 2.8, Theorem 3.4 and Theorem 2 from [6], we can prove the following result.

THEOREM 3.5. *Let $\varphi, \psi \in \mathbf{B}_p$ be such that $\varphi(\mathbf{D}) \subseteq \mathbf{D}$. Also, suppose that the measure ν_q is a vanishing q-Carleson measure for $\mathbf{B}_q$. Suppose $W_{\varphi,\psi}$ is bounded from $\mathbf{B_p}$ into $\mathbf{B_q}$. Then there is an absolute constant $C \geq 1$ such that*

$$\limsup_{|a|\to 1} \int_{\mathbf{D}} \left(\frac{1-|a|^2}{|1-\overline{a}\omega|^2} \right)^q \mathrm{d}\mu_q \leq \left\| W_{\varphi,\psi\varphi'} \right\|_e^q \leq C \limsup_{|a|\to 1} \int_{\mathbf{D}} \left(\frac{1-|a|^2}{|1-\overline{a}\omega|^2} \right)^q \mathrm{d}\mu_q \,.$$

4. WEIGHTED COMPOSITION OPERATORS BETWEEN S^p SPACES

In this section, we find estimates for the essential norm of weighted composition operators.

Take $f \in H^p$. Then by Fatou's theorem, the radial limits $f^*\left(\mathrm{e}^{\mathrm{i}\theta}\right) = \lim_{r\to 1^-} f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)$ exists almost everywhere on $\partial\mathbf{D}$ and $f^* \in L^p(\partial\mathbf{D}, \mathrm{d}m)$, where $\mathrm{d}m(z)$ is the normalized measure on $\partial\mathbf{D}$. We also denote this radial limit by f. Take $\varphi : \mathbf{D} \to \mathbf{D}$ and $\psi \in H(\mathbf{D})$ such that $\psi\varphi' \in H^q$. We define the measure $\mu_{\varphi,\psi\varphi',q}$ on $\overline{\mathbf{D}}$ by

$$\mu_{\varphi,\psi\varphi',q}(E) = \int_{\varphi^{-1}(E)\cap\partial\mathbf{D}} \left|\psi\varphi'\right|^q \mathrm{d}m \,,$$

where E is a measurable subset of the closed unit disk $\overline{\mathbf{D}}$.

THEOREM 4.1. ([3]) *Take $1 \leq p, q \leq \infty$. Let $\varphi \in H(\mathbf{D})$ be such that $\varphi(\mathbf{D}) \subset \mathbf{D}$ and $\psi \in S^q$. Then $W_{\varphi,\psi}$ exists as a bounded operator from S^p into S^q if and only if $W_{\varphi,\psi\varphi'}$ exists as a bounded operator from H^p into H^q.*

Moreover, if $(p, q) \neq (1, \infty)$, then $W_{\varphi,\psi} : S^p \to S^q$ is compact if and only if $W_{\varphi,\psi\varphi'} : H^p \to H^q$ is compact.

By using Theorem 4.1 and Theorem 4 of [6], we can prove the following result.

THEOREM 4.2. *Take $1 \leq p \leq q < \infty$. Let $\varphi \in H(\mathbf{D})$ be such that $\varphi(\mathbf{D}) \subset \mathbf{D}$ and $\psi \in S^q$. Then the weighted composition operator $W_{\varphi,\psi}$ defines a bounded operator from S^p into S^q if and only if*

$$\sup_{a\in\mathbf{D}}\int_{\partial\mathbf{D}}\left(\frac{1-|a|^2}{|1-\overline{a}\omega|^2}\right)^{\frac{q}{p}}\mathrm{d}\mu_{\varphi,\psi\varphi',q}(\omega)<\infty\,.$$

By using Theorem 4.1 and Theorem 5 of [6], the following results follow.

THEOREM 4.3. *Take $1\le p\le q<\infty$. Let $\varphi\in H(\mathbf{D})$ be such that $\varphi(\mathbf{D})\subset\mathbf{D}$ and $\psi\in S^q$. Let $W_{\varphi,\psi}$ be bounded from S^p into S^q. Then there is an absolute constant $C\ge 1$ such that*

$$\limsup_{|a|\to 1}\int_{\partial\mathbf{D}}\left(\frac{1-|a|^2}{|1-\overline{a}\omega|^2}\right)^{\frac{q}{p}}\mathrm{d}\mu_{\varphi,\psi\varphi',q}(\omega)$$
$$\le\|W_{\varphi,\psi\varphi'}\|_e^q\le C\limsup_{|a|\to 1}\int_{\partial\mathbf{D}}\left(\frac{1-|a|^2}{|1-\overline{a}\omega|^2}\right)^{\frac{q}{p}}\mathrm{d}\mu_{\varphi,\psi\varphi',q}(\omega)\,.$$

THEOREM 4.4. *Take $1\le p\le q<\infty$. Let $\varphi\in H(\mathbf{D})$ be such that $\varphi(\mathbf{D})\subset\mathbf{D}$ and $\psi\in S^q$. Then the weighted composition operator $W_{\varphi,\psi}$ defines a bounded operator from S^p into S^q if and only if*

$$\sup_{a\in\mathbf{D}}\int_{\partial\mathbf{D}}\left(\frac{1-|a|^2}{|1-\overline{a}\omega|^2}\right)^{\frac{q}{p}}\mathrm{d}\mu_{\varphi,\psi\varphi',q}(\omega)<\infty\,.$$

By using Theorem 4.1 and Proposition 2 of [6], we can prove the following theorem.

THEOREM 4.5. *Take $1\le q<p<\infty$. Let $\varphi\in H(\mathbf{D})$ be such that $\varphi(\mathbf{D})\subset\mathbf{D}$ and $\psi\in S^q$. Then $W_{\varphi,\psi}$ is bounded from S^p into S^q if and only if*

$$\int_0^{2\pi}\left(\int_{\Gamma(\theta)}\frac{\mathrm{d}\mu_{\varphi,\psi\varphi',q}(\omega)}{1-|\omega|^2}\right)^{\frac{p}{p-q}}\mathrm{d}\theta<\infty\,,$$

where $\Gamma(\theta)$ is the Stolz angle at θ, which is defined for real θ as the convex hull of the set $\left\{\mathrm{e}^{\mathrm{i}\theta}\right\}\cup\left\{z\,:\,|z|<\sqrt{1/2}\right\}$.

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